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Optimal Control of Cancer Growth

Jens Christian Larsen

Vanløse Alle 50 2. mf. tv, 2720 Vanløse, Copenhagen, Denmark
Email: jlarsen.math@hotmai.com

Abstract

The purpose of the present paper is to apply the Pontryagin Minimum Principle to mathematical models of cancer growth. In [1], I presented a discrete affine model $T$ of cancer growth in the variables $C$ for cancer, $GF$ for growth factors and $GI$ for growth inhibitors. One can sometimes find an affine vector field $X$ on $\mathbb{R}^3$ whose time one map is $T$. It is to this vector field we apply the Pontryagin Minimum Principle. We also apply the Discrete Pontryagin Minimum Principle to the model $T$. So we prove that maximal chemotherapy can be optimal and also that it might not depending on the spectral properties of the matrix $A$, (see below). In section five we determine an optimal strategy for chemo or immune therapy.

Keywords

Cancer, Models of Cancer Growth, Pontryagin Minimum Principle

1. Introduction

Our reference to optimal control theory of ODEs is [2] and to control theory of discrete systems [3]. There is a review of papers in optimal control theory applied to cancer [4]. The model we consider here is from [1] and is defined by

$$A = \begin{pmatrix} \alpha + \beta & \gamma \\ \delta & 1 + \mu_f \\ \sigma & 0 \\ 1 + \mu_i \end{pmatrix}, \quad T(y) = Ay + g$$

where $y = (C, GF, GI)^T$, $g = (g_C, g_F, g_I)^T \in \mathbb{R}^3$, and $T$ denotes transpose. Here $\alpha \in \mathbb{R}_+, \beta \in \mathbb{R}_+, \delta, \sigma \in \mathbb{R}, \mu_f, \mu_i < 0$. We assume that $\alpha \delta + \beta \sigma \neq 0$.

The matrix $A$ has characteristic polynomial $-p(\lambda)$ where

$$p(\lambda) = \lambda^3 - \lambda^2 \left(3 + \gamma + \mu_f + \mu_i\right) - \lambda \left(\alpha \delta + \beta \sigma - (1 + \mu_f)(1 + \mu_i)\right)$$

$$-\left(1 + \gamma\right)\left(2 + \mu_f + \mu_i\right) + \alpha \delta (1 + \mu_f) + \beta \sigma (1 + \mu_i) - \left(1 + \gamma\right)(1 + \mu_f)(1 + \mu_i).$$
and this polynomial can have (i) three real roots or (ii) one real root and two imaginary roots. It turns out that asking \( \mu = \mu_r = \mu_i \) simplifies matters considerably. Then
\[
p(\lambda) = -(1 + \mu - \lambda)\left(\lambda^2 - (2 + \gamma + \mu)\lambda + (1 + \mu)(1 + \mu) - a\delta - \beta\sigma\right)
\]
(4)

So the eigenvalues are \( 1 + \mu \) and
\[
\lambda_{\pm} = 1 + \frac{\gamma + \mu \pm \sqrt{(\gamma - \mu)^2 + 4(\alpha\delta + \beta\sigma)}}{2}
\]
(5)

We assume that \( 1 + \mu > 0, \mu < 0 \). In case (i) if the eigenvalues of \( A \), \( \lambda_1, \lambda_2, \tilde{\lambda} \) are positive and distinct and we assume this, then you can find an affine vector field \( X \) on \( \mathbb{R}^3 \) such that the time one map is
\[
\Phi_i^X = T
\]
(6)

see [1] and below. In case (ii) if the eigenvalues of \( A \) are \( 1 + \mu, a \pm ib, a > 0, 1 + \mu > 0, b \neq 0 \) and we assume this, you can also find an affine vector field \( X \) on \( \mathbb{R}^3 \) such that
\[
\Phi_i^X = T
\]
(7)

In case (i) define a matrix of eigenvectors
\[
D = \begin{pmatrix}
1 + \mu - \lambda_1 & 1 + \mu - \lambda_2 & 0 \\
-\delta & -\delta & -\beta \\
-\sigma & -\sigma & \alpha
\end{pmatrix}
\]
(8)

and in case (ii) define
\[
U = \begin{pmatrix}
1 + \mu - a & -b & 0 \\
-\delta & 0 & -\beta \\
-\sigma & 0 & \alpha
\end{pmatrix}
\]
(9)

Then in case (i)
\[
\Lambda = D^{-1}AD = \begin{pmatrix}
\lambda_1 & 0 & 0 \\
0 & \lambda_2 & 0 \\
0 & 0 & \tilde{\lambda}
\end{pmatrix}
\]
(10)

if the eigenvalues are distinct and positive and in case (ii)
\[
\Lambda = U^{-1}AU = \begin{pmatrix}
a & b & 0 \\
-b & a & 0 \\
0 & 0 & 1 + \mu
\end{pmatrix}
\]
(11)

see [1]. Here \( D^{-1} \) denotes the inverse to \( D \). To find an \( X \) in case (i) define when the eigenvalues are real, positive and distinct, an affine vector field
\[
Y(x) = \begin{pmatrix}
\ln \lambda_1 & 0 & 0 \\
0 & \ln \lambda_2 & 0 \\
0 & 0 & \ln \tilde{\lambda}
\end{pmatrix}x + d
\]
(12)

Then with the right choice of \( d \)
\[
\Phi_i^X(x) = D^{-1}Adx + D^{-1}g
\]
(13)
hence if we let

$$X(y) = DY\left(D^{-1}(y)\right)$$  \hspace{1cm} (14)$$

we get

$$\Phi_{1}^{X} = T$$  \hspace{1cm} (15)$$

And to find an $X$ in case (ii) when $a > 0, b \neq 0, 1 + \mu > 0$, define

$$Y(x) = \begin{pmatrix} a_1 & b_1 & 0 \\ -b_1 & a_1 & 0 \\ 0 & 0 & \ln(1 + \mu) \end{pmatrix} + \tilde{d}$$  \hspace{1cm} (16)$$

Then with the right choice of $a_i, b_i \in \mathbb{R}, \tilde{d} \in \mathbb{R}^3$ we get

$$\Phi_{1}^{Y}(x) = U^{-1}AU(x) + U^{-1}g$$  \hspace{1cm} (17)$$

hence with

$$X(y) = UY\left(U^{-1}(y)\right)$$  \hspace{1cm} (18)$$

we have that the time one map is

$$\Phi_{1}^{Y}(y) = T\left(y\right)$$  \hspace{1cm} (19)$$

See [1] for details of the above and also below.

In section two we solve the problem:

minimize $C(T), \ T > 0$ (fixed) subject to

$$\left( \begin{array}{c} C' \\ GF \\ GI \end{array} \right) = \left( \begin{array}{c} C \\ GF \\ GI \end{array} \right) + e_2 u(t)$$  \hspace{1cm} (20)$$

$u(t) \in [0, g_0], g_0 > 0$, by first solving it for $Y$ and then infer the solution for $X$. Here

$$e_2 = B = (0, 0, 1)^T$$  \hspace{1cm} (21)$$

and $u(t)$ is piecewise continuous. In section three we apply the discrete Pontryagin minimum principle to the difference equation

$$x_{k+1} = Ax_k + Bu_k + g$$

where

$$x_k = \left(C(k), GF(k), GI(k)\right)^T$$  \hspace{1cm} (22)$$

$u_k \in [0, g_0], g_0 > 0$. $k = 0, 1, \cdots, N - 1$, where $N \in \mathbb{N}, N \geq 2, x_0 = x$, with the objective to minimize $C(N)$. There are again two cases to consider (i) and (ii) above. If $\mu = \mu_f = \mu_i$ and the eigenvalues are positive and distinct, maximal chemo therapy is optimal. But in case (ii) it is not always optimal.

When $\mu_f \neq \mu_i$ and the eigenvalues are real and distinct, we produce a counter example to maximal chemo therapy being optimal, see section four. Some solid tumors grow like Gompertz functions, see [5]. There are several important monographs in mathematics and medicine, see [6]-[11]. [12]-[19] are

In section five we consider optimality of the discrete model $T$ when $\mu_T \neq \mu_I$. Here we also determine optimal control of the map $T$.

### 2. Optimal Control of $X$

The purpose of this section is to minimize $C(T)$, subject to

$$
\begin{pmatrix}
C \\
GF \\
GI
\end{pmatrix} = X
\begin{pmatrix}
C \\
GF \\
GI
\end{pmatrix} + e, u(t)
$$

where $T > 0$ fixed and with $\mu = \mu_T = \mu_I$. Let us consider (ii) first. We assume that there are a real eigenvalue $1 + \mu$ and two imaginary eigenvalues $a \pm ib, a > 0, 1 + \mu > 0, b \neq 0$.

Now define the two by two matrix

$$
L_1 = \begin{pmatrix}
a & b \\
-b & a
\end{pmatrix}
$$

where

$$
b_1 = \tan^{-1}\left(\frac{b}{a}\right)
$$

$$
a_i = \ln(a^2 + b^2)
$$

This will imply, that

$$
\exp\begin{pmatrix}
a & b \\
-b & a
\end{pmatrix} = \begin{pmatrix}
a & b \\
-b & a
\end{pmatrix} = L
$$

Also let

$$
\tilde{B} = \begin{pmatrix}
a & b & 0 \\
-b & a & 0 \\
0 & 0 & \ln(1 + \mu)
\end{pmatrix}
$$

and define the vector field

$$
Y(x, v) = \tilde{B}x + d + U^{-1}e_3v = \tilde{Y}(x) + U^{-1}e_3v
$$

which is affine when $v = 0$, where $e_1, e_2, e_3$ is the canonical basis in $\mathbb{R}^3$. Also $x, d \in \mathbb{R}^3, v \in [0, g_3^0], g_3^0 > 0$. Put

$$
\tilde{X}(y) = U\tilde{Y}U^{-1}(y)
$$

Let

$$
X(y, v) = \tilde{X}(y) + e_3v = YY(U^{-1}(y), v)
$$

Define $d_1, d_2, d_3$ by

$$
U^{-1}(g)_{i,2} = L_1(L - \text{id})d_{i,2}
$$
and

\[ \frac{\mu}{\ln(1+\mu)} d_3 = \left( U^{-1} g \right)_3 \]  
(33)

Then

\[ \Phi^X_1 = T \]  
(34)

when \( v = 0 \). To this vector field with \( v \in \left[ 0, g^0 \right] \) associate the Hamiltonian

\[ H(x, p, v, t) = p^1 Y(x, v) \]  
(35)

Then we have the adjoint equations

\[ p'_1 = -\frac{\partial H}{\partial x_1} = -p_a a_1 + p_b b_1 \]  
(36)

\[ p'_2 = -\frac{\partial H}{\partial x_2} = -p_a b_1 - p_b a_1 \]  
(37)

\[ p'_3 = -\frac{\partial H}{\partial x_3} = -p_3 \ln(1+\mu) \]  
(38)

So

\[ p'_{1,2} = \begin{pmatrix} -a_1 & b_1 \\ -b_1 & -a_1 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \]  
(39)

which has flow

\[ p_{1,2}(t) = \exp(-a_1 t) \begin{pmatrix} \cos(b_1 t) & \sin(b_1 t) \\ -\sin(b_1 t) & \cos(b_1 t) \end{pmatrix} p_{1,2}(0) \]  
(40)

Define

\[ d(t) = L_1^{-1} (\exp(L t) - \text{id}) d_{1,2} \]  
(41)

Then the flow of \( Y \) is for \( v = 0 \)

\[ \Phi^Y(t, x)_{1,2} = \exp(L t) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + d(t) \]  
(42)

\[ \Phi^Y(t, x)_3 = \exp(\ln(1+\mu) t) x_3 + \frac{\exp(\ln(1+\mu) t) - 1}{\ln(1+\mu)} d_3 \]  
(43)

Define

\[ S_1(x) = (1+\mu-a)x_1 - bx_2 \]  
(44)

