Bounds for Goal Achieving Probabilities of Mean-Variance Strategies with a No Bankruptcy Constraint

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ABSTRACT
We establish, through solving semi-infinite programming problems, bounds on the probability of safely reaching a desired level of wealth on a finite horizon, when an investor starts with an optimal mean-variance financial investment strategy under a non-negative wealth restriction.

Keywords: First Passage-Time; Mean-Variance Portfolios; Semi-Infinite Programming

1. Introduction
In probability theory, the first passage-time problem is the study of the first moment when a stochastic process reaches a certain threshold. This problem often arises in financial mathematics and particularly in portfolio management. For example, consider a risky strategy on a horizon \([0, T]\), the investor may encounter a specific instant \(t\) when the amount of wealth \(x(t)\) is sufficient enough so that he may, at this point, safely reinvest all of his money in a simple bank account with (deterministic) interest rate \(r(t)\) and the resulting terminal wealth \(x(T)\) will attain his financial goal \(z\). So we consider the following stopping time random variable:

\[
\tau_z = \inf \left\{ 0 \leq t \leq T : x(t) e^{\int_0^t r(s) \, ds} = z \right\}
\]  

and we naturally want to compute the probability \(P(\tau_z \leq T)\) of such an event. If \(x_0 > 0\) is his initial wealth then we will assume \(z > x_0 \exp \left( \int_0^T r(s) \, ds \right)\) so that the investor cannot achieve his financial goal by simply placing his initial investment in a bank account.

2. Market Model
In order to investigate this goal-achieving problem, we must first define a mathematical setting for the dynamics of the financial market. We will consider here the celebrated Black-Scholes model that we next describe. The first asset is a bank account whose price at time \(t\), \(P_b(t)\), is the solution to the following ordinary differential equation (ODE):

\[
d P_b(t) = r(t) P_b(t) \, dt. \tag{2}
\]

The next assets consist of \(m\) stocks whose prices \(\{P_i(t), \ldots, P_m(t)\}\) at time \(t\) are the solutions to the following SDEs (stochastic differential equations):

\[
d P_i(t) = P_i(t) \left[ b_i(t) \, dt + \sum_{j=1}^{m} \sigma_{ij}(t) \, dW_j(t) \right] \tag{3}
\]

where \(\{W(t) \geq 0\}\) is a standard \(m\)-dimensional Brownian motion.

We will assume that the interest rate \(r(t)\), stock appreciation rates \(b_i(t)\) and stock volatilities \(\sigma_{ij}(t)\) are deterministic functions and that

\[
\sigma(t) = \begin{bmatrix} \sigma_{11}(t) & \cdots & \sigma_{1m}(t) \\ \vdots & \ddots & \vdots \\ \sigma_{m1}(t) & \cdots & \sigma_{mm}(t) \end{bmatrix} \tag{4}
\]

is invertible.

Let \(u(t) = (u_1(t), \ldots, u_m(t))^T, 0 \leq t \leq T\) be a financial strategy (or portfolio) where \(u_i(t)\) is the amount placed in the \(i^{th}\) stock. If we assume that all strategies \(u(t)\) are self-financed (no outside injection of funds to the investors) and with no transaction costs then the wealth dynamic at time \(t\) is given by the following stochastic differential equation (SDE):

\[
dx(t) = \left[ r(t) x(t) + B(t) u(t) \right] \, dt + u(t)^T \sigma(t) \, dW(t) \tag{5}
\]

where \(B(t) = (b_1(t) - r(t), \ldots, b_m(t) - r(t))^T\).

Finally, among all the possible strategies, we will fo-
cus on the one generated by a family of stochastic control problems defined by
\[
\min \text{VAR}(x(T)) \text{ s.t. } E\left(x(T)\right) = z. \tag{6}
\]

These are known as mean-variance problems and are considered the cornerstone of modern portfolio management theory which originated with the work of Nobel Prize laureate H. Markowitz.

3. Goal Achieving Probabilities

3.1. Case 1: Unconstrained and No Short-Selling Restriction

In this context, the optimal wealth process has the following form
\[
x(t) = y_0 e^{\alpha t} + \beta e^{\gamma t} r(s) dt + \sigma \int_0^t e^{\gamma (s-t)} dW(s) \tag{7}
\]
with \(y_0 < 0, \beta > 0\) and \(\alpha(t) > 0\) having specific values for the unconstrained and no short-selling (no borrowing stocks) case respectively. The computation of the probability for \(P(\tau_x \leq T)\), following a stochastic time change, can be reduced to the calculation of the probability of the first passage time of a Brownian motion with drift through a fixed level, more precisely the probability is given by:
\[
P(\tau_x \leq T) = \Phi\left(\frac{1}{2} \sqrt{\lambda} \int_0^T \sigma^2(s) ds\right) + e^{\gamma T} \int_0^\infty e^{-\frac{\lambda}{2} z^2} \Phi\left(-\frac{5}{2} \sqrt{\frac{\lambda}{2}} \sigma^2(s) ds\right) \tag{8}
\]
where
\[
\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-z^2/2} dz \tag{9}
\]
is the cumulative density function of a standardized normal distribution.

