$L^\infty$-Error Estimate of Schwarz Algorithm for Noncoercive Variational Inequalities

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ABSTRACT

The Schwarz method for a class of elliptic variational inequalities with noncoercive operator was studied in this work. The author proved the error estimate in $L^\infty$-norm for two domains with overlapping nonmatching grids using the geometrical convergence of solutions and the uniform convergence of subsolutions.

KEYWORDS

Variational Inequalities; Schwarz Method; Subsolutions; $L^\infty$-Error Estimates

1. Introduction

More than one hundred years ago, Schwarz algorithms were proposed for proving the solvability of PDEs on a complicated domain. With parallel calculators, this rediscovery of these methods as algorithms of calculations was based on a modern variational approach. Pierre-Louis Lions was the starting point of an intense research activity to develop this tool of calculation, see, e.g., [1,2] and the references therein [3-9].

In this paper, we give a new approach to the finite element approximation for the problem of variational inequality with noncoercive operator. This problem arises in stochastic control (see [10]). We consider a domain which is the union of two overlapping sub-domains where each sub-domain has its own generated triangulation. To prove the main result of this work, we construct two sequences of subsolutions and estimate the errors between Schwarz iterates and the subsolutions. The proof stands on a Lipschitz continuous dependency with respect to the source term for variational inequality, while in [5] the proof stands on a Lipschitz continuous dependency with respect to the boundary condition.

The paper is organized as follows. In Sections 2, we introduce the continuous and discrete obstacle problem as well as Schwarz algorithm with two sub-domains and give the geometrical convergence theorem. In Section 3, we establish two sequences of subsolutions and their error estimates and prove a main result concerning the error estimate of solution in the $L^\infty$-norm, taking into account the combination of geometrical convergence and uniform convergence [11,12] of finite element approximation.

2. Schwarz Algorithm for Variational Inequalities with Noncoercive Operator

2.1. Notations and Assumptions

Let’s consider functions

$$a_i, a_j, a_0 \in C^2(\Omega); 1 \leq i, j \leq 2$$

such that
\[
\sum_{i\leq j\leq 2} a_{ij} \xi_i \xi_j \geq \alpha \sum_{i\leq j\leq 2} \xi_i^2; \xi \in \mathbb{R}^2, \alpha > 0
\]
\[
a_{ij}(x) = a_{ij}(x); a_0(x) \geq a_0 > 0
\]

where \( \Omega \) is a connected bounded domain in \( \mathbb{R}^2 \) with sufficiently regular boundary \( \partial \Omega \).

We define a second order differential operator
\[
A = -\sum_{i\leq j\leq 2} \frac{\partial}{\partial x_j} \left( a_{ij} \frac{\partial}{\partial x_i} \right) + \sum_{i\leq j\leq 2} a_{ij} \frac{\partial}{\partial x_j} + a_0
\]
where the bilinear form associated:
\[
a(u,v) = \int_{\Omega} \left( \sum_{i\leq j\leq 2} a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i} + \sum_{i\leq j\leq 2} a_{ij} \frac{\partial u}{\partial x_j} v + a_0 uv \right) dx
\]

Let \( f \) be a function in \( L^\infty(\Omega) \)
an obstacle
\[
\psi \in W^{2,\infty}(\Omega)
\]
a regular function \( g \) defined on \( \partial \Omega \) such that
\[
g \in W^{2-p}(\Omega), 2 \leq p < \infty, g \leq \psi \text{ on } \partial \Omega
\]

AM and a nonempty convex set
\[
K_{(\psi,g)} = \{ v \in H^1(\Omega) / v-g \in H^1_0(\Omega), v \leq \psi \text{ in } \Omega \}.
\]

We assume there exists \( \lambda > 0 \) large enough and a constant \( \beta > 0 \) such that
\[
\beta \|v\|_{L^2(\Omega)}^2 \leq a(v,v) + \lambda \int_{\Omega} v^2 dx.
\]

Putting
\[
b(u,v) = a(u,v) + \lambda \int_{\Omega} uv dx \ \forall u,v \in H^1(\Omega)
\]
then the bilinear form \( b(\cdot,\cdot) \) is strongly coercive.