Then we have the transversality conditions

\[ p_1(T) = \frac{\partial S_1}{\partial x_1} = 1 + \mu - a \]  
(45)

\[ p_2(T) = \frac{\partial S_1}{\partial x_2} = -b \]  
(46)

\[ p_3(T) = \frac{\partial S_1}{\partial x_3} = 0 \]  
(47)
This means that \( p_3(t) = 0 \) and
\[
p_{1,2}(T) = \exp(-aT) \begin{pmatrix} \cos(h_T) & \sin(h_T) \\ -\sin(h_T) & \cos(h_T) \end{pmatrix} \begin{pmatrix} 1 + \mu - a \\ -b \end{pmatrix}
\]
(48)

The two by two matrix in this equation has inverse
\[
\begin{pmatrix} \cos(h_T) & -\sin(h_T) \\ \sin(h_T) & \cos(h_T) \end{pmatrix}
\]
(49)
So
\[
p_{1,2}(0) = \exp(aT) \begin{pmatrix} \cos(h_T) & -\sin(h_T) \\ \sin(h_T) & \cos(h_T) \end{pmatrix} \begin{pmatrix} 1 + \mu - a \\ -b \end{pmatrix}
\]
(50)
But then we have
\[
p_{1,2}(t) = \exp(a(T - t)) \begin{pmatrix} \cos(h_T(t-T)) & \sin(h_T(t-T)) \\ -\sin(h_T(t-T)) & \cos(h_T(t-T)) \end{pmatrix} \begin{pmatrix} 1 + \mu - a \\ -b \end{pmatrix}
\]
(51)
Notice that we have
\[
U_{31} = b\beta
\]
(52)
\[
U_{32} = (1 + \mu - a)\beta
\]
(53)
If
\[
H(x^*(t), p(t), u^*(t), t) \leq H(x^*(t), p(t), u, t)
\]
for all \( u \in U = [0, g^a] \), then \( (x^*(t), u^*(t)) \) is optimal, see Equations (55) to (60). But this amounts to the inequality
\[
\left( p_1(t)b\beta u + p_2(t)(1 + \mu - a)\beta u \right) \frac{1}{\det(U)}
\]
(55)
\[
= \exp(a(T - t)) \left[ (1 + \mu - a) - b\sin(h_T(t-T)) \right] b\beta u
\]
(56)
\[
+ \left[ -\sin(h_T(t-T))(1 + \mu - a) - b\cos(h_T(t-T)) \right] (1 + \mu - a)\beta u \frac{1}{\det(U)}
\]
(57)
\[
= \exp(a(T - t)) \sin(h_T(t-T)) \left[ -b^2 - (1 + \mu - a)^2 \right] \beta u \frac{1}{\det(U)}
\]
(58)
\[
= \exp(a(T - t)) (\alpha\delta + \beta\sigma) \beta u \sin(h_T(t-T)) \frac{1}{\det(U)}
\]
(59)
\[
\leq \exp(a(T - t)) \sin(h_T(t-T)) \frac{\beta u}{-b}
\]
(60)
where \( \det(U) \) denotes the determinant of \( U \). We have used that
\[
1 + \mu - a = 1 + \mu - \left( 1 + \frac{\gamma + \mu}{2} \right) = \frac{\mu - \gamma}{2}
\]
(61)
and
\[
\lambda = 1 + \frac{\gamma + \mu}{2} + \frac{1}{2}\sqrt{\left(\gamma - \mu\right)^2 + 4(\alpha\delta + \beta\sigma)} = a + ib
\]
(62)
so that
\[(1 + \mu - a)^2 + b^2 = -(\alpha \delta + \beta \sigma) \tag{63}\]
and also that \(\det \mathbf{U} = -b(\alpha \delta + \beta \sigma)\). So if
\[\frac{\sin(b_1(t-T))}{-b} > 0 \tag{64}\]
then
\[u^*(t) = g_i^0 \tag{65}\]
is optimal and if the reverse inequality holds then
\[u^*(t) = 0 \tag{66}\]
is optimal. We shall now consider the case where all eigenvalues \(\lambda_1, \lambda_2, \lambda_3\) are real, positive and distinct and that \(F = \mu = \mu_i = \mu_j\). Here we let
\[Y(x,u) = \tilde{\Lambda}x + d + D^{-1}e, u \tag{67}\]
where
\[\tilde{\Lambda} = \text{diag}(\ln \lambda_1, \ln \lambda_2, \ln (1 + \mu)) \tag{68}\]
Also define \(d\) in
\[\frac{\lambda_1 - 1}{\ln \lambda_1}d_1 = \left(D^{-1}g\right)_1 \tag{69}\]
\[\frac{\lambda_2 - 1}{\ln \lambda_2}d_2 = \left(D^{-1}g\right)_2 \tag{70}\]
\[\frac{\mu}{\ln (1 + \mu)}d_3 = \left(D^{-1}g\right)_3 \tag{71}\]
\[d = (d_1, d_2, d_3)^\top \in \mathbb{R}^3. \tag{72}\]
Now define the Hamiltonian
\[H(x,p,u,t) = p^TY(x,u) \tag{72}\]
Then we get the adjoint equations
\[p_1' = -\frac{\partial H}{\partial x_1} = -\ln \lambda_1 p_1 \tag{73}\]
\[p_2' = -\frac{\partial H}{\partial x_2} = -\ln \lambda_2 p_2 \tag{74}\]
\[p_3' = -\frac{\partial H}{\partial x_3} = -\ln (1 + \mu) p_3 \tag{75}\]
see [2]. Now define
\[S_i(x) = (1 + \mu - \lambda_i)x_i + (1 + \mu - \lambda_i)x_2 \tag{76}\]
Then we have the transversality conditions
\[p(T) = \frac{\partial S_i}{\partial x} = \begin{pmatrix} 1 + \mu - \lambda_i \\ 1 + \mu - \lambda_i \\ 0 \end{pmatrix} \tag{77}\]
by [2] and see below. Now observe, that
\[
\det(D) = (\lambda_\alpha - \lambda_\beta)(\alpha \delta + \beta \sigma)
\]  
(78)

Because
\[
1 + \mu - \lambda_\alpha = \frac{\mu - \gamma + 1}{2} \sqrt{(\mu - \gamma)^2 + 4(\alpha \delta + \beta \sigma)}
\]  
(79)

we find
\[
(1 + \mu - \lambda_\alpha)(1 + \mu - \lambda_\beta) = -(\alpha \delta + \beta \sigma)
\]  
(80)

We shall need
\[
D^{-1} = \begin{pmatrix}
-\alpha \delta - \beta \sigma & -\alpha(1 + \mu - \lambda_\alpha) & -\beta(1 + \mu - \lambda_\alpha) \\
\alpha \delta + \beta \sigma & \alpha(1 + \mu - \lambda_\alpha) & \beta(1 + \mu - \lambda_\alpha) \\
0 & -\sigma(\lambda_\alpha - \lambda_\beta) & \delta(\lambda_\alpha - \lambda_\beta)
\end{pmatrix} \frac{1}{\det D}
\]  
(81)

We have
\[
\begin{pmatrix}
C(T) \\
0 \\
0
\end{pmatrix} = D \begin{pmatrix}
x_1(T) \\
x_3(T)
\end{pmatrix}
\]  
(82)

hence the definition of \( S_1 \). Now
\[
p(T) = \frac{\partial S_1}{\partial x}
\]  
(83)

thus
\[
p_1(T) = p_1(0) \exp(-\lambda_\alpha t) = p_1(0) \exp(-\ln(\lambda_\alpha) T) = 1 + \mu - \lambda_\alpha
\]  
(84)

\[
p_2(T) = p_2(0) \exp(-\lambda_\beta t) = p_2(0) \exp(-\ln(\lambda_\beta) T) = 1 + \mu - \lambda_\beta
\]  
(85)

\[
p_3(T) = p_3(0) \exp(-(1 + \mu) t) = p_3(0) \exp(-\ln(1 + \mu) T) = 0
\]  
(86)

So
\[
p_1(t) = (1 + \mu - \lambda_\alpha) \exp(\ln(\lambda_\alpha)(T - t))
\]  
(87)

\[
p_2(t) = (1 + \mu - \lambda_\beta) \exp(\ln(\lambda_\beta)(T - t))
\]  
(88)

\[
p_3(t) = 0
\]  
(89)

If
\[
H(\dot{x}(t), p(t), u^*(t), t) \leq H(\dot{x}(t), p(t), u, t)
\]  
(90)

for all \( u \in \mathbb{U} \), \( (\dot{x}(t), u^*(t)) \) is optimal. And this is equivalent to
\[
p(t)^T D^{-1} e_j u^*(t) \leq p(t)^T D^{-1} e_j u
\]  
(91)

which again is equivalent to
\[
(1 + \mu - \lambda_\alpha) \exp(\ln(\lambda_\alpha)(T - t))(-\beta)(1 + \mu - \lambda_\beta) \frac{u^*(t)}{\det(D)}
\]  
(92)

\[
+(1 + \mu - \lambda_\beta) \exp(\ln(\lambda_\beta)(T - t))\beta(1 + \mu - \lambda_\alpha) \frac{u^*(t)}{\det(D)}
\]  
(93)
for all $u \in \mathbb{U}$. It follows that

$$u^*(t) = g_i^0$$  \hspace{1cm} (96)$$
is optimal. We have the two Hamiltonians

$$H^Y(x, p, u, t) = p^T Y(x, u) = \langle p, Y(x, u) \rangle$$  \hspace{1cm} (97)$$

$$H^X(y, q, v, t) = q^T X(y, v) = \langle q, X(y, v) \rangle$$  \hspace{1cm} (98)$$

where $\langle \cdot, \cdot \rangle$ denotes the canonical inner product. It follows that when the eigenvalues are $a \pm ib, 1 + \mu > 0, a > 0, b \neq 0$

$$y(t) = Ux(t)$$  \hspace{1cm} (99)$$

$$p(t) = U^T q(t)$$  \hspace{1cm} (100)$$

if

$$y(0) = Ux(0)$$  \hspace{1cm} (101)$$

$$p(T) = U^T q(T)$$  \hspace{1cm} (102)$$

where $y(t)$ is an integral curve of $X$ and $x(t)$ an integral curve of $Y$. Because

$$q' = \frac{\partial H^X}{\partial y} = -(U^{-1})^T \tilde{B}^T U^T q$$  \hspace{1cm} (103)$$
hence

$$U^T q' = -\tilde{B}^T U^T q$$  \hspace{1cm} (104)$$
and

$$p' = -\tilde{B}^T p$$  \hspace{1cm} (105)$$

We now get, that

$$H^X(y(t), q(t), v(t), t) = \langle q(t), X(y(t), v(t)) \rangle$$  \hspace{1cm} (106)$$

$$= \langle q(t), UY(U^{-1}y(t), v(t)) \rangle$$  \hspace{1cm} (107)$$

$$= \langle U^T q(t), Y(U^{-1}(y(t)), v(t)) \rangle$$  \hspace{1cm} (108)$$

$$= \langle p(t), Y(x(t), v(t)) \rangle$$  \hspace{1cm} (109)$$

$$= H^X(x(t), p(t), v(t), t)$$  \hspace{1cm} (110)$$

So

$$H^X(y(t), q(t), v(t), t) = H^X(x(t), p(t), v(t), t)$$  \hspace{1cm} (111)$$

When the eigenvalues of $A$ are real, distinct and positive

$$y(t) = Dx(t)$$  \hspace{1cm} (112)$$
\[ p(t) = D^T q(t) \] (113)

if
\[ y(0) = Dx(0) \] (114)
\[ p(T) = D^T q(T) \] (115)

We also have
\[ q' = -\left(D^{-1}\right)^T \Lambda^T D^T q \] (116)

and
\[ \left(D^T q\right)' = -\Lambda^T D^T q \] (117)

thus
\[ p' = -\Lambda^T p \] (118)

Hence
\[ H^x (y(t), q(t), v(t), t) = H^T (x(t), p(t), v(t), t) \] (119)

We need the following theorem, which is well known.

