The Sum and Difference of Two Constant Elasticity of Variance Stochastic Variables

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ABSTRACT

We have applied the Lie-Trotter operator splitting method to model the dynamics of both the sum and difference of two correlated constant elasticity of variance (CEV) stochastic variables. Within the Lie-Trotter splitting approximation, both the sum and difference are shown to follow a shifted CEV stochastic process, and approximate probability distributions are determined in closed form. Illustrative numerical examples are presented to demonstrate the validity and accuracy of these approximate distributions. These approximate probability distributions can be used to valuate two-asset options, e.g. spread options and basket options, where the CEV variables represent the forward prices of the underlying assets. Moreover, we believe that this new approach can be extended to study the algebraic sum of \( N \) CEV variables with potential applications in pricing multi-asset options.

Keywords: Constant Elasticity of Variance Stochastic Variables; Probability Distribution Functions; Backward Kolmogorov Equation; Lie-Trotter Splitting Approximation

1. Introduction

Recently Lo [1] proposed a new simple approach to tackle the long-standing problem: “Given two correlated lognormal stochastic variables, what is the stochastic dynamics of the sum or difference of the two variables?”, or equivalently, “What is the probability distribution of the sum or difference of two correlated lognormal stochastic variables?” The solution to this problem has wide applications in many fields including financial modelling, actuarial sciences, telecommunications, biosciences and physics [2-15]. By means of the Lie-Trotter operator splitting method [16], Lo showed that both the sum and difference of two correlated lognormal stochastic processes could be approximated by a shifted lognormal stochastic process, and approximate probability distributions were determined in closed form. Unlike previous studies which treat the sum and difference in a separate manner [2-5, 8,13,15,17-27], Lo’s method provides a new unified approach to model the dynamics of both the sum and difference of the two stochastic variables. In addition, in terms of the approximate solutions, Lo presented an analytical series expansion of the exact solutions, which can allow us to improve the approximation systematically.

In this communication we extend Lo’s approach to study the dynamics of both the sum and difference of two correlated constant elasticity of variance (CEV) stochastic processes. The CEV process was first proposed by Cox and Ross [28] as an alternative to the lognormal stochastic movements of stock prices in the Black-Scholes model. The CEV process, which is defined by the stochastic differential equation

\[ dS = \sigma S^{(\beta/2)} dZ \quad \text{for } 0 \leq \beta < 2 \] (1)

with \( dZ \) being a standard Weiner process, has been receiving much attention because it has the ability to give rise to a volatility skew and to explain the volatility smile [29-41]. In the limiting case \( \beta = 2 \), the CEV model returns to the conventional Black-Scholes model, whilst it is reduced to the Ornstein-Uhlenbeck model in the case of \( \beta = 0 \). By the Lie-Trotter operator splitting method, we show that both the sum and difference of two correlated CEV processes can be modelled by a shifted CEV process. Approximate probability distributions of both the sum and difference are determined in closed form, and illustrative numerical examples are presented to demonstrate the validity and accuracy of these approximate distributions.
\[ \text{2. Lie-Trotter Operator Splitting Method} \]

We consider two CEV stochastic variables \( S_1 \) and \( S_2 \), which are described by the stochastic differential equations:

\[
dS_i = \sigma_i S_i^{\beta/2} dZ_i \quad i = 1,2
\]

for \( 0 \leq \beta < 2 \). Here \( dZ \) denotes a standard Weiner process associated with \( S_i \), and the two Weiner processes are correlated as \( dZ_1 dZ_2 = \rho dt \). Without loss of generality, we also assume that \( \sigma_1 > \sigma_2 \). The joint probability distribution function \( P(S_1, S_2, t; S_{10}, S_{20}, t_0) \) of the two correlated CEV variables obeys the backward Kolmogorov equation [42-45]

\[
\left\{ \frac{\partial}{\partial t_0} + \hat{L}_0 \right\} P(S_1, S_2, t; S_{10}, S_{20}, t_0) = 0 \quad (3)
\]