Let \( u \in K_{(\psi,g)} \) be the solution of variational inequality (V.I)
\[
a(u,v-u) \geq (f,v-u) \ \forall v \in K_{(\psi,g)}
\]
which is equivalent to
\[
\begin{cases}
\text{Find } u \in K_{(\psi,g)} \text{ solution of} \\
b(u,v-u) \geq (f+\lambda u,v-u) \ \forall v \in K_{(\psi,g)}
\end{cases}
\]

\((\cdot,\cdot)\) denotes the usual inner product in \( L^2(\Omega) \).

We define \( \pi = \sigma(f+\lambda w) \in K_{(\psi,g)} \) the solution of the following V.I
\[
b(\pi,v-\pi) \geq (f+\lambda w,v-\pi) \ \forall v \in K_{(\psi,g)}
\]

where \( w \in L^\infty(\Omega) \) and \( \sigma \) is a mapping from \( L^\infty(\Omega) \) into itself.

**Remark 1.** We call quasi-variational inequality \( (Q.V.I) \) if the right hand side \( (f+\lambda u) \) depends of solution \( u \), in the contrary case we call variational inequality \( (V.I) \).

### 2.2. Some Preliminary Results on the V.I Noncoercive

Thanks to [10], the problem (12) has one and only one solution, moreover \( u \) satisfies the regularity property
\[
u \in W^{2,p}(\Omega), 2 \leq p < \infty.
\]

We give a monotonicity property of the solution with respect to both the source term, the boundary condition
and the obstacle. Let \((f, \psi, g); (\tilde{f}, \tilde{\psi}, \tilde{g})\) be a pair of data and \(u = \partial (f, \psi, g); \tilde{u} = \partial (\tilde{f}, \tilde{\psi}, \tilde{g})\) the corresponding solution of V.I (12).

**Lemma 1** [10] Under the preceding notations and assumptions (1) to (11), if \(f \geq \tilde{f}, \psi \geq \tilde{\psi}\) and \(g \geq \tilde{g}\), then \(\partial (f, \psi, g) \geq \partial (\tilde{f}, \tilde{\psi}, \tilde{g})\).

Let \(X\) be the set of sub-solutions of the Q.V.I, i.e., all the \(\overline{w} \in K_{(\nu, \delta)}\) such that

\[
b(\overline{w}, v) \leq (f + \lambda \overline{w}, v), \quad v \geq 0, v \in H^1_0(\Omega)
\]

that is equivalent to

\[
a(\overline{w}, v) \leq (f, v), \quad v \geq 0, v \in H^1_0(\Omega).
\]

**Lemma 2** [10] Under the preceding notations and assumptions (1) to (11), the solution \(u\) of problem (12) is the maximum element of the set \(X\).

We show the Lipschitz property, which gives the continuous dependence to the data \(f\).

**Lemma 3** Under the preceding notations and assumptions (1) to (11), we have

\[
\| u - \tilde{u} \|_{C^0(\Omega)} \leq \max \left\{ c \left\| f - \tilde{f} \right\|_{C^0(\Omega)}, \left\| g - \tilde{g} \right\|_{C^0(\Omega)}, \left\| \psi - \tilde{\psi} \right\|_{C^0(\Omega)} \right\}
\]

where \(c\) is an independent constant of data.

**Proof** Firstly, let

\[
\Phi = \max \left\{ \frac{1}{\alpha_0} \left\| f - \tilde{f} \right\|_{C^0(\Omega)}, \left\| g - \tilde{g} \right\|_{C^0(\Omega)}, \left\| \psi - \tilde{\psi} \right\|_{C^0(\Omega)} \right\}
\]

we have

\[
a(\Phi, v - u) = \Phi(a_0, v - u)
\]

then

\[
a(u + \Phi, v - u) \geq (f + a_0 \Phi, v - u)
\]

and

\[
a(u + \Phi, (v + \Phi) - (u + \Phi)) \geq (f + a_0 \Phi, (v + \Phi) - (u + \Phi))
\]

if we put

\[
u + \Phi = \tilde{v} \quad \text{and} \quad v + \Phi = \tilde{v}
\]

then

\[
a(\tilde{u}, \tilde{v} - \tilde{u}) \geq (f + a_0 \Phi, \tilde{v} - \tilde{u})
\]

therefore

\[
\partial (f + a_0 \Phi, \psi + \Phi, g + \Phi) = \tilde{u} = \partial (f, \psi, g) + \Phi.
\]