**Theorem 1** Now The following statements about a \( C^2 \) function from \( \mathbb{R}^n \to \mathbb{R} \) to the reals, where \( n \) is a positive integer, \( n \in \mathbb{N} \) are equivalent:

(i) \( f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y) \) where \( \lambda \in [0,1] \);

(ii) \( f(y) \geq f(x) + \frac{\partial f}{\partial x}(x)(y-x) \);

(iii) \( \sum_{i,j=1,...,n} \gamma_{ij} \frac{\partial^2 f}{\partial x_i \partial x_j}(x) \geq 0 \)

where \( x, y, \gamma \in \mathbb{R}^n \).

\( (x(t), u(t)) \) is admissible by definition if \( 0 \leq u(t) \leq g_t^0 \) and
\[ x'(t) = Y(x(t), u(t)), \quad x(0) = x_0 \in \mathbb{R}^3 \] (121)

To see that
\[ S_1(x^*(T)) \leq S_1(x(T)) \] (122)

for \( (x^*(t), u^*(t)) \) optimal candidate and \( (x(t), u(t)) \) admissible argue as in [2]

\[ \Delta = S_1(x^*(T)) - S_1(x(T)) \] (123)
\[ = S_1(x^*(T)) - S_1(x(T)) \] (124)
\[ \int_0^T \left( H^x (x^*(t), p(t), u^*(t), t) - p(t)^T x''(t) \right) dt \] (125)

We have the following inequality, which follows from theorem 1.
\[
H^T(x^*, p, u^*, t) - H^T(x, p, u, t) \\
\leq \frac{\partial H^T}{\partial x}(x^*, p, u^*, t)(x^* - x) + \frac{\partial H^T}{\partial u}(x^*, p, u^*, t)(u^* - u)
\]

(126)

We also have

\[
p^* = -\frac{\partial H^T}{\partial x}(x^*, p, u^*, t)
\]

(127)

We can now estimate

\[
\Delta \leq S_1(x'(T)) - S_1(x(T)) + \int_0^T p^T(x - x^*) + p^T(x^* - x^*) \, dt
\]

(128)

\[
+ \int_0^T \frac{\partial H^T}{\partial u}(x^*, p, u^*, t)(u^* - u) \, dt
\]

(129)

\[
\leq \int_0^T \frac{d}{dt}(p^T(x - x^*)) \, dt + S_1(x'(T)) - S_1(x(T))
\]

(130)

\[
\leq \frac{\partial S_1}{\partial x}(x'(T))(x(T) - x^*(T)) + S_1(x'(T)) - S_1(x(T))
\]

(131)

\[
\leq 0
\]

(132)

because \( S_1 \) is convex, by theorem 1. We have used, that we have arranged, that

\[
\frac{\partial H}{\partial u}(x^*, p, u^*, t)(u^* - u) \leq 0
\]

(133)

for all \( u \in U \), by the mean value theorem.

We have optimality.

**3. Optimal Control of T**

In this section we consider the problem: minimize \( C(N) \) subject to

\[
y_{k+1} = Ay_k + Bu_k + g = f(y_k, u_k)
\]

(134)

\[
k = 1, \cdots, N - 1, N \in \mathbb{N}, N \geq 2, u(k) \in U = [0, g^0], g^0 > 0, \mu = \mu_k = \mu_i \text{ where } A
\]

is as in the introduction.

Also

\[
y_k = (C(k), GF(k), GI(k)), g \in \mathbb{R}^3
\]

(135)

Here

\[
B = (0, 0, 1)\,^T, \quad g = (g_c, g_s, g_i)\,^T
\]

(136)

Assume (i) the eigenvalues of \( A \) are real and distinct.

In the Discrete Pontryagin Minimal Principle applied to \( T \) you define the Hamiltonian by (138) and then you find

\[
\frac{\partial H}{\partial u_k}(x_k^*, u_k^*, \lambda_k)(u_k^* - u_k)
\]

(137)

and minimize it to find the optimal control \( u_k^* \). It is optimal due to computations (157) to (163) below.
Define then the Hamiltonian
\[ H(x_k, u_k, \lambda_k) = \lambda_k^T(Ax_k + \hat{B}u_k + D^{-1}g) \]  
where \( \lambda_k \in \mathbb{R}^3 \) and
\[ \hat{B} = D^{-1}B \]
\[ \hat{A} = D^{-1}AD \]

Then we have the adjoint equation
\[ \lambda_{k-1} = \frac{\partial H}{\partial x_k}(x_k, u_k, \lambda_k) = \hat{A}^T \lambda_k \]  
Inductively
\[ \lambda_{N-1} = \left( \hat{A}^T \right)^{N-1} \lambda_N \]  
In particular
\[ \lambda_0 = \left( \hat{A}^T \right)^{N-1} \lambda_N \]  
For \( k = 0 \) we have
\[ \begin{aligned}
H(x_0^*, u_0^*, \lambda_0) &= (\lambda_0)^T(Ax_0^* + \hat{B}u_0^* + D^{-1}g) \\
&\leq H(x_0^*, u_0^*, \lambda_0) = (\lambda_0)^T(Ax_0^* + \hat{B}u + D^{-1}g)
\end{aligned} \]  
which is equivalent to
\[ (\lambda_0)^T \hat{B}u_0^* \leq (\lambda_0)^T \hat{B}u \]  
Define
\[ S_j(x) = F(x) = (1 + \mu - \lambda_j)x_1 + (1 + \mu - \lambda_1)x_2 \]  
Now
\[ \lambda_0 = \left( \hat{A}^T \right)^{N-1} \lambda_N \]  
where
\[ \lambda_{N-1} = \frac{\partial F}{\partial x_j}(x_N) \]  
Thus
\[ \begin{aligned}
(\lambda_0)^T \hat{B}u_0^* &= \left( \left( \hat{A}^T \right)^{N-1} \lambda_N \right)^T \hat{B}u_0^* = \left( \lambda_{N-1} \right)^T \hat{A}^{N-1} \hat{B}u_0^* = \frac{\partial F^T}{\partial x_j}(x_N) \hat{A}^{N-1} \hat{B}u_0^*
\end{aligned} \]  
Assume that (i) holds and \( \lambda_1, \lambda_2, \lambda_3, 1 + \mu \in \mathbb{R} \) are distinct, when \( \mu = \mu_f = \mu_f \).
Then
\[ \hat{B} = \begin{pmatrix} D_{31} \\ D_{32} \\ D_{33} \end{pmatrix} \frac{1}{\det(D)} = \begin{pmatrix} -\beta(1 + \mu - \lambda_3) \\ \beta(1 + \mu - \lambda_2) \\ \delta(\lambda_1 - \lambda_2) \end{pmatrix} \frac{1}{\det(D)} = D^{-1}e_3 \]  
So now we get, that (146) amounts to
\[
(1 + \mu - \lambda, 1 - \mu - \lambda, 0)
\begin{pmatrix}
\hat{\lambda}_k^{N-1} & 0 & 0 \\
0 & \hat{\lambda}_k^{N-1} & 0 \\
0 & 0 & (1 + \mu)^{N-1}
\end{pmatrix}
\hat{B} = (152)
\]

\[
= -\beta(-\alpha\delta - \beta\sigma)\lambda_k^{N-1}
\frac{\det(D)}{\det(D)} + \beta(-\alpha\delta - \beta\sigma)\lambda_k^{N-1}
\]

\[
= \frac{\alpha\delta + \beta\sigma}{(\lambda - \lambda_0)(\alpha\delta + \beta\sigma)}(\lambda_k^{N-1} - \lambda_k^{N-1})
\]

\[
= \frac{\beta}{\lambda - \lambda_0}(\lambda_k^{N-1} - \lambda_k^{N-1}) \leq 0
\]

Similarly for \( k = 1, \cdots, N - 2 \)

\[
\lambda_k^T \hat{B} u_k^* = (1 + \mu - \lambda, 1 + \mu - \lambda, 0)
\begin{pmatrix}
\lambda_k^{N-k} & 0 & 0 \\
0 & \lambda_k^{N-k} & 0 \\
0 & 0 & (1 + \mu)^{N-k}
\end{pmatrix}
\hat{B} u_k^* < 0 \quad (156)
\]

For \( k = N - 1 \) we have, that \((1, 0, 0) \cdot \hat{B} = 0\). This means that maximal chemotherapy is optimal. Because similar to (128) to (132), we get

\[
\Delta = S_i \left( x_i^* \right) - S_i \left( x_i \right)
\]

\[
= S_i \left( x_i^* \right) - S_i \left( x_i \right) + \sum_{k=0}^{N-1} \left( H \left( x_i^*, u_i^*, \lambda_k \right) - H \left( x_i, u_i, \lambda_k \right) - \lambda_k^T x_{i+1} + \lambda_k^T x_{i+1} \right)
\]

\[
= S_i \left( x_i^* \right) - S_i \left( x_i \right) + \sum_{k=0}^{N-1} \frac{\partial H}{\partial x_k} \left( x_i^*, u_i^*, \lambda_k \right) \left( x_i^* - x_i \right)
\]

\[
= S_i \left( x_i^* \right) - S_i \left( x_i \right) + \sum_{k=0}^{N-1} \frac{\partial H}{\partial u_k} \left( x_i^*, u_i^*, \lambda_k \right) \left( u_i^* - u_i \right)
\]

\[
\leq S_i \left( x_i^* \right) - S_i \left( x_i \right) + \sum_{k=0}^{N-1} \lambda_k^T x_{i+1} + \lambda_k^T x_{i+1}
\]

\[
\leq S_i \left( x_i^* \right) - S_i \left( x_i \right) + \sum_{k=0}^{N-1} \lambda_k^T x_{i+1} - \lambda_k^T x_{i+1}
\]

\[
= S_i \left( x_i^* \right) - S_i \left( x_i \right) - \frac{\partial S_i}{\partial x} \left( x_i^* \right) \left( x_i^* - x_i \right) \leq 0
\]

by theorem 1 and since \( S_i \) is convex.

Then we have

\[
C_i \left( N \right) - C \left( N \right) = S_i \left( x_i^* \right) - S_i \left( x_i \right) \leq 0
\]

As above we get

\[
\hat{H} \left( y_k, u_k, \xi_k \right) = \xi_k^T \left( Ay_k + g + v_k \right)
\]

\[
y_{k+1} = Ay_k + g + v_k
\]

Also

\[
\xi_{k-1} = A^T \xi_k
\]
and
\[
\lambda_{k-1} = A^T \lambda_k
\]  
(168)

Thus
\[
\lambda_{k-1} = D^\dagger \zeta_{k-1}
\]  
(169)

When
\[
y_0 = Dx_0
\]  
(170)

then
\[
y_k = Dx_k
\]  
(171)

Hence
\[
\tilde{H}(y_k, v_k, \zeta_k) = H(x_k, v_k, \lambda_k)
\]  
(172)

Now consider the case where there are imaginary eigenvalues
\[ a \pm ib, a > 0, b \neq 0, 1 + \mu > 0 \]  
for \( A \). We need the following well known formulas for
\[
\begin{pmatrix} a & b \\ -b & a \end{pmatrix}^{2p+1} = \begin{pmatrix} A_p & B_p \\ -B_p & A_p \end{pmatrix}
\]  
(173)

which are well known, where \( p \in \mathbb{N}_0 \) and
\[
A_p = \sum_{q=0}^{p} \binom{2p+1}{2q} (-1)^q b^{2q} a^{2p-2q}
\]  
(174)

\[
B_p = \sum_{q=0}^{p} \binom{2p+1}{2q+1} (-1)^q b^{2q+1} a^{2p-2q}
\]  
(175)

and \( p \in \mathbb{N} \)
\[
\begin{pmatrix} a & b \\ -b & a \end{pmatrix}^{2p} = \begin{pmatrix} C_p & D_p \\ -D_p & C_p \end{pmatrix}
\]  
(176)

where
\[
C_p = \sum_{q=0}^{p} \binom{2p}{2q} (-1)^q b^{2q} a^{2p-2q}
\]  
(177)

\[
D_p = \sum_{q=0}^{p-1} \binom{2p}{2q+1} (-1)^q b^{2q+1} a^{2p-1-2q}
\]  
(178)