where

\[
\hat{L}_0 = \frac{1}{2} \sigma_1^2 S_{10}^2 \frac{\partial^2}{\partial S_{10}^2} + \rho \sigma_1 \sigma_2 S_{10}^{\beta/2} S_{20}^{\beta/2} \frac{\partial^2}{\partial S_{10} \partial S_{20}} + \frac{1}{2} \sigma_2^2 S_{20}^{\beta/2} \frac{\partial^2}{\partial S_{20}^2}
\]

subject to the boundary condition

\[
P(S_1, S_2, t; S_{10}, S_{20}, t_0 \rightarrow t) = \delta(S_1 - S_{10}) \delta(S_2 - S_{20}). \quad (5)
\]

This joint probability distribution function tells us how probable the two CEV variables assume the values \( S_1 \) and \( S_2 \) at time \( t > t_0 \), provided that their values at \( t_0 \) are given by \( S_{10} \) and \( S_{20} \). Once \( P(S_1, S_2, t; S_{10}, S_{20}, t_0) \) is found, the probability distribution of the sum or difference, namely \( S^+ = S_1 \pm S_2 \), of the two correlated CEV variables can be obtained by evaluating the integral

\[
\hat{P}_\pm \left( S^+, t; S_{10}, S_{20}, t_0 \rightarrow t \right) = \int_{0}^{\infty} dS_1 dS_2 \delta(S_1 \pm S_2 - S^+). \quad (6)
\]

Unfortunately, the joint probability distribution function is not available in closed form, except for the case of \( \rho = 0 \). Hence, we must resort to numerical methods, e.g. the finite-difference method or Monte Carlo simulation. Nevertheless, the numerically exact solution does not provide any information about the stochastic dynamics of the sum or difference explicitly.

It is observed that the probability distribution of the sum or difference of the two correlated CEV variables, i.e. \( \hat{P}_\pm \left( S^+, t; S_{10}, S_{20}, t_0 \right) \), also satisfies the same backward Kolmogorov equation given in Equation (3), but with a different boundary condition [43]

\[
\hat{P}_\pm \left( S^+, t; S_{10}, S_{20}, t_0 \rightarrow t \right) = \delta(S_{10} \pm S_{20} - S^+). \quad (7)
\]

To solve for \( \hat{P}_\pm \left( S^+, t; S_{10}, S_{20}, t_0 \right) \), we first rewrite the backward Kolmogorov equation in terms of the new variables \( S^\pm = S_{10} \pm S_{20} \)

\[
\left\{ \frac{\partial}{\partial t_0} + \hat{L}_0 + \hat{L}_0 + \hat{L}_0 \right\} \hat{P}_\pm \left( S^+, t; S^\pm, S_0, t_0 \right) = 0 \quad (8)
\]

where

\[
\hat{L}_0 = \frac{1}{2} \left( S^+_0 \right)^\beta \left[ \sigma_1^2 \left( 1 + \frac{S_+}{S_0} \right)^\beta + 2 \rho \sigma_1 \sigma_2 \left( 1 - \left( \frac{S_+}{S_0} \right)^\beta \right) \right] + \sigma_2^2 \left( 1 - \frac{S^-}{S_0} \right)^\beta \frac{\partial^2}{\partial S^-^2}
\]

\[
\hat{L}_0 = \frac{1}{2} \left( S^-_0 \right)^\beta \left[ \sigma_1^2 \left( 1 + \frac{S_+}{S_0} \right)^\beta - 2 \rho \sigma_1 \sigma_2 \left( 1 - \left( \frac{S_+}{S_0} \right)^\beta \right) \right] + \sigma_2^2 \left( 1 - \frac{S^-}{S_0} \right)^\beta \frac{\partial^2}{\partial S^-^2}
\]

\[
\hat{L}_0 = \left\{ \sigma_1^2 \left( \frac{S^+_0 + S^-_0}{2} \right) - \sigma_2^2 \left( \frac{S^+_0 - S^-_0}{2} \right) \right\} \frac{\partial^2}{\partial S^+_0 \partial S^-_0}
\]

\[
\quad \sigma_\pm = \sqrt{\sigma_1^2 + \sigma_2^2 \pm 2 \rho \sigma_1 \sigma_2}. \quad (12)
\]