Secondly, it is clear that

\[
\tilde{f} \leq f + \left\| f - \tilde{f} \right\|_{C^0(\Omega)} \leq f + \frac{a_0}{\alpha_0} \left\| f - \tilde{f} \right\|_{C^0(\Omega)} \leq f + a_0 \Phi
\]

and

\[
\tilde{\psi} \leq \psi + \Phi
\]

\[
\tilde{g} \leq g + \Phi
\]

so, due to lemma 1, we get

\[
\partial (\tilde{f}, \tilde{\psi}, \tilde{g}) \leq \partial (f + a_0 \Phi, \psi + \Phi, g + \Phi) = \partial (f, \psi, g) + \Phi
\]

which gives

\[
\partial (\tilde{f}, \tilde{\psi}, \tilde{g}) - \partial (f, \psi, g) \leq \Phi
\]

by changing the roles of \((f, \psi, g)\) and \((\tilde{f}, \tilde{\psi}, \tilde{g})\), we obtain
which completes the proof.

**Remark 2** If \( \psi = \bar{\psi} \) and \( g = \bar{g} \), then we have

\[
\|u - \bar{u}\|_{\infty, \Omega} \leq c \|f - \bar{f}\|_{\infty, \Omega}.
\]

Let \( \Omega \) be decomposed into triangles and let \( \tau^h \) denote the set of those elements; \( h > 0 \) is the mesh-size. We assume the triangulation \( \tau^h \) is regular and quasi-uniform. Let \( V_h \) denote the standard piecewise linear finite element space and by \( \varphi_i, i = 1, 2, \ldots, m(h) \), the basis functions of the space \( V_h \). Let \( r_h \) be the usual restriction operator in \( \Omega \). The discrete counterpart of (13) consists of finding \( u_h \in K_h \) solution of

\[
b(u_h, v_h - u_h) \geq (f + \lambda u_h, v_h - u_h) \quad \forall v_h \in K_h
\]

where

\[
K_h = \{ v_h \in V_h / v_h = \pi_h g \text{ on } \partial\Omega, v_h \leq r_h v \text{ in } \tau^h \}
\]

\( \pi_h \) is an interpolation operator on \( \partial\Omega \).

We shall assume that the matrix \( B \) defined by

\[
B_{ij} = b(\varphi_i, \varphi_j) = a(\varphi_i, \varphi_j) + \lambda \int_{\Omega} \varphi_i \varphi_j \, dx
\]

is \( M \)-matrix [13] (i.e. angles of triangles of \( \tau^h \) are \( \leq \pi/2 \)).

### 2.3. The Continuous Schwarz Algorithm

Consider the model obstacle problem: find \( u \in K_{(0,0)} \) such that

\[
b(u, v - u) \geq (f + \lambda u, v - u) \quad \forall v \in K_{(0,0)}
\]

where \( K_{(0,0)} \) defined in (9) with \( g = 0 \).

We decompose \( \Omega \) into two overlapping polygonal subdomains \( \Omega_i \) and \( \Omega_j \) such that

\[
\Omega = \bigcup \Omega_i, \quad \Omega_i \cap \Omega_j \neq \emptyset
\]

and \( u \) satisfies the local regularity property

\[
u_{\Omega_i} \in W^{2,p}(\Omega_i), \quad 2 \leq p < \infty
\]

we denote \( \partial\Omega_i \) the boundary of \( \Omega_i \), and \( \Gamma_i = \partial\Omega_i \cap \partial\Omega_j \). The intersection of \( \Gamma_i \) and \( \Gamma_j \); \( i \neq j \) is assumed to be empty. We will always assume to simplify that \( \Gamma_i, \Gamma_j \) are smooth.