You can prove them by induction. We have
\[
H(x_k, v_k, \lambda_k) = \langle \lambda_k, Bx_k + U^{-1} v_k + U^{-1} g \rangle
\]
\[
= \tilde{H}(y_k, v_k, \zeta_k) = \langle \zeta_k, A v_k + g + e_y v_k \rangle
\]  
(179)

where
\[
\tilde{B} = \begin{pmatrix} a & b & 0 \\ -b & a & 0 \\ 0 & 0 & 1 + \mu \end{pmatrix}
\]  
(180)

As above
\[ \lambda_x = \left( \tilde{B}^T \right)^{N-1} \tilde{\lambda}_{N-1} = \left( \tilde{B}^T \right)^{N-1} \frac{\partial F}{\partial x} \]  

where

\[ S_i(x) = F(x) = (1 + \mu - a)x_1 - bx_2 \]  

So

\[ \lambda_i^T \tilde{B} = (1 + \mu - a, -b, 0) \tilde{B}^{N-1} \begin{pmatrix} \beta b \, \beta (1 + \mu - a)^T \end{pmatrix} \frac{1}{\det(U)} \]  

For \( k \geq 0 \) we find when \( N - k - 1 = 2p + 1, k \neq N - 1 \)

\[
\frac{\partial H}{\partial v_k} = (1 + \mu - a, -b) \begin{pmatrix} A_p & B_p \\ -B_p & A_p \end{pmatrix} B \begin{pmatrix} b \\ 1 + \mu - a \end{pmatrix} \frac{1}{\det(U)} \]  

\[ = \beta \begin{pmatrix} (1 + \mu - a) \, b \, (1 + \mu - a) \, B_p \end{pmatrix} - b \left( -b \right) \frac{1}{\det(U)} \]  

\[ = \beta \begin{pmatrix} (1 + \mu - a)^2 + b^2 \end{pmatrix} B_p \frac{1}{\det(U)} \]  

\[ = \frac{\beta}{b} B_p \]  

Here \( \det(U) = -b \left( \alpha \delta + \beta \sigma \right) \). When \( N - k - 1 = 2p, k \neq N - 1 \)

\[ \frac{\partial H}{\partial v_k} = (1 + \mu - a, -b) \begin{pmatrix} C_p & D_p \\ -D_p & C_p \end{pmatrix} B \begin{pmatrix} b \\ 1 + \mu - a \end{pmatrix} \frac{1}{\det(U)} = \frac{\beta}{b} D_p \]  

If

\[ \frac{\beta}{b} B_p < 0 \]  

let

\[ \nu_k^* = g_i^0 \]  

and

\[ \nu_k^* = 0 \]  

if the reverse inequality holds. If

\[ \frac{\beta}{b} D_p < 0 \]  

let

\[ \nu_k^* = g_i^0 \]  

and

\[ \nu_k^* = 0 \]  

if the reverse inequality holds. Then \( \left( x_e^*, v_k^* \right) \) is optimal, by (157) to (163). For
we have $(1,0,0) \cdot B = 0$.

4. A Counter Example

We shall now present a counter example to optimality of maximal chemo therapy when the eigenvalues are real and $\mu_F \neq \mu_I$ for the model

$$T(y) = Ay + g$$

of the introduction.

Remember the definition of the discriminant $\Delta$ of a cubic polynomial

$$p(\lambda) = \lambda^3 + a_1 \lambda^2 + a_2 \lambda + a_3$$

Namely

$$-108 \Delta = a_1^2 a_2^2 - 27 a_1^2 a_3^2 - 4 a_1^3 a_2 a_3 + 18 a_1 a_2 a_3$$

Notice that the degree of $1 + \gamma$ is four in $a_1^2 a_2^2$ and $-4a_1^3 a_3$ while it is only two or three in $-27 a_1^2 a_3^2, -4a_1^3 a_3, 18a_1 a_2 a_3$. We thus get

$$-108 \Delta = a_1^2 a_2^2 - 4a_1^3 a_3 + \text{lower order terms in } (1 + \gamma)$$

(199)

and this is

$$\Delta < 0$$

(209)

so that there are three real distinct roots, see Uspensky [20]. But

$$A_{12}^2 = \beta (1 + \gamma) + \beta (1 + \mu_I) > 0$$

(210)

when $-\gamma$ is large. So maximal chemo therapy is not optimal for $N = 3$. In fact
\[
x_3 = A^3 x_0 + A^2 (Bu_0 + g) + A (Bu_i + g) + Bu_2 + g
\]

So \( u_0 = 0, u_i = g_i^0 \) gives an optimal trajectory.

5. Optimality of \( T \) When \( \mu_F \neq \mu_I \)

Consider the model \( T \) from the introduction

\[
T(y) = Ay + g + w_{e_i}
\]

where

\[
A = \begin{pmatrix}
1 + \gamma & \alpha & \beta \\
\delta & 1 + \mu_F & 0 \\
\sigma & 0 & 1 + \mu_i
\end{pmatrix}
\]

and

\[
y = (C, GF, GI)^T, \quad g = (g_c, g_F, g_I)^T \in \mathbb{R}^3
\]

Assume (i) the eigenvalues \( \lambda_-, \lambda_+, \tilde{\lambda} = 1+\mu \) of \( A \) are real and distinct, when \( \mu = \mu_F = \mu_I \).

**Theorem 2** There exists a Euclidean open ball \( B_\rho(\mu, \mu), \rho > 0 \) in \( \mathbb{R}^2 \), such that for \( (\mu_F, \mu_I) \in B_\rho(\mu, \mu) \) maximal chemotherapy is optimal.

Also let

\[
D = \begin{pmatrix}
(1 + \mu_F - \lambda_-)(1 + \mu_i - \lambda_+) & (1 + \mu_F - \lambda_-)(1 + \mu_i - \lambda_-) & -\beta(1 + \mu_F - \tilde{\lambda}) \\
-\delta(1 + \mu_i - \lambda_-) & -\delta(1 + \mu_i - \lambda_-) & \delta\beta \\
-\sigma(1 + \mu_F - \lambda_-) & -\sigma(1 + \mu_F - \lambda_-) & -\alpha\delta + (1 + \gamma - \tilde{\lambda})(1 + \mu_F - \tilde{\lambda})
\end{pmatrix}
\]

be a matrix with column eigenvectors to the eigenvalues \( \lambda_-, \lambda_+, \tilde{\lambda} \) of \( A \). We have \( \delta \neq 0 \)

\[
\det(D)_{\mu_F - \mu_I - \mu} = (1 + \mu - \lambda_-)(1 + \mu - \lambda_+)\begin{vmatrix}
1 + \mu - \lambda_- & 1 + \mu - \lambda_+ & 0 \\
-\delta & -\delta & \beta\delta \\
-\sigma & -\sigma & -\alpha\delta
\end{vmatrix}
\]

\[
= (1 + \mu - \lambda_-)(\delta^2\alpha + \beta\sigma\delta)(-\alpha\delta - \sigma\beta) \quad (217)
\]

\[
= (1 + \mu - \lambda_-)(\alpha\delta + \beta\sigma)^2 \delta 
\]

Define the Hamiltonian

\[
H(x_i, u_x, \lambda_x) = \lambda_x^T f_x(x_i, u_x) = \lambda_x^T (Ax_i + g + Bu_x)
\]

\( k = 0, \cdots, N-1 \) and

\[
F(x) = (1 + \mu_F - \lambda_-)(1 + \mu_i - \lambda_-) x_1 + (1 + \mu_F - \lambda_-)(1 + \mu_i - \lambda_-) x_2 \\
- \beta(1 + \mu_F - \tilde{\lambda}) x_3
\]

We have

\[
\lambda_{N-k-1} = (A^T)^k \lambda_{N-1} \quad (222)
\]
In particular
\[
\lambda_0 = (A^T)^{N-1} \lambda_{N-1} = (A^T)^{N-1} \frac{\partial F}{\partial x}(x_N^*)
\]  
(223)

Now consider the system
\[
y_{k+1} = D^{-1}ADy_k + D^{-1}g + D^{-1}e_ju_k
\]  
(224)

which is conjugate to the $x_k$ system. Now observe that with
\[
\tilde{H}(y_k, v_k, \xi_k) = \xi_k^T \left( A\xi_k + D^{-1}g + D^{-1}e_jv_k \right)
\]  
(225)

and
\[
\Lambda = D^{-1}AD
\]  
(226)

we have that
\[
\xi_k = \left( A^T \right)^{N-k-1} \xi_{N-1}
\]  
(227)

and
\[
\xi_0 = \left( A^T \right)^{N-1} \xi_{N-1}
\]  
(228)

\[
\tilde{H}(y_0^*, v_0^*, \xi_k) \leq \tilde{H}(y_0^*, v_0^*, \xi_0)
\]  
(229)

is equivalent to
\[
\xi_0^T D^{-1}Bv_0^* \leq \xi_0^T D^{-1}Bv_0
\]  
(230)

So if
\[
\Delta = \xi_0^T D^{-1}B < 0
\]  
(231)

then maximal chemo therapy is optimal, by a computation like (157) to (163).

Here
\[
\left( \zeta_{N-1} \right)^T = \frac{\partial F}{\partial x}(x_N^*)^T = \left( (1+\mu_r - \lambda_0) \left( (1+\mu_r - \lambda_0) \right) \right)
\]  
(232)

But this amounts to
\[
\Delta = (1+\mu_f - \lambda_a) \left( 1+\mu_r - \lambda_0 \right) \lambda_{N-1}^{N-1} \frac{D_{11}}{\det(D)}
\]  
(233)

\[
+ (1+\mu_f - \lambda_a) \left( 1+\mu_r - \lambda_0 \right) \lambda_{N-1}^{N-1} \frac{D_{12}}{\det(D)}
\]  
(234)

\[
- \beta \left( 1+\mu_f - \lambda_a \right) \zeta_{N-1}^{N-1} \frac{D_{22}}{\det(D)}
\]  
(235)

where $D_{ij}$ are complements in $D$. Hence
\[
D_{11} = (1+\mu_f - \lambda_a) \left( 1+\mu_r - \lambda_0 \right) \delta \beta - \delta \beta \left( 1+\mu_r - \lambda_0 \right) \left( 1+\mu_f - \lambda_a \right)
\]  
(236)

\[
D_{12} = -\left( 1+\mu_f - \lambda_a \right) \left( 1+\mu_r - \lambda_0 \right) \delta \beta + \delta \beta \left( 1+\mu_r - \lambda_0 \right) \left( 1+\mu_f - \lambda_a \right)
\]  
(237)

\[
D_{22} = (1+\mu_f - \lambda_a) \left( 1+\mu_r - \lambda_0 \right) \left( -\delta \right) \left( 1+\mu_r - \lambda_0 \right)
\]  
(238)
Inserted into (233) to (235) we obtain, that $\Delta$ becomes

$$\frac{1}{\det(D)}(1 + \mu_F - \lambda_s)(1 + \mu_I - \lambda_s)\lambda_{N-1}^{N-1}(1 + \mu_F - \lambda_s)(1 + \mu_I - \lambda_s)\delta \beta$$

$$- \delta \beta (1 + \mu_F - \lambda_s)(1 + \mu_I - \lambda_s)\lambda_{N-1}^{N-1}$$

$$+ \frac{1}{\det(D)}(1 + \mu_F - \lambda_s)(1 + \mu_I - \lambda_s)\lambda_{N-1}^{N-1}(-1 + \mu_F - \lambda_s)(1 + \mu_I - \lambda_s)\delta \beta$$

$$+ \delta \beta (1 + \mu_F - \lambda_s)(1 + \mu_I - \lambda_s)\lambda_{N-1}^{N-1}$$

Now observe that when $\mu_F = \mu_I = \mu$

$$\Delta = \frac{1}{\det(D)}(1 + \mu - \lambda_s)(1 + \mu - \lambda_s)(1 + \mu - \lambda_s)(1 + \mu - \lambda_s)\beta \delta \left(\lambda_{N-1}^{N-1} - \lambda_{N-1}^{N-1}\right)$$

$$= \beta \frac{(\alpha \delta + \beta \sigma)^2}{(\lambda_s - \lambda_s)(\alpha \delta + \beta \sigma)^2} \left(\lambda_{N-1}^{N-1} - \lambda_{N-1}^{N-1}\right)$$