Detailed proofs can be found in Li and Zhou [1] and Scott and Watier [2].

3.2. Case 2: No Bankruptcy Restriction

In this case, unfortunately, the optimal wealth process has a more complex expression, according to Bielecki et al. [3] it is given by
\[
x_N(t) = \lambda e^{-\gamma t} \int_0^t \theta(s) dW(s) \tag{10}
\]
where
\[
f(t,Z) = \Phi\left(-d_+(t,y(t,Z))\right) - \frac{y(t,Z)}{\lambda} e^{\gamma t} r(s) \Phi\left(-d_-(t,y(t,Z))\right) \tag{11}
\]
and \(\lambda > z\) and \(\mu > 0\) are Lagrange multipliers obtained by solving the nonlinear system of equations:
\[
E\left[(\lambda - \mu \rho(T))^2\right] = z \tag{15}
\]
\[
E\left[\rho(T)(\lambda - \mu \rho(T))^2\right] = x_0 \tag{16}
\]
with
\[
\rho(T) = e^{-\int_0^T \sigma^2(t) dt} \int_0^T \sigma^2(s) ds \tag{17}
\]
Evidently, an explicit form for the corresponding goal-achieving probability \(P(\tau^{NB}_x \leq T)\) as in the cases discussed in Section 3.1 appears unrealistic. However, we will show that we can obtain precise bounds for this probability through solving (deterministic) semi-infinite programming (SIP) problems.

The basic idea is to convert the original passage-time problem of this complex stochastic process with a fixed barrier into an equivalent passage-time problem for a simple Gaussian Markovian process but with a time-varying boundary.

To this end, the following result will be useful. Let \(A > 0\), then
\[
g(x) = \Phi(x + A) - e^{-A^2} \frac{1}{2} x^2 \Phi(x) \tag{18}
\]
is a strictly increasing function on the real line that takes on values in \([0,1]\).

The proof is straightforward since clearly
\[
\lim_{x \to \infty} g(x) = 0 \quad \text{and} \quad \lim_{x \to -\infty} g(x) = 1, \quad \text{while}
\]
\[
\frac{d}{dx} g(x) = Ae^{-A^2} \frac{1}{2} x \Phi(x) > 0 \tag{19}
\]
From this property we have that, for each fixed \(t \in [0,T]\),
\[
f(t,\int_0^T \theta(s) dW(s)) = \frac{z}{\lambda} \tag{20}
\]
therefore, if

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Therefore the family of Daniels curves appears to be excellent candidates for obtaining explicit upper and lower bounds for our original goal-achieving problem. Finally, in order to generate the tightest bounds possible, we are naturally led to solve the following SIP problems:

\[
\begin{align*}
\sup_{\alpha>0, 0, c_2 \in \mathbb{R}} & \left\{ \frac{-S(T)}{\sqrt{h(T)}} + c_1 \frac{S(T) - \alpha}{\sqrt{h(T)}} \right\} \\
& + c_2 \frac{S(T) - 2\alpha}{\sqrt{h(T)}} \\
\text{s.t.} & \ S(t) \geq f^{-1}\left( t, \frac{z}{\lambda} \right) \text{for all} \ t \in [0, T]
\end{align*}
\]

and

\[
\begin{align*}
\inf_{\alpha>0, 0, c_2 \in \mathbb{R}} & \left\{ \frac{-S(T)}{\sqrt{h(T)}} + c_1 \frac{S(T) - \alpha}{\sqrt{h(T)}} \right\} \\
& + c_2 \frac{S(T) - 2\alpha}{\sqrt{h(T)}} \\
\text{s.t.} & \ S(t) \leq f^{-1}\left( t, \frac{z}{\lambda} \right) \text{for all} \ t \in [0, T]
\end{align*}
\]

For inquiries on efficient techniques for solving these SIP problems we refer the reader to Lopez and Still [5] and Reemtsen and Rückmann [6].

4. Numerical Examples

In order to illustrate that the solutions to the 3-parameter SIP problems can produce tight bounds, let us reprise the one stock market model example in Bielecki et al. that is

\[
\begin{align*}
\rho = 0.6, \ \beta = 0.12, \ \sigma(t) = 0.15, \ x_t = 1, \ T = 1
\end{align*}
\]

but with different wealth objective \( z \). Table 1 sums up the results.

Finally, we can easily show that the 80% rule (i.e. \( P(\tau \leq T) > 0.80 \), for all possible values of the market parameters) obtained by Li and Zhou and, Scott and Watier unfortunately does not hold in general for a no-bankruptcy optimal mean-variance strategy. For example, if we set \( z = 2.0 \), by solving (29), we have \( P(\tau \leq T) < 0.65 \).

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Table 1. Goal achieving probability bounds.
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REFERENCES