For \( w \in C^0(\Gamma_j) \), we define

\[
V_i^w = \{ v \in H^1(\Omega_i) / v = 0 \text{ on } \partial\Omega_i \cap \partial\Omega_j, v = w \text{ on } \Gamma_j \}; \quad i = 1, 2.
\]

We associate with problem (19) the following system: find \( (u_1, u_2) \in V_1 \times V_2 \) solution of

\[
\begin{align*}
b_1(u_1, v_1 - u_1) &\geq (f_1 + \lambda u_1, v_1 - u_1) \quad \forall v_1 \in V_1, \\
u_1 &\leq \psi, v_1 \leq \psi \quad \text{in } \Omega_1 \\
b_2(u_2, v_2 - u_2) &\geq (f_2 + \lambda u_2, v_2 - u_2) \quad \forall v_2 \in V_2, \\
u_2 &\leq \psi, v_2 \leq \psi \quad \text{in } \Omega_2
\end{align*}
\]

where

\[
b_i(u, v) = \int_{\Omega_i} \left( \sum_{j \neq i} a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_j} + \sum_{j \neq i} a_{ij} \frac{\partial u}{\partial x_j} + a_{ii} \nu v + \lambda uv \right) \, dx; \quad i = 1, 2
\]

\[
f_i = \int_{\Omega_i} u_i = u_{\mid \Omega_i}.
\]

Starting from \( u^0 = \psi \), we define the continuous Schwarz sequences \( (u_i^{(n)}) \) on \( \Omega_i \) such that \( u_i^{(n+1)} \in V_i^{(n)} \) solves
and \( \left( u^{n+1}_2 \right) \) on \( \Omega_2 \) such that \( u^{n+1}_2 \in V_2^{n+1} \) solves

\[
\begin{aligned}
\mathcal{B}_2 \left( u^{n+1}_2, v - u^{n+1}_2 \right) &\geq \left( f_2 + \lambda u^n_2, v - u^{n+1}_2 \right) \quad \forall v \in V_2^{n+1} \\
u^{n+1}_2 &\leq \psi, v \leq \psi \quad \text{in} \ \Omega_2
\end{aligned}
\]

(22)

where

\[
u^0_1 = u^0 \quad \text{in} \ \Omega_1, \quad u^0_2 = u^0 \quad \text{in} \ \Omega_2,
\]

\[
u^{n+1}_1 = 0 \quad \text{in} \ \bar{\Omega} \setminus \bar{\Omega}_1 \quad \text{and} \quad u^{n+1}_2 = 0 \quad \text{in} \ \bar{\Omega} \setminus \bar{\Omega}_2.
\]

The following geometrical convergence is due to (2), pages 51-63

**Theorem 1** The sequences \( \left( u^{n+1}_i \right) \) and \( \left( u^{n+1}_2 \right), n \geq 0 \) of the Schwarz algorithm converge geometrically to the solution of the problem (20). More precisely, there exist two constants \( k_1, k_2 \in [0,1] \) such that for all \( n > 0 \)

\[
\begin{aligned}
\| u^n_1 - u^{n+1}_1 \|_{L^2(\Omega_1)} &\leq k_1^n \| u^0 - u \|_{L^2(\Gamma_1)} \\
\| u^n_2 - u^{n+1}_2 \|_{L^2(\Omega_2)} &\leq k_2^n \| u^0 - u \|_{L^2(\Gamma_2)}
\end{aligned}
\]

(23)

2.4. The Discretization

For \( i = 1, 2 \); let \( \tau^h \) be a standard regular and quasi-uniform finite element triangulation in \( \Omega_i \), \( h^i \) being the mesh size. We assume that the two triangulations are mutually independent on \( \bar{\Omega} \cap \bar{\Omega}_i \), where a triangle belonging to one triangulation does not necessarily belong to the other. Let \( V^h_i = V_i^h \left( \Omega_i \right) \) be the space of continuous piecewise linear functions on \( \tau^h_i \) which vanish on \( \partial \Omega_i \cap \partial \Omega_i \). For \( w \in C^0 \left( \Gamma_i \right) \) we define

\[
V_{\Omega}^{h,w} = \left\{ v_h \in V_h / v_h = 0 \text{ on} \ \partial \Omega \cap \partial \Omega_i, v_h = \pi_h \left( w \right) \text{ on} \ \Gamma_i \right\}; \ i = 1, 2
\]

where \( \pi_h \) denotes a suitable interpolation operator on \( \Gamma_i \). We give the discrete counterparts of Schwarz algorithm defined in (21) and (22) as follows.