So maximal chemo therapy is optimal in this case as we have seen above. Now compute

$$\Delta = \frac{1}{\det(D)}\delta \beta (1 + \mu_F - \lambda_s)(1 + \mu_I - \lambda_s)(1 + \mu_F - \lambda_s)(1 + \mu_I - \lambda_s)\left(\lambda_{N-1}^{N-1} - \lambda_{N-1}^{N-1}\right)$$

$$+ \frac{1}{\det(D)}\delta \beta (1 + \mu_F - \lambda_s)(1 + \mu_I - \lambda_s)(-1 + \mu_F - \lambda_s)(1 + \mu_I - \lambda_s)\left(\lambda_{N-1}^{N-1} - \lambda_{N-1}^{N-1}\right)$$

$$+ \frac{1}{\det(D)}\delta \beta (1 + \mu_F - \lambda_s)(1 + \mu_I - \lambda_s)(1 + \mu_F - \lambda_s)(1 + \mu_I - \lambda_s)\left(\lambda_{N-1}^{N-1} - \lambda_{N-1}^{N-1}\right)$$

Notice that for $k = 0, \cdots, N-2$

$$c_k = \frac{1}{\det(D)}\delta \beta (1 + \mu_F - \lambda_s)(1 + \mu_I - \lambda_s)(1 + \mu_F - \lambda_s)(1 + \mu_I - \lambda_s)\left(\lambda_{N-1}^{N-1} - \lambda_{N-1}^{N-1}\right) < 0$$

when $\mu = \mu_F = \mu_I$, due to the assumptions. Observe also that (246) = 0, (247) = 0, when $\mu = \mu_F = \mu_I$, because then $\lambda = 1 + \mu$. Now take $\rho > 0$ such that

$$\left|\frac{1}{\det(D)}\delta \beta (1 + \mu_F - \lambda_s)(1 + \mu_I - \lambda_s)(-1 + \mu_F - \lambda_s)(1 + \mu_I - \lambda_s)\left(\lambda_{N-1}^{N-1} - \lambda_{N-1}^{N-1}\right)\right| < -\frac{c_k}{3}$$

and

$$\left|\frac{1}{\det(D)}\delta \beta (1 + \mu_F - \lambda_s)(1 + \mu_I - \lambda_s)(1 + \mu_F - \lambda_s)(1 + \mu_I - \lambda_s)\left(\lambda_{N-1}^{N-1} - \lambda_{N-1}^{N-1}\right)\right| < -\frac{c_k}{3}$$
And finally
\[
-\frac{1}{\det(D)} \delta(1+\mu_{p}-\lambda_{+})(1+\mu_{i}-\lambda_{+})(1+\mu_{p}-\lambda_{-})(1+\mu_{i}-\lambda_{-})
\]
\[
\left(\lambda_{+}^{N-k-1} - \lambda_{-}^{N-k-1}\right) > -\frac{2}{3} \tilde{c}_{k}
\]
(250)

But then $\Delta < 0$, and maximal chemo therapy is optimal. We have used, that the roots of a polynomial depend continuously on the coefficients of the polynomial, see [21]. For $k = N-1$ notice that $(1,0,0) \cdot B = 0$. So $u_{N-1} = g^{0}_{i}$ is optimal.

When (ii) define the following $U$ below by computing
\[
\left(1+\mu_{p}-a-ib\right)(1+\mu_{i}-a-ib)
\]
\[
-\delta(1+\mu_{i}-a-ib)
\]
\[
-\sigma(1+\mu_{p}-a-ib)
\]
\[
= \left(1+\mu_{p}-a\right)(1+\mu_{i}-a)b^{2}+i\left(-\left(1+\mu_{i}-a\right)b-\left(1+\mu_{p}-a\right)b\right)
\]
\[
-\delta(1+\mu_{i}-a)+i\delta b
\]
\[
-\sigma(1+\mu_{p}-a)+i\sigma b
\]
\[
= v_{i} = v_{i} + iv_{2}
\]
(253)

So define $U$ to be
\[
\left(1+\mu_{p}-a\right)(1+\mu_{i}-a)b^{2} - (1+\mu_{i}-a)b - (1+\mu_{p}-a)b - \beta\left(1+\mu_{p} - \lambda_{+}\right)
\]
\[
-\delta(1+\mu_{i}-a)
\]
\[
\delta b
\]
\[
-\sigma(1+\mu_{p}-a)
\]
\[
\sigma b
\]
\[
-\alpha\delta\left(1+\gamma - \lambda_{+}\right)\left(1+\mu_{p} - \lambda_{+}\right)
\]
(254)

Then
\[
U^{-1} AU = \begin{pmatrix} a & b & 0 \\ -b & a & 0 \\ 0 & 0 & \tilde{\lambda} \end{pmatrix}
\]
(255)

Define
\[
\tilde{F}(x) = \left(\left(1+\mu_{p}-a\right)(1+\mu_{i}-a)b^{2}\right)x_{1} + \left(-b(a-1-\mu_{i}) - b(a-1-\mu_{p})\right)x_{2}
\]
\[
-\beta\left(1+\mu_{p} - \lambda_{+}\right)x_{3}
\]
(256)

**Theorem 3** Suppose (ii) i.e. eigenvalues of $A$ are $a+ib$, $b \neq 0$. If

$N - k - 1 = 2p + 1, k = 0, \ldots , N - 1$ and

\[
\left\langle e_{i} , A^{N-k-1} B \right\rangle = \frac{\partial F}{\partial \alpha} \begin{pmatrix} A_{p} & B_{p} & 0 \\ -B_{p} & A_{p} & 0 \\ 0 & 0 & 2^{2p} \end{pmatrix} U^{-1} B < 0
\]
(257)

let $v_{i} = g^{0}_{i}$ and if the reverse inequality holds let $v_{i} = 0$. If

$N - k - 1 = 2p, k = 0, \ldots , N - 1$ and

\[
\left\langle e_{i} , A^{N-k-1} B \right\rangle = \frac{\partial F}{\partial \alpha} \begin{pmatrix} C_{p} & D_{p} & 0 \\ -D_{p} & C_{p} & 0 \\ 0 & 0 & 2^{2p} \end{pmatrix} U^{-1} B < 0
\]
(258)

let $v_{i} = g^{0}_{i}$ and if the reverse inequality holds $v_{i} = 0$. Then $\left(\tilde{x}_{i}, v_{i}^*\right)$ is optim-
al.

**Proof.** Define the Hamiltonian

\[
\tilde{H}(\mathbf{y}_k, \mathbf{v}_k, \zeta_k) = \zeta_k^T \left( U^{-1} A \mathbf{y}_k + U^{-1} g + U^{-1} B \mathbf{v}_k \right)
\]  

(259)

Then with

\[
\tilde{B} = U^{-1} A U = \begin{pmatrix} a & b & 0 \\ -b & a & 0 \\ 0 & 0 & \tilde{\lambda} \end{pmatrix}
\]

(260)

we find

\[
\tilde{\zeta}_k = \left( \tilde{B}^T \right)^{N-k-1} \tilde{\zeta}_{N-1} = \left( \tilde{B}^T \right)^{N-k-1} \frac{\partial \tilde{F}}{\partial \mathbf{x}}
\]

(261)

So

\[
\frac{\partial \tilde{H}}{\partial \mathbf{v}_k} = \frac{\partial \tilde{F}^T}{\partial \mathbf{x}} \tilde{B}^{N-k-1} U^{-1} B = \frac{\partial \tilde{F}^T}{\partial \mathbf{x}} \begin{pmatrix} A_p & B_p & 0 \\ -B_p & A_p & 0 \\ 0 & 0 & \tilde{\lambda}^{2p+1} \end{pmatrix} U^{-1} B
\]

(262)

when \( N-k-1 = 2p+1, k = 0, \cdots, N-1 \) and when \( N-k-1 = 2p, k = 0, \cdots, N-1 \)

\[
\frac{\partial \tilde{F}^T}{\partial \mathbf{x}} \begin{pmatrix} C_p & D_p & 0 \\ -D_p & C_p & 0 \\ 0 & 0 & \tilde{\lambda}^{2p} \end{pmatrix} U^{-1} B
\]

(263)

Optimality follows from a computation like (157) to (163).

**Theorem 4** Suppose (i) and the eigenvalues of \( A \) are real and distinct. If

\[
\left\{ e_l, A^{N-k-1} B \right\} = \frac{\partial \tilde{F}^T}{\partial \mathbf{x}} \begin{pmatrix} \tilde{\lambda}_l^{N-k-1} & 0 & 0 \\ 0 & \tilde{\lambda}_l^{N-k-1} & 0 \\ 0 & 0 & \tilde{\lambda}_l^{N-k-1} \end{pmatrix} D^{-1} B < 0
\]

(264)

let \( v_k^* = g_l^* \) and if the reverse inequality holds let \( v_k^* = 0 \). Then \( (v_k^*, v_k^*) \) is optimal.

**Proof.** Define the Hamiltonian

\[
\tilde{H}(\mathbf{y}_k, \mathbf{v}_k, \zeta_k) = \zeta_k^T \left( D^{-1} A \mathbf{y}_k + D^{-1} g + D^{-1} B \mathbf{v}_k \right)
\]

(265)

Then with

\[
\Lambda = D^{-1} A
\]

(266)

we get

\[
\tilde{\zeta}_k = \left( \Lambda^T \right)^{N-k-1} \tilde{\zeta}_{N-1} = \left( \Lambda^T \right)^{N-k-1} \frac{\partial \tilde{F}}{\partial \mathbf{x}}
\]

(267)

Then the partial derivative of \( \tilde{H} \) with respect to \( \mathbf{v}_k \) is

\[
\frac{\partial \tilde{F}^T}{\partial \mathbf{x}} \begin{pmatrix} \tilde{\lambda}_l^{N-k-1} & 0 & 0 \\ 0 & \tilde{\lambda}_l^{N-k-1} & 0 \\ 0 & 0 & \tilde{\lambda}_l^{N-k-1} \end{pmatrix} D^{-1} B
\]

(268)

Optimality follows from a computation like (157) to (163).
6. Summary

In the present paper we applied the discrete Pontryagin Minimal Principle to a discrete model $T$ of cancer growth and the Pontryagin Minimal Principle to an affine vector field that generates $T$. When $\mu = \mu_\mu = \mu$ and the eigenvalues of $T$ are real and distinct, maximal chemotherapy is optimal for the discrete model, while this is not necessarily so when the eigenvalues of $A$ are $1 + \mu, a + ib, b \neq 0, 1 + \mu > 0$.

For the affine vector field that generates $T$, we have proven similar statements, when $\mu = \mu_\mu = \mu$. Maximal chemotherapy is optimal, when the eigenvalues of $A$ are real, positive and distinct and this is not necessarily so, when there are imaginary eigenvalues. We finally considered what happens in the discrete model, when $\mu_\mu \neq \mu$. In particular we have derived an optimal strategy to give chemotherapy or immune therapy.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

References


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Threshold Dynamics of a Vector-Borne Epidemic Model for Huanglongbing with Impulsive Control

Jianping Wang*, Feifei Feng, Zhicai Guo, Hengmin Lv, Juanjuan Wang

The Department of Basic Course Education, Ji’an College, Ji’an, China
Email: *jpwmath@126.com

Abstract
In this paper, the basic reproduction number is calculated for Huanglongbing (HLB) model with impulses which is a vector-borne epidemic model with impulses. For controlling HLB, farmers’ experience is replanting of healthy plants and removing infected plants. To reflect the real world, we construct an impulsive control model which considers replanting of healthy plants and removing infected plants at one fixed time. By analyzing the model, we conclude that the disease-free equilibrium is globally asymptotically stable if the basic reproduction number $R_0 < 1$, and we prove that the HLB is permanence if the basic reproduction number $R_0 > 1$.