The corresponding boundary condition now becomes

\[
\hat{P}_\pm \left( S^+, t; S^\pm, S_0, t_0 \rightarrow t \right) = \delta(S^+_0 - S^-). \quad (13)
\]

Accordingly, the formal solution of Equation (8) is readily given by

\[
\hat{P}_\pm \left( S^+, t; S^\pm, S_0, t_0 \right) = \exp \left\{ (t - t_0) \left( \hat{L}_0 + \hat{L}_0 + \hat{L}_0 \right) \right\} \delta(S^+_0 - S^-). \quad (14)
\]

Since the exponential operator \( \exp \left\{ (t - t_0) \left( \hat{L}_0 + \hat{L}_0 + \hat{L}_0 \right) \right\} \) is difficult to evaluate, we apply the Lie-Trotter operator splitting method\(^1\) to ap-
approximate the operator by [16,46-49]
\[ \hat{O}_x^{LT} = \exp \left\{ (t-t_0) \frac{d}{dt} \right\} \exp \left\{ (t-t_0) \left( \frac{d}{dt} + \frac{d}{dx} \right) \right\}, \]
and obtain an approximation to the formal solution \[ P_x^{LT} (s^+, t; x^+_0, x^-_0, t_0), \]
\[ = \hat{O}_x^{LT} \delta (x^+_0 - x^-_0) = \exp \left\{ (t-t_0) \frac{d}{dt} \right\} \delta (x^+_0 - x^-_0) \]
where the relation
\[ \exp \left\{ (t-t_0) \left( \frac{d}{dt} + \frac{d}{dx} \right) \right\} \delta (x^+_0 - x^-_0) \] is utilized. For \( (x^+_0/x^-_0)^2 \approx 1 \), which is normally valid unless \( x^+_0 \) and \( x^-_0 \) are both close to zero, the operators \( \hat{L}_x \) and \( \hat{L}_z \) can be approximately expressed as
\[ \hat{L}_x \approx \frac{1}{2} \sigma_0^2 S_0 \frac{\partial^2}{\partial S_0^2} \]
in terms of the two new variables:
\[ \tilde{S}^+_0 = S^+_0 + \left( \frac{\sigma^2_1 - \sigma^2_2}{\sigma^2} \right) S^-_0 \]
\[ \tilde{S}^-_0 = S^-_0 + \left( \frac{\sigma^2_2 - \sigma^2_1}{\sigma^2} \right) S^+_0. \]
Here the parameters \( \tilde{\sigma}_x \) and \( \tilde{\sigma}_z \) are defined by
\[ \tilde{\sigma}_x = \frac{\sigma_x}{\sqrt{2}} \]
\[ \tilde{\sigma}_z = \sigma_z \left( \frac{\sigma^2_1 - \sigma^2_2}{2} \right)^{1/2}. \]
Obviously, both \( \tilde{S}^+ \) and \( \tilde{S}^- \) are CEV random variables defined by the stochastic differential equations
\[ d\tilde{S}^\pm = \tilde{\sigma}^\pm d\tilde{S}^\pm dz^\pm, \]
and their closed-form probability distribution functions are given by
\[
\begin{align*}
& f^{CEV} (\tilde{S}^+, t; \tilde{S}^+_0, t_0) \\
& = 2 \left( \frac{\tilde{S}^+_0}{\tilde{S}^+_0} \right)^{(1-\beta)/2} \frac{(2-\beta)}{(2-\beta)} \sigma^2 \left( t-t_0 \right) ^{(1-\beta)/2} I_\nu \left( \frac{\tilde{S}^+_0}{\tilde{S}^+_0} \right) ^{(2-\beta)/2} \left( t-t_0 \right) ^{(2-\beta)/2} \\
& \times \exp \left\{ - \frac{2 \left[ (\tilde{S}^+_0)^{2-\beta} + (\tilde{S}^-_0)^{2-\beta} \right]}{(2-\beta) \tilde{\sigma}^2 \left( t-t_0 \right) ^{1-\beta}} \right\}
\end{align*}
\]
for \( t > t_0 \), where \( I_\nu (\cdot) \) denotes the modified Bessel function of order \( \nu \). As a result, it can be inferred that within the Lie-Trotter splitting approximation both \( S^+ \) and \( S^- \) are governed by a shifted CEV process. It should be noted that for the Lie-Trotter splitting approximation to be valid, \( \tilde{\sigma}_x^2 (t-t_0) \) needs to be small.