Starting from \( u^{h_1}_0 = r_1 \psi \), we define the discrete Schwarz sequence \( u^{n+1}_{1h} \) on \( \Omega_1 \) such that \( u^{n+1}_{1h} \in V_{1h}^{n+1} \) solves

\[
\begin{aligned}
\mathcal{B}_{h_1} \left( u^{n+1}_{1h}, v - u^{n+1}_{1h} \right) &\geq \left( f_1 + \lambda u^n_{1h}, v - u^{n+1}_{1h} \right) \quad \forall v \in V_{1h}^{n+1} \\
u^{n+1}_{1h} &\leq \psi, v \leq \psi \quad \text{in} \ \Omega_1
\end{aligned}
\]

(23)

and on \( \Omega_2 \) the sequence \( u^{n+1}_{2h} \in V_{2h}^{n+1} \) solves

\[
\begin{aligned}
\mathcal{B}_{h_2} \left( u^{n+1}_{2h}, v - u^{n+1}_{2h} \right) &\geq \left( f_2 + \lambda u^n_{2h}, v - u^{n+1}_{2h} \right) \quad \forall v \in V_{2h}^{n+1} \\
u^{n+1}_{2h} &\leq \psi, v \leq \psi \quad \text{in} \ \Omega_2
\end{aligned}
\]

(24)

We will also always assume that the respective matrices resulting from problems (23) and (24) are \( M \) -matrices.

3. Error Analysis

This section is devoted to the proof of the main result of this work. For that, we begin by introducing two auxiliary sequences.

3.1. Auxiliary Schwarz Sequences

To simplify the notation, we take
Let $\overline{u}^{n+1} = \sigma_h \left( f_i + \lambda u^n \right) \in V_h \left( \Omega \right)$ be the solution of discrete V.I

$$
\begin{aligned}
& b_h \left( \overline{u}^{n+1}_h, v_h - \overline{u}^{n+1}_h \right) \geq \left( f_i + \lambda u^n, v_h - \overline{u}^{n+1}_h \right) \\
& \forall v_h \in V_h \left( \Omega \right), \overline{u}^{n+1}_h \leq r_i \psi, v_h \leq r_i \psi \text{ in } \tau^i, \\
& \overline{u}^{n+1}_h = \pi_h u^{n+1}_h, v_h = \pi_h u^{n+1}_h \text{ on } \Gamma_1
\end{aligned}
$$

(25)

where $u^{n+1}_i = \sigma \left( f_i + \lambda u^n \right)$; $i = 1, 2$ is the solution of continuous V.I (21) (resp. (22)) and let $\overline{u}^{(h),n+1}_i = \sigma \left( f_i + \lambda u^n \right) \in V_i$ be the solution of continuous V.I

$$
\begin{aligned}
& b_h \left( \overline{u}^{(h),n+1}_i, v - \overline{u}^{(h),n+1}_i \right) \geq \left( f_i + \lambda u^n, v - \overline{u}^{(h),n+1}_i \right) \\
& \forall v \in H^1 \left( \Omega \right), \overline{u}^{(h),n+1}_i \leq \psi, v \leq \psi \text{ in } \Omega, \\
& \overline{u}^{(h),n+1}_i = u^{n+1}_i, v = u^{n+1}_i \text{ on } \Gamma_1
\end{aligned}
$$

(26)

where $u^{n+1}_i = \sigma_h \left( f_i + \lambda u^n \right)$; $i = 1, 2$ is the solution of discrete V.I (23) (resp. (24)).

It is clear that $\overline{u}^{n+1}_h$ is the finite element approximation of $u^n$. Then, as $\|f_i + \lambda u^n\| \leq c$ (independent of $n$), therefore, we apply the error estimate for variational inequality (see [11,12]), we get

$$
\|u^n - \overline{u}^n\| \leq ch^2 \|\log h\|^3
$$

(27)

similarly, we have

$$
\|u^n - \overline{u}^{(h),n}\| \leq ch^2 \|\log h\|^3.
$$

(28)

3.2. Sequences of Sub-Solutions

The following theorems will play an important role in proving the main result of this paper.

3.2.1. Part One—Discrete Sub-Solution

We construct a discrete function $\alpha^n_h$ near $u^n$ such that: $\alpha^n_h \leq u^n_h$.