Keywords
Huanglongbing, Impulsive Control, Basic Reproduction Number, Disease-Free Periodic Solution, Permanence

1. Introduction
Huanglongbing (HLB) is one of the most serious problems of citrus worldwide which caused by the bacteria Candidatus Liberibacter spp., whose name in Chinese means “yellow dragon disease”, was first reported from southern China in 1919 and is now known to occur in next to 40 different Asian, African, Oceanian, South and North American countries [1]. HLB has no cure and affects all citrus varieties, reducing the productivity of orchards because the fruits of infected plants have poor quality and, in extreme cases, infection leads to plant death [2]. HLB symptoms are virtually the same wherever the disease occurs. Infected trees show a blotchy mottle condition of the leaves that result in the development of yellow shoots, the early and very characteristic symptom of the disease [1].
we all know, HLB can be spread efficiently by vector psyllids to all commercial cultivars of citrus [3] [4].

Mathematical models play an important role in understanding the epidemiology of vector-transmitted plant diseases. Applications of mathematical approach to plant epidemics were reviewed by Van der Plank [5] and Kranz [6]. There are many authors establish continuous mathematical models to describe the transmission of HLB. Chiyaka et al. [7] proposed a compartmental model of ordinary differential equations for the HLB transmission dynamics within a citrus tree considering 10 dimensions. In [2], Raphael et al. constructed a 6 dimensions model of ordinary differential equations with delay time. However the dynamic behaviors of these models are studied only by using computer simulations.

But, in our real world, farmers’ experiences have led to development of integrated management concepts for virus diseases that combine available host resistance, cultural, chemical and biological control measures. A cultural control strategy including replanting, and/or removing (rouging) diseased plants is a widely accepted treatment for plant epidemics which involves periodic inspections followed by removal of the detected infected plants [8] [9] [10] [11] [12]. Periodic replanting of healthy plants or removing (rouging) infected plants in plant-virus disease epidemics is widely used to minimize losses and maximize returns [12]. There are only a few countries have been able to control Asian HLB. São Paulo State (SPS) might be one of the first to be successful. In SPS, encouraging results have been obtained in the control of HLB by tree removal and insecticide treatments against psyllids [13]. Monocrotophos has a short residual effect on psyllid, repeated application is often required to suppress psyllid, which can cause pesticide resistance. Pesticides pollution is also recognized as a major health hazard to human beings and beneficial insects. To deal with these questions, we propose model dealing in detail with the killing efficiency rate and decay rate of pesticides. The residual effects of pesticides (i.e. killing efficiency rate and decay rate) on the threshold conditions are also addressed.

A model for the temporal spread of an epidemic in a closed plant population with periodic removals of infected plants has been considered by Fishman et al. [8]. Integrated management has been found to be more effective at eliminating epidemics. In this paper, according to the above biological background, we develop a hybrid impulsive control model, in which replanting of healthy plants and removing infected plants at one fixed moment and pesticide spraying at another fixed moment are considered, to propose optimal control strategy.

The paper is organized as follows. In Section 2, we formulate the impulsive epidemic model and also simplify the original system (2.1). In Section 3, we introduce some useful lemmas and the basic reproduction number of the model. In Sections 4 and 5, we proved the global stability of the disease-free equilibrium and permanence of the model, respectively. In the finally section, a brief discussion is given.
2. Model Formulation

Let \( S_h(t), I_h(t) \) denote susceptible citrus host and infected citrus host, respectively, and \( S_v(t), I_v(t) \) represent susceptible psyllid and infected psyllid, respectively. We give the following system:

\[
\begin{aligned}
\frac{dS_h(t)}{dt} &= -\lambda S_h(t) I_v(t) - d_1 S_h(t), \\
\frac{dI_h(t)}{dt} &= \lambda S_h(t) I_v(t) - d_1 I_h(t) - v I_h(t), \\
\frac{dS_v(t)}{dt} &= \lambda - \beta S_v(t) I_h(t) - d_2 S_v(t), \\
\frac{dI_v(t)}{dt} &= \beta S_v(t) I_h(t) - d_2 I_v(t), \\
S_h(t^+) &= S_h(t) + \delta, \\
I_h(t^+) &= (1 - \phi) I_h(t), \\
S_v(t^+) &= S_v(t), \\
I_v(t^+) &= I_v(t),
\end{aligned}
\]  

(2.1)

with initial condition

\[
S_h(0^+) > 0, I_h(0^+) > 0, S_v(0^+) > 0, I_v(0^+) > 0.
\]  

(2.2)

The model is satisfied with the following assumptions.

- \( S_h, I_h, S_v \) and \( I_v \) are left continuous, that is, \( S_h(t) = S_h(t^-) \), \( I_h(t) = I_h(t^-) \), \( S_v(t) = S_v(t^-) \) and \( I_v(t) = I_v(t^-) \) for all \( t \geq 0 \).
- \( \lambda \geq 0 \) is the infected rate of citrus host. \( d_1 > 0, v > 0 \) are the nature death rate and disease induced death rate of citrus, respectively.
- \( \Lambda > 0 \) is constant recruitment rate of psyllid.
- \( \beta \geq 0, d_2 > 0 \) are the infected rate and nature death rate of psyllid, respectively.
- \( \delta \geq 0, 0 < \phi < 1 \) are the recruitment rate of citrus and removing rate of infected citrus by impulses, respectively.
- \( T > 0 \) is the interpulse time, i.e., the time between two consecutive pulse replanting and removing.

The following lemma is obvious.

**Lemma 2.1.** If \( S_h(0^+) > 0, I_h(0^+) > 0, S_v(0^+) > 0 \) and \( I_v(0^+) > 0 \), then \( S_h(t) > 0, I_h(t) > 0, S_v(t) > 0 \) and \( I_v(t) > 0 \) for every \( t > 0 \).

Denote \( G := \left\{ (S_h, I_h, S_v, I_v) \in \mathbb{R}^4_+ \mid 0 \leq S_h + I_h \leq N_h^*, 0 \leq S_v + I_v \leq N_v^* \right\} \), where

\[
N_h^* = \frac{\delta}{1 - e^{-\alpha T}}, \quad N_v^* = \frac{\Lambda}{d_2}.
\]

**Theorem 2.1.** The solutions of system (2.1) with initial condition (2.2) eventually enter into \( G \) and \( G \) is positively invariant for system (2.1).

**Proof:** Let \( N_h = S_h + I_h, N_v = S_v + I_v \). By system (2.1), we have
By the first and third equations of (2.3), we get
\[
\begin{cases}
\frac{dN_h(t)}{dt} = -d_iN_h(t) - vl_h(t), & t \neq kT, \\
N_h(t^+) = N_h(t) - \phi I_h(t) + \delta, & t = kT.
\end{cases}
\] (2.3)

Thus, we have \( N_h \leq \frac{\delta}{1 - e^{-d_iT}}. \)

From the second and fourth equations of (2.3), we have
\[
\frac{dN_v(t)}{dt} \leq \Lambda - d_zN_v.
\]

Then, we have \( N_v \leq \frac{\Lambda}{d_z}. \)

Then, from the above analysis, which implies that \( G \) is positively invariant.

### 3. The Basic Reproduction Number of (2.1)

Let \((R^n, R^n)\) be the standard ordered \(n\)-dimensional Euclidean space with a norm \(|\cdot|\). For \(u, v \in R^n\), we write \(u \geq v\) if \(u - v \in R^n\), \(u > v\) if \(u - v \in R^n \setminus \{0\}\), \(u \gg v\) if \(u - v \in Int(R^n)\), respectively.

Set \(A(t)\) be cooperative, irreducible and periodic \(n \times n\) matrix function with period \(\omega > 0\), \(P\) be a \(n \times n\) constant matrix, \(T\) be a pulse period satisfying \(\omega |T| = q, q \in N\). Then \(\Phi_{A(t)}(t)\) is the fundamental solution matrix of the linear differential equation
\[
\frac{dy(t)}{dt} = A(t)y,
\]
and \(r(P^\omega \Phi_{A(t)}(\omega))\) is the spectral radius of \(P^\omega \Phi_{A(t)}(\omega)\). By Perron-Frobenius theorem, \(r(P^\omega \Phi_{A(t)}(\omega))\) is the principal eigenvalue of \(\Phi_{A(t)}(\omega)\) in the sense that it is simple and admits an eigenvector \(v^* \gg 0\).

Firstly, we introduce some lemmas which will be useful for our further arguments.

**Lemma 3.1.** [14] Let \(\mu = \frac{1}{\omega} \ln r(P^\omega \Phi_{A(t)}(\omega))\). Then there exists a positive, \(\omega\)-periodic function \(v(t)\) such that \(e^{\omega t} v(t)\) is a solution of
\[
\begin{cases}
\frac{dy(t)}{dt} = A(t)y, & t \neq kT, k \in N, \\
y(t^+) = P y(t), & t = kT, k \in N.
\end{cases}
\] (3.1)

In what follows, we give the basic reproduction number \(R_0\) for system (2.1).
Similar to Yang and Xiao [15].

An impulsive periodic differential mathematical model in which impulses occur at fixed times may be described as follows:

\[
\begin{align*}
\frac{dx(t)}{dt} &= f(t, x), \quad t \neq kT, \\
x(t^+) &= x(t) + I(t), \quad t = kT,
\end{align*}
\]

where \( f : \mathbb{R}^n \times \Omega \rightarrow \mathbb{R}^n \) is \( \omega \)-periodic function and \( q = \omega/T \), for \( t = kt \), \( x(kt^+) = \lim x(kt + h), k \in \mathbb{N}, N = 1, 2, \cdots \) and \( \Omega \) is an open set.

Let \( \mathcal{F}(t, x(t)) \) be the input rate of newly infected individuals in the \( i \)-th compartment, and \( \mathcal{V}(t, x(t)) = \mathcal{V}_1(t, x(t)) - \mathcal{V}_2(t, x(t)) \) where \( \mathcal{V}_1(t, x(t)) \) be the input rate of individuals by other means, and \( \mathcal{V}_2(t, x(t)) \) be the rate of transfer of individuals out of compartment \( \xi \) then \( \mathcal{V}(t, x(t)) \) denotes the net transfer rate out of compartments. We suppose that \( x(t_i) \) immediately after pulses equals

\[
x(kt^+) = x(kt) + I(kt) = \psi(x(kt)),
\]

where \( \psi : \Omega \rightarrow \Omega, \Omega \in \mathbb{R}^n, \psi \in C^1(\Omega, \Omega) \).

Denote

\[
x = (x_1, x_2, \cdots, x_n)^T,
\]

\[
\mathcal{F}(t, x) = (\mathcal{F}_1(t, x), \mathcal{F}_2(t, x), \cdots, \mathcal{F}_n(t, x))^T,
\]

\[
\mathcal{V}(t, x) = (\mathcal{V}_1(t, x), \mathcal{V}_2(t, x), \cdots, \mathcal{V}_n(t, x))^T,
\]

where \( A^T \) denotes the transpose of \( A \), and \( x_1, x_2, \cdots, x_n \) are \( n \) homogeneous compartments in a heterogeneous population, with each \( x_i \geq 0 \) being the number of individuals in each compartment. Assume that the compartments sort by two types, with the first \( m \) compartments \( x_1, x_2, \cdots, x_m \) the infected individual, and \( x_{m+1}, \cdots, x_n \) the uninfected individuals. Denote

\[
X = (x_1, \cdots, x_m), \quad Y = (x_{m+1}, \cdots, x_n),
\]

\[
\psi = (h, g)^T, \quad h = (\psi_1, \cdots, \psi_m), \quad g = (\psi_{m+1}, \cdots, \psi_n),
\]

\[
h \in C^1(\Omega, \mathbb{R}^m), \quad g \in C^1(\Omega, \mathbb{R}^{n-m}).
\]

Now, system (3.2) can be written as

\[
\begin{align*}
\frac{dx(t)}{dt} &= \mathcal{F}(t, x(t)) - \mathcal{V}(t, x(t)), \quad t \neq kT, \\
x(t^+) &= h(x(t)), \quad t = kT.
\end{align*}
\]

Define \( X_\delta \) to be the set of all disease-free states:

\[
X_\delta = \{ x \geq 0 \mid x_i = 0, i = 1, \cdots, m \}.
\]

Furthermore, assume that

\[
\tilde{x}(t) = (0, \cdots, 0, \tilde{x}_{m+1}(t), \cdots, \tilde{x}_n(t))^T.
\]
be a disease-free periodic solution over the \( k \)-th time interval \((kT,(k+1)T]\) with \( \bar{x}_i(t) > 0 \), \( m+1 \leq i \leq n \), for all \( t \geq 0 \).