3. Illustrative Numerical Examples

In Figure 1 we plot the approximate closed-form probability distribution function of the sum \( S^+ \) of two uncorrelated CEV variables (i.e. \( \rho = 0 \)) given in Equation (23) for different values of the input parameters. We start with \( S_{10} = 110 \), \( S_{20} = 105 \), \( \sigma_1 = 0.5 \) and \( \sigma_2 = 0.3 \) in Figure 1(a). Then, in order to examine the effect of \( S_{20} \), we decrease its value to 85 in Figure 1(b) and to 65 in Figure 1(c). In Figures 1(d)-(f) we repeat the same investigation with a new set of values for \( \sigma_1 \) and \( \sigma_2 \), namely \( \sigma_1 = 0.3 \) and \( \sigma_2 = 0.2 \). Without loss of generality, we set \( t-t_0 = 1 \) for simplicity. The (numerically) exact results which are obtained by numerical integrations are also included for comparison. It is clear that the proposed approximation can provide accurate estimates for the exact values.

In order to have a clearer picture of the accuracy, we plot the corresponding errors of the estimation in Figure 2. We can easily see that major discrepancies appear around the peak of the probability distribution function and that the estimation deteriorates as the elasticity parameter \( \beta \) increases. It should be noted that the proposed approximation is exact in the special case of \( \beta = 0 \), i.e. the Ornstein-Uhlenbeck model. We also observed that the errors increase with the ratio \( S_0/S_0^* \) and \( \sigma^*_0 \) (or equivalently, \( \sigma_1 \) and \( \sigma_2 \)) as expected.

Next, we apply the same sequence of analysis to the approximate closed-form probability distribution function of the difference \( S^- \) given in Equation (23). Similar observations about the accuracy of the proposed approximation can be made for the difference \( S^- \), too (see Figures 3 and 4).

4. Conclusion

In this communication we have applied a new unified approach proposed by Lo (2012) to model the dynamics of both the sum and difference of two correlated CEV stochastic variables. By the Lie-Trotter operator splitting method, both the sum and difference are shown to follow a shifted CEV stochastic process, and approximate probability distributions are determined in closed form. Illustrative numerical examples are presented to demonstrate the validity and accuracy of these approximate distributions. These approximate probability distributions can be used to validate two-asset options, e.g. the spread options, where the CEV variables represent the forward prices of the underlying assets. Moreover, we believe that this new approach can be extended to study the
Figure 1. Probability density vs. $S_1 + S_2$: the dotted lines denote the distributions of the approximate shifted CEV process, and the solid lines show the exact results. (a) $S_{10} = 110, S_{20} = 105, \sigma_1 = 0.5$ and $\sigma_2 = 0.3$; (b) $S_{10} = 110, S_{20} = 85, \sigma_1 = 0.5$ and $\sigma_2 = 0.3$; (c) $S_{10} = 110, S_{20} = 65, \sigma_1 = 0.5$ and $\sigma_2 = 0.3$; (d) $S_{10} = 110, S_{20} = 105, \sigma_1 = 0.3$ and $\sigma_2 = 0.2$; (e) $S_{10} = 110, S_{20} = 85, \sigma_1 = 0.3$ and $\sigma_2 = 0.2$; (f) $S_{10} = 110, S_{20} = 65, \sigma_1 = 0.3$ and $\sigma_2 = 0.2$. 