Theorem 2 Let $\overline{u}^{n+1}_i$ be the solution of (25). Then there exists a function $\alpha^n_h$ and a constant $c$ independent of $h$ and $n$, such that

$$
\alpha^n_h \leq u^n_h \\
\|u^n_h - \alpha^n_h\| \leq ch^2 \|\log h\|^3
$$

Proof Let us give the proof for $i = 1$. The one for $i = 2$ is similar. Indeed, $\overline{u}^{n+1}_i$ being the solution of V.I (25) for $i = 1$, it is easy to show that $\overline{u}^{n+1}_i$ is also a subsolution, i.e.

$$
\begin{aligned}
& b_h \left( \overline{u}^{n+1}_i, \varphi_i \right) \geq \left( f_i + \lambda u^n, \varphi_i \right) \forall \varphi_i \geq 0; l = 1, \ldots, m(\ell) \\
& \overline{u}^{n+1}_i \leq r_i \psi \text{ in } \tau^i, \overline{u}^{n+1}_i = \pi_h u^{n+1}_h \text{ on } \Gamma_1
\end{aligned}
$$

then

$$
\begin{aligned}
& b_h \left( \overline{u}^{n+1}_i, \varphi_i \right) \geq \left( f_i + \lambda \|u^n - \overline{u}^{n+1}_i\| + \lambda \overline{u}^{n+1}_i, \varphi_i \right) \forall \varphi_i \geq 0 \\
& l = 1, \ldots, m(\ell), \overline{u}^{n+1}_i \leq r_i \psi \text{ in } \tau^i, \overline{u}^{n+1}_i = \pi_h u^{n+1}_h \text{ on } \Gamma_1
\end{aligned}
$$

so, due to lemma 2 (discrete case), it follows that

$$
\overline{u}^{n+1}_i \leq \tilde{u}^{n+1}_i = \partial_h \left( \tilde{f}_i, r_i \psi, \pi_h u^{n+1}_h \right)
$$

(29)
where
\[ \tilde{f}_i = f_i + \lambda \left\| u_i^n - \tilde{u}_i^n \right\| \]
setting \( u_i^{n+1} = \partial_h \left( f_i, r_i \psi, \pi_h u_{2h} \right) \) and using both remark2 (discrete case) and estimate (27), we get
\[
\left\| u_i^{n+1} - \tilde{u}_i^{n+1} \right\| \leq c \left\| \tilde{f}_i - f_i \right\| \leq c \lambda \left\| u_i^n - \tilde{u}_i^n \right\| \leq c h^2 \left| \log h \right|^3
\]
(30)
which combined with (29) yields
\[
\tilde{u}_i^{n+1} \leq u_i^{n+1} + ch^2 \left| \log h \right|^3
\]
Thus, we choose
\[
\alpha_i^{n+1} = \tilde{u}_i^{n+1} - ch^2 \left| \log h \right|^3
\]
then
\[
\alpha_i^{n+1} \leq u_i^{n+1}
\]
and
\[
\left\| \alpha_i^{n+1} - u_i^{n+1} \right\| \leq \left\| u_i^{n+1} - \tilde{u}_i^{n+1} - ch^2 \left| \log h \right|^3 \right\| \leq \left\| u_i^{n+1} - u_i^{n+1} + ch^2 \left| \log h \right|^3 \right\| 
\leq ch^2 \left| \log h \right|^3 + ch^2 \left| \log h \right|^3 \leq ch^2 \left| \log h \right|^3
\]

3.2.2. Part Two—Continuous Sub-Solution
We construct a continuous function \( \beta_i^{(h),n} \) near \( u_i^n \) such that: \( \beta_i^{(h),n} \leq u_i^n \).

**Theorem 3** Let \( \tilde{u}_i^{(h),n+1} \) be the solution of (26). Then there exists a function \( \beta_i^{(h),n} \) and a constant \( c \) independent of \( h \) and \( n \) such that
\[
\beta_i^{(h),n} \leq u_i^n \\
\left\| \beta_i^{(h),n} - u_i^n \right\| \leq ch^2 \left| \log h \right|^3
\]