Let
\[
F(t) = \left( \frac{\partial \mathcal{F}(t,x(t))}{\partial x_j} \right)_{1 \leq i, j \leq m}, \quad V(t) = \left( \frac{\partial \mathcal{V}(t,x(t))}{\partial x_j} \right)_{1 \leq i, j \leq m},
\]
\[
M(t) = \left( \frac{\partial \psi_i(t,x(t))}{\partial x_j} \right)_{m+1 \leq i, j \leq m}, \quad Q = \left( \frac{\partial \psi_i(t,x(t))}{\partial x_j} \right)_{m+1 \leq i, j \leq m},
\]
and
\[
P = \left( \frac{\partial \psi_i(t,x(t))}{\partial x_j} \right)_{1 \leq i, j \leq m},
\]
where \( \mathcal{F}(t,x(t)) \), \( \mathcal{V}(t,x(t)) \), \( f_i(x(t)) \), \( x \), and \( \psi \) are the \( i \)-th component of \( \mathcal{F}(t,x(t)) \), \( \mathcal{V}(t,x(t)) \), \( f_i(x(t)) \), \( x \) and \( \psi \), respectively.

We make the following assumptions, which are the same biological meanings as those by Wang and Zhao [16] and Yang and Xiao [15].

(H1) If \( x \geq 0 \), then \( \mathcal{F}_i, \mathcal{V}^+, \mathcal{V}^- \geq 0 \) for \( i = 1, \ldots, n \).

(H2) If \( x_i = 0 \), then \( \mathcal{V}^- = 0 \). In particular, if \( x \in X_i \), then \( \mathcal{V}^- = 0 \) for \( i = 1, \ldots, m \).

(H3) \( \mathcal{F}_i = 0 \) if \( i > m \).

(H4) If \( x \in X_j \), then \( \mathcal{F}_i = 0 \) and \( \mathcal{V}^+_i = 0 \) for \( i = 1, \ldots, m \).

(H5) The pulses on the infected compartments must be uncoupled with the uninfected compartments; that is, \( h(x(nT)) \) is essentially \( h(X(nT)) \).

(H6) It holds that \( h(0) = 0 \).

(H7) \( r(Q^\ast \Phi_M(T)) < 1 \), where \( \Phi_M(t) \) is the fundamental solution matrix of the system
\[
\frac{dZ(t)}{dt} = M(t)Z.
\]

(H8) \( r(P^\ast \Phi_M(T)) < 1 \).

In the following, we study the threshold dynamics of system (2.1) and show that its basic reproduction number can be defined as the spectral radius of the so-called next infection operator as that in impulsive and periodic environment [16].

Let \( Y(t,s), t \geq s \) be the evolution operator of the linear impulsive periodic system
\[
\begin{align*}
\frac{dy(t)}{dt} &= -V(t)y, \quad t \neq nT, \\
y(t^+) &= P_y(t), \quad t = nT,
\end{align*}
\]
where the explicit expression of \( Y(t,s) \) can be found in [17], we omit it here. By assumption (H1)-(H8), we also know that the periodic solution of system (3.4) is asymptotically stable.

Now, we define the so-called next infection operator \( L \) as follows:
\[
(L\phi)(t) = \lim_{\alpha \to \infty} \int_a^T Y(t,s)F(s)\phi(s)ds, \quad \phi \in C_o,
\]
where $C_{\omega}$ is defined as the ordered Banach space of all $\omega$-periodic functions from $\mathbb{R}$ to $\mathbb{R}^m$, equipped with the maximum norm $\| \|_{\infty}$, and the positive cone 

$C_{\omega}^+ = \{ \phi \in C_{\omega} : \phi(t) \geq 0, \forall t \in \mathbb{R} \}; \phi(s)$ is the initial distribution of infectious individuals.

The limit as $a$ goes to infinity does exist, and the next infection operator $L$ is well defined, continuous, positive and compact on the domain. We now define the basic reproductive number as the spectral radius of $L$ is $R_0 \hat{=} \rho(L)$.

From above discussion, we have the following results.

**Lemma 3.3.** Assume that (H1)-(H8) hold, Then the following statements are valid:

1) $R_0 = 1$ if and only if $r\left(P^t \Phi_{F-x}(\omega)\right) = 1$.
2) $R_0 > 1$ if and only if $r\left(P^t \Phi_{F-x}(\omega)\right) > 1$.
3) $R_0 < 1$ if and only if $r\left(P^t \Phi_{F-x}(\omega)\right) < 1$.

The proof in detail is similar to periodic systems in [15].

**Lemma 3.4.** If $R_0 < 1$ the disease-free periodic solution $\hat{x}(t)$ is asymptotically stable, and unstable if $R_0 > 1$.

**Proof:** Observe that the linearized system of system (3.3) at the disease-free periodic solution is

$$
\begin{align*}
\frac{dx(t)}{dt} &= \begin{pmatrix} F(t) - V(t) & 0 \\ -J(t) & M(t) \end{pmatrix} x(t), \quad t \neq kT, \\
\hat{x}(t^+) &= \begin{pmatrix} P & 0 \\ \Gamma & Q \end{pmatrix} \hat{x}(t), \quad t = kT.
\end{align*}
$$

(3.5)

Then the monodromy matrix of the impulsive system (3.5) equals

$$
\begin{pmatrix} P & 0 \\ \Gamma & Q \end{pmatrix} \Phi_{F-x}(T) \begin{pmatrix} 0 \\ \Phi_M(T) \end{pmatrix},
$$

where * represents a non-zero block matrix. Then the Floquet multipliers of system (3.3) are the eigenvalues of $r\left(P^t \Phi_{F-x}(T)\right)$ and $r\left(Q^t \Phi_M(T)\right)$. By assumption (H7), that is, $r\left(Q^t \Phi_M(T)\right) < 1$, it then follows that the disease-free periodic solution is asymptotically stable if $r\left(P^t \Phi_{F-x}(T)\right) < 1$, and unstable if $r\left(P^t \Phi_{F-x}(T)\right) > 1$. This completes the proof.

Following, we demonstrate the existence of the disease-free periodic solution. Set $I_a(t) = 0, I_s(t) = 0$ for all $t > 0$. Under this condition, we have the following system:

$$
\begin{align*}
\frac{dS_a(t)}{dt} &= -d_s S_a(t), \quad t \neq kT, \\
\frac{dS_s(t)}{dt} &= \Lambda - d_s S_s(t), \\
S_a(t^+) &= S_a(t) + \delta, \\
S_s(t^+) &= S_s(t), \\
S_s(t) &= S_s(t), \quad t = kT.
\end{align*}
$$

(3.6)

From the first and third equations of system (3.6), we have
\[
\begin{align*}
\frac{dS_h(t)}{dt} &= -d_t S_h(t), \quad t \neq kT, \\
S_h(t^+) &= S_h(t) + \delta, \quad t = kT.
\end{align*}
\]

(3.7)

Then, over the \( k \)-th impulsive interval,
\( S_h(t) = S_h(kT^+)e^{-d_t(t-kT)}, t \in (kT,(k+1)T] \). By the impulsive condition, we have
\( S_h((k+1)T^+) = S_h(kT^+)e^{-d_t} + \sigma \). The unique fixed point of this system equals
\( S_h^*(t) = \frac{\sigma}{1 - e^{-d_t}} \).

Accordingly, the impulsive periodic solution of the system (3.7) is
\( S_h^*(t) = \frac{\sigma}{1 - e^{-d_t}}e^{-d_t(t-kT)} , \quad t \in (kT,(k+1)T] \).

Obviously, \( S_h^*(t) \) is globally asymptotically stable.

From system (3.6), we know that \( S_h(t) \) is not affected by impulse, and we have
\( \lim_{t \to \infty} S_h(t) = \Lambda d_2^{-1} \). Hence, system (2.1) has a unique disease-free periodic solution
\( E_0 = (S_h^*(t),0,S_h^*(t),0) \).

Obviously, by Lemma 3.4, we have that \( (S_h^*(t),0,S_h^*(t),0) \) of system (2.4) is asymptotically stable if \( R_0 < 1 \), and unstable if \( R_0 > 1 \).

We denote \( x(t) = (I_h(t), I_s(t), S_h(t), S_s(t))^T \), then for system (2.1), we have
\[
\mathcal{F}(t,x(t)) = \begin{bmatrix}
\lambda S_h(t)I_v(t) \\
\beta S_h(t)I_b(t) \\
0 \\
0
\end{bmatrix},
\]

(3.8)

\[
\mathcal{V}^+(t,x(t)) = \begin{bmatrix}
\begin{smallmatrix}
0 \\
0 \\
\Lambda
\end{smallmatrix} & \begin{smallmatrix}
d_1I_v(t) \\
d_2I_s(t) \\
\lambda S_h(t)I_v(t) + d_1S_h(t) \\
\beta S_h(t)I_b(t) + d_2S_s(t)
\end{smallmatrix}
\end{bmatrix},
\mathcal{V}^-(t,x(t)) = \begin{bmatrix}
d_1I_v(t) \\
d_2I_s(t) \\
\lambda S_h(t)I_v(t) + d_1S_h(t) \\
\beta S_h(t)I_b(t) + d_2S_s(t)
\end{bmatrix},
\]

(3.9)

and \( \mathcal{V}(t,x(t)) = \mathcal{V}^+(t,x(t)) - \mathcal{V}^-(t,x(t)) \).

Furthermore, we denote \( X = (I_h,I_s),Y = (S_h,S_s) \). By [15], suppose that \( x(t) \) immediately after pulses equals
\[
x(t^+) = \psi(x(t)), \quad t = nT.
\]

(3.10)

For the system (2.1), we have
\[
\psi = (h,g)^T, \quad h = (\psi_1,\psi_2) = ((1-\phi)I_h(t),I_s(t)),
\]
\[
g = (\psi_3,\psi_4) = (S_h(t) + \delta,S_s(t)).
\]

Clearly, conditions (H1)-(H6) are satisfied for system (2.1). There are only (H7) and (H8) should be verified in the following.

\( x(t) = (0,0,S_h^*(t),S_s^*(t)) \) is the disease-free periodic solution for system (2.1). We define \( f(t,x(t)) = \mathcal{F}(t,x(t)) - \mathcal{V}(t,x(t)) \),
\[
M(t) = \left[ \frac{\partial f_i(t,x(t))}{\partial x_j} \right]_{3\leq i,j\leq 4} \quad \text{and} \quad Q = \left[ \frac{\partial \psi_i(t,x(t))}{\partial x_j} \right]_{3\leq i,j\leq 4}, \quad \text{where} \quad f_i(t,x(t)), \quad x_i
and \( \psi_i \) are the \( i \)-th component of \( f(t,x(t)) \), \( x \) and \( \psi \), respectively.

Then, from (3.8) and (3.9), we obtain
\[
M(t) = \begin{pmatrix} -d_i & 0 \\ 0 & -d_x \end{pmatrix}.
\]

From (3.10), we have
\[
Q = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix},
\]
and hence, \( r(Q \Phi_{st}(T)) < 1 \). Therefore, (H7) holds.