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Figure 2. Error vs. $S_1 + S_2$: the error is calculated by subtracting the approximate estimate from the exact result. (a) $S_{10} = 110$, $S_{20} = 105$, $\sigma_1 = 0.5$ and $\sigma_2 = 0.3$; (b) $S_{10} = 110$, $S_{20} = 85$, $\sigma_1 = 0.5$ and $\sigma_2 = 0.3$; (c) $S_{10} = 110$, $S_{20} = 65$, $\sigma_1 = 0.5$ and $\sigma_2 = 0.3$; (d) $S_{10} = 110$, $S_{20} = 105$, $\sigma_1 = 0.3$ and $\sigma_2 = 0.2$; (e) $S_{10} = 110$, $S_{20} = 85$, $\sigma_1 = 0.3$ and $\sigma_2 = 0.2$; (f) $S_{10} = 110$, $S_{20} = 65$, $\sigma_1 = 0.3$ and $\sigma_2 = 0.2$. 
Figure 3. Probability density vs. \( S_1 - S_2 \): the dotted lines denote the distributions of the approximate shifted CEV process, and the solid lines show the exact results. (a) \( S_{10} = 110, S_{20} = 105, \sigma_1 = 0.5 \) and \( \sigma_2 = 0.3 \); (b) \( S_{10} = 110, S_{20} = 85, \sigma_1 = 0.5 \) and \( \sigma_2 = 0.3 \); (c) \( S_{10} = 110, S_{20} = 65, \sigma_1 = 0.5 \) and \( \sigma_2 = 0.3 \); (d) \( S_{10} = 110, S_{20} = 105, \sigma_1 = 0.3 \) and \( \sigma_2 = 0.2 \); (e) \( S_{10} = 110, S_{20} = 85, \sigma_1 = 0.3 \) and \( \sigma_2 = 0.2 \); (f) \( S_{10} = 110, S_{20} = 65, \sigma_1 = 0.3 \) and \( \sigma_2 = 0.2 \).
Figure 4. Error vs. $S_1 - S_2$: the error is calculated by subtracting the approximate estimate from the exact result. (a) $S_{10} = 110$, $S_{20} = 105$, $\sigma_1 = 0.5$ and $\sigma_2 = 0.3$; (b) $S_{10} = 110$, $S_{20} = 85$, $\sigma_1 = 0.5$ and $\sigma_2 = 0.3$; (c) $S_{10} = 110$, $S_{20} = 65$, $\sigma_1 = 0.5$ and $\sigma_2 = 0.3$; (d) $S_{10} = 110$, $S_{20} = 105$, $\sigma_1 = 0.3$ and $\sigma_2 = 0.2$; (e) $S_{10} = 110$, $S_{20} = 85$, $\sigma_1 = 0.3$ and $\sigma_2 = 0.2$; (f) $S_{10} = 110$, $S_{20} = 65$, $\sigma_1 = 0.3$ and $\sigma_2 = 0.2$. 

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REFERENCES


http://dx.doi.org/10.3905/jod.2012.19.3.077

http://dx.doi.org/10.1016/0304-405X(76)90023-4


http://dx.doi.org/10.1142/S0219024900000814

http://dx.doi.org/10.1287/mnsc.47.7.949.9804

http://dx.doi.org/10.1287/mnsc.48.7.917.2815

http://dx.doi.org/10.1016/S0304-4076(03)00107-6

http://dx.doi.org/10.1111/j.0960-1627.2005.00224.x

http://dx.doi.org/10.1016/j.insmatheco.2006.04.007

http://dx.doi.org/10.1016/j.matcom.2007.09.012

http://dx.doi.org/10.1142/S0219091509001605

http://dx.doi.org/10.1016/j.insmatheco.2009.01.005


http://dx.doi.org/10.1080/10920277.2011.10597628

http://dx.doi.org/10.1016/j.insmatheco.2011.10.013


http://dx.doi.org/10.2140/pjm.1958.8.887

http://dx.doi.org/10.1063/1.526596

http://dx.doi.org/10.1103/PhysRevE.57.1284