**Proof** Let us give the proof for \( i = 1 \). The one for \( i = 2 \) is similar. Indeed, \( \tilde{u}_i^{(h),n+1} \) being the solution of V.I (26) for \( i = 1 \), it is also a subsolution, i.e.
\[
\begin{cases}
\left( \beta_i^{(h),n+1}, w \right) \leq f_i + \lambda u_i^n, w \forall w \in H_0^1(\Omega_1), w \geq 0 \\
\tilde{u}_i^{(h),n+1} \leq \psi \text{ in } \Omega_1, \tilde{u}_i^{(h),n+1} = u_2^n \text{ on } \Gamma_1
\end{cases}
\]
then
\[
\begin{cases}
\left( \beta_i^{(h),n+1}, w \right) \leq f_i + \lambda \left\| u_i^n - \tilde{u}_i^n \right\| + \lambda \tilde{u}_i^{(h),n}, w \\
\forall w \in H_0^1(\Omega_1), w \geq 0, \beta_i^{(h),n+1} \leq \psi \text{ in } \Omega_1 \\
\tilde{u}_i^{(h),n+1} = u_2^n \text{ on } \Gamma_1
\end{cases}
\]
so, making use of lemma 2, we obtain
\[
\tilde{u}_i^{(h),n+1} \leq u_i^{n+1} = \partial \left( \tilde{f}_i, \psi, u_2^n \right)
\]
(31)
where
\[
\tilde{f}_i = f_i + \lambda \left\| u_i^n - \tilde{u}_i^n \right\|
\]
Setting \( u_i^{n+1} = \partial \left( f_i, \psi, u_2^n \right) \) and using both Remark 2 and estimate (28), we get
\[
\left\| u_i^{n+1} - u_i^{n+1} \right\| \leq c \left\| \tilde{f}_i - f_i \right\| \leq ch^2 \left| \log h \right|^3
\]
(32)
so, combining (31) with estimate (32) yields
\[
\tilde{u}_i^{(h),n+1} \leq u_i^{n+1} + ch^2 \left| \log h \right|^3
\]
Finally, choosing
\[ \beta_i^{(h),n+1} = u_i^{(h),n+1} - ch^2 \| \log h \|^3 \]
we get immediately the results.

### 3.3. $L^\infty$-Error Estimate

**Theorem 4** *(Main result)* Let \( u_i^{(n+1)} \) (resp. \( u_i^{n+1} \)) be the solution of (21), (22) (resp. (23), (24)). Then there exists a constant \( c \) independent of \( h \) and \( n \), such that
\[
\| u_i - u_i^{n+1} \| \leq c h^2 \| \log h \|^3 \\
\| u_i - u_i^{n+1} \|_{H^1(\Omega_i)} \leq c h \| \log h \|^3.
\]

**Proof** Thanks to theorem 2 and theorem 3, we have
\[
\| u_i^{n+1} - u_i^{n+1} \| \leq c h^2 \| \log h \|^3 \\
\| u_i^{n+1} - u_i^{n+1} \|_{H^1(\Omega_i)} \leq c h \| \log h \|^3
\]
therefore
\[
\| u_i^{n+1} - u_i^{n+1} \| \leq c h^2 \| \log h \|^3; \quad i = 1, 2
\]
moreover
\[
\| u_i - u_i^{n+1} \| \leq \| u_i - u_i^{n+1} \| + \| u_i^{n+1} - u_i^{n+1} \|
\]
let \( k = \max (k_1, k_2) \), then making use of Theorem 1 and estimate (33), we get
\[
\| u_i - u_i^{n+1} \| \leq k^{2n} \| u_0 - u \|_{C^0(\Omega_i)} + ch^2 \| \log h \|^3 \leq k^{2n} \| u - u \|_{C^0(\Omega_i)} + ch^2 \| \log h \|^3
\]
we choose \( n \) such that
\[
k^{2n} \leq h^2
\]
then
\[
\| u_i - u_i^{n+1} \| \leq c h^2 + ch^2 \| \log h \|^3 \leq c h^2 \| \log h \|^3
\]
and by inverse inequality, we get
\[
\| u_i - u_i^{n+1} \|_{H^1(\Omega_i)} \leq c h \| \log h \|^3
\]

### 4. Conclusion

We have established a convergence order of Schwarz algorithm for two overlapping subdomains with non-matching grids. This approach developed in this paper relies on the geometrical convergence and the error estimate between the continuous and discrete Schwarz iterates. The constant \( c \) in error estimate is independent of Schwarz iterate \( n \).

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