We further denote \( F(t), V(t) \) and \( P \) are \( 2 \times 2 \) matrices defined by
\[
F(t) = \begin{pmatrix} \frac{\partial F_i(t,x(t))}{\partial x_j} \\ \frac{\partial F_i(t,x(t))}{\partial x_j} \end{pmatrix}_{ij=1,2},
\]
\[
V(t) = \begin{pmatrix} \frac{\partial V_i(t,x(t))}{\partial x_j} \\ \frac{\partial V_i(t,x(t))}{\partial x_j} \end{pmatrix}_{ij=1,2}
\]
\[
P = \begin{pmatrix} \frac{\partial \psi_i(x(t))}{\partial x_j} \\ \frac{\partial \psi_i(x(t))}{\partial x_j} \end{pmatrix}_{ij=1,2},
\]
where \( F_i(t,x) \) and \( V_i(t,x) \) are the \( i \)-th component of \( F(t,x) \) and \( V(t,x) \), respectively. Then from (3.8), (3.9) and (3.10), it follows that
\[
F(t) = \begin{pmatrix} 0 \\ \beta S_i^*(t) \end{pmatrix},
\]
\[
V(t) = \begin{pmatrix} d_i + \nu & 0 \\ 0 & d_x \end{pmatrix},
\]
and
\[
P = \begin{pmatrix} 1 - \phi \\ 0 \\ 0 \\ 1 \end{pmatrix}.
\]

It is easy to see that \( r(P^\omega \Phi_{st}(\omega)) < 1 \) satisfied. (H8) is hold.

Thus, the Lemma 3.3 is right for system (2.1).


In this section, we prove that the disease-free periodic solution \( E_0 = (S_h^*(t), 0, S^*_v(t), 0) \) is globally asymptotically stable, if \( R_0 < 1 \) and hence, the disease extinct.

Firstly, we need to prove the following lemma.

**Lemma 4.1.** For the system (2.1), it holds that
\[
\lim_{t \to +\infty} \left( N_i (t) - S_i^* (t) \right) = 0,
\]
where
\[
N_i (t) = S_h (t) + I_i (t), N_v (t) = S_v (t) + I_v (t).
\]

**Proof:** Let \( y_i (t) = N_i (t) - S_i^* (t) \), from Theorem 2.1, we have
\[
\frac{dy_i (t)}{dt} = \frac{dN_i (t)}{dt} - \frac{dS_i^* (t)}{dt} = -d_i N_i (t) - \nu N_i (t) - d_i S_i^* (t)
\]
\[
\leq -d_i \left( N_i (t) - S_i^* (t) \right) = -d_i y_i (t),
\]
and
\[
y_i (t^+) = N_i (t) - S_i^* (t^+)
\]
\[
= N_i (t) - \psi I_i (t) + \delta - S_i^* (t) - \delta
\]
\[
\leq N_i (t) - S_i^* (t) = y_i (t),
\]
\[
\lim_{t \to +\infty} \left( N_i (t) - S_i^* (t) \right) = 0.
\]
for \( t_z > t_1 \).

Obviously, by (4.1), (4.2) and the comparison principle of impulsive differential equations in [17], we have

\[
\lim_{t \to +\infty} y_1(t) = \lim_{t \to +\infty} \left( N_i(t) - S_i^*(t) \right) = 0.
\]

In similar method, we can prove

\[
\lim_{t \to +\infty} \left( N_i(t) - S_i^*(t) \right) = 0,
\]

for \( t_z > t_1 \).

Hence, the proof is completed.

**Theorem 4.1.** For any solution of system (2.1), if \( R_0 < 1 \), then the disease-free periodic solution \((S_i^*(t), 0, S_i^*(t), 0)\) is globally asymptotically stable and if \( R_0 > 1 \), then it is unstable.

**Proof:** By Lemma 3.3, if \( R_0 > 1 \), then \((S_i^*(t), 0, S_i^*(t), 0)\) is unstable and if \( R_0 < 1 \), then \((S_i^*(t), 0, S_i^*(t), 0)\) is locally stable. Hence, it is sufficient to show that the global attractivity of \((S_i^*(t), 0, S_i^*(t), 0)\) for \( R_0 < 1 \).

Now, we prove the global attractivity of the disease-free solution.

From Lemma 4.1, there exist a \( t_1 \geq t_2 \) and a positive constant \( \epsilon_1 \) such that \( S_i(t) \leq S_i^*(t) + \epsilon_1 \), \( S_i(t) \leq S_i^*(t) + \epsilon_1 \).

By the second, fourth, sixth and eighth equations of system (2.1), we have

\[
\begin{align*}
\begin{cases}
\frac{dI_h}{dt}(t) &\leq \lambda \left( S_h^*(t) + \epsilon_1 \right) I_c(t) - (d_1 + v) I_h(t), \\
\frac{dI_c}{dt}(t) &\leq \beta \left( S_c^*(t) + \epsilon_1 \right) I_h(t) - d_2 I_c(t), \\
I_h(t^+) &= (1 - \phi) I_h(t^+), \\
I_c(t^+) &= I_c(t),
\end{cases}
\end{align*}
\]

for \( t \geq t_1 \).

Set \( M_{\epsilon_1}(t) \) be the \( 2 \times 2 \) matrix function such that

\[
M_{\epsilon_1}(t) = \begin{pmatrix} 0 & \epsilon_1 \\ \epsilon_1 & 0 \end{pmatrix}.
\]

By Lemma 3.3, we have \( r \left( P^t \Phi_{P \cdot \cdot \cdot P} (\omega) \right) < 1 \), we restrict \( \epsilon_1 > 0 \), such that \( r \left( P^t \Phi_{P \cdot \cdot \cdot P + M_{\epsilon_1}} (\omega) \right) < 1 \). Let us consider the following system

\[
\begin{align*}
\begin{cases}
\frac{dI_h}{dt}(t) &= \lambda \left( S_h^*(t) + \epsilon_1 \right) I_c(t) - (d_1 + v) I_h(t), \\
\frac{dI_c}{dt}(t) &= \beta \left( S_c^*(t) + \epsilon_1 \right) I_h(t) - d_2 I_c(t), \\
I_h(t^+) &= (1 - \phi) I_h(t^+), \\
I_c(t^+) &= I_c(t),
\end{cases}
\end{align*}
\]

for \( t \neq kT \),

\[
\begin{align*}
\begin{cases}
\frac{dI_h}{dt}(t) &= \lambda \left( S_h^*(t) + \epsilon_1 \right) I_c(t) - (d_1 + v) I_h(t), \\
\frac{dI_c}{dt}(t) &= \beta \left( S_c^*(t) + \epsilon_1 \right) I_h(t) - d_2 I_c(t), \\
I_h(t^+) &= (1 - \phi) I_h(t^+), \\
I_c(t^+) &= I_c(t),
\end{cases}
\end{align*}
\]

for \( t = kT \).

By Lemma 3.1 and the standard comparison principle, there exists a positive \( T \)-periodic function \( V(t) = (V_i(t), V_2(t)) \) such that \( J_1(t) \leq V(t) \exp(pt) \)
where \( J_i(t) = (I_a(t), I_v(t))^T \) and \( p_i = \frac{1}{T} \ln \left( r \left( F_{r-F,M_0} \right)(T) \right) < 0 \). Then, we see that \( \lim_{t \to \infty} I_a(t) = 0 \) and \( \lim_{t \to \infty} I_v(t) = 0 \).

Moreover, we obtain that \( \lim_{t \to \infty}(S_a(t) - S'_a(t)) = 0 \), \( \lim_{t \to \infty}(S_v(t) - S'_v(t)) = 0 \). Hence, the disease-free periodic solution \((S'_a(t), 0, S'_v(t), 0)\) is globally attractive. This completes the proof.

### 5. Permanence

In this section, we show that if \( R_0 > 1 \), then the disease persists.

Let \( \bar{X} \) be a matrix space, \( f : \bar{X} \to \bar{X} \) be a continuous map, and \( X_0 \subseteq \bar{X} \) be an open set. Define

\[
\mathcal{E}X_0 := \bar{X} \setminus X_0, M_\delta := \left\{ x \in \mathcal{E}X_0 : f^n(x) \in \mathcal{E}X_0, n \geq 0 \right\}.
\]

\( A_\delta \) is a maximal compact invariant set of \( f \) in \( \mathcal{E}X_0 \). A finite sequence \( \{M_i, \cdots, M_k\} \) are disjoint, compact, and invariant subsets of \( \mathcal{E}X_0 \), and each of them is isolated in \( \mathcal{E}X_0 \).

We present persistence theory [18] as follows:

**Lemma 5.1.** Assume that

1) \( f(X_0) \subset X_0 \) and \( f \) has a global attractor \( A \);
2) The maximal compact invariant set \( A_\delta = A \cap M_\delta \) of \( f \) in \( \mathcal{E}X_0 \), possibly empty, has an acyclic covering \( \tilde{M} \) and where \( \tilde{M} = \{M_i, \cdots, M_k\} \) with the following properties:
   a) \( M_i \) is isolated in \( \bar{X} \); b) \( W^r(M_i) \cap X_0 = \emptyset \) for each \( 1 \leq i \leq k \).

Then, \( f \) is uniformly persistent with respect to \((X_0, \mathcal{E}X_0)\), i.e., there is \( \eta > 0 \) such that for any compact internally chain transitive set \( L \) with \( L \not\subseteq M_i \) for all \( 1 \leq i \leq k \), \( \inf_{x \in L} d(x, \mathcal{E}X_0) > \eta \).

Define Poincaré map \( \tilde{P} : R^+_0 \to R^+_0 \) associated with system (2.1), satisfying \( \tilde{P}(x^0) = u(\omega^0, X^0), \forall x^0 \in R^+_0 \), where \( u(t, x^0) \) is the unique solution of system (2.1) with \( u(0, x^0) = x^0 \). Now, we denote \( \bar{X} = \{(S_a, I_a, S_v, I_v) \in R^+_0 : X_0 = \{(S_a, I_a, S_v, I_v) \in \mathcal{E}X_0, (S_a, 0) \leq S_a \leq 0, I_a > 0, S_v > 0, I_v > 0) \} \) and \( \mathcal{E}X_0 = \bar{X} \setminus X_0 \).

**Theorem 5.1.** Suppose that \( R_0 > 1 \), then system (2.1) exists a positive constant \( \epsilon > 0 \) such that for all \( (S'_a(0), I'_a(0), S'_v(0), I'_v(0)) = (S^0_a, I^0_a, S^0_v, I^0_v) \in X_0 \),

\[
\liminf_{t \to +\infty} I'_a(t) \geq \epsilon, \quad \liminf_{t \to +\infty} I'_v(t) \geq \epsilon.
\]

**Proof:** Firstly, we prove that \( \tilde{P} \) is uniformly persistent with respect to \((X_0, \mathcal{E}X_0)\). From Theorem 2.1, it is obvious that \( \bar{X} \) and \( X_0 \) are positively invariant. We also know that \( \tilde{P} \) is point dissipative on \( R^+_0 \) from Lemma 4.1.

Denote \( M_\delta = \{(S_a^0, I_a^0, S_v^0, I_v^0) \in \mathcal{E}X_0 : \tilde{P}^n(S_a^0, I_a^0, S_v^0, I_v^0) \in \mathcal{E}X_0, \forall n \geq 0 \} \).

Next, we need to show that \( M_\delta = \{(S_a^0, 0, S_v^0, 0) \mid S_a^0 \geq 0, S_v^0 \geq 0 \} \).

Obviously, \( \{(S_a^0, 0, S_v^0, 0) \mid S_a^0 \geq 0, S_v^0 \geq 0 \} \subseteq M_\delta \). We now need to prove that \( M_\delta \setminus \{(S_a^0, 0, S_v^0, 0) \mid S_a^0 \geq 0, S_v^0 \geq 0 \} = \emptyset \). Suppose it’s not hold. For any \( (S_a^0, I_a^0, S_v^0, I_v^0) \in M_\delta \setminus \{(S_a^0, 0, S_v^0, 0) \mid S_a^0 \geq 0, S_v^0 \geq 0 \} = \emptyset \). For the case \( I_v^0 = 0, I_a^0 > 0 \), it is obvious that \( I_v(t) > 0 \) and \( S_v(t) > 0 \) for all \( t > 0 \). From second and sixth equations of system (2.4), we have

\[
J_i(t) = (I_a(t), I_v(t))^T \quad \text{and} \quad p_i = \frac{1}{T} \ln \left( r \left( F_{r-F,M_0} \right)(T) \right) < 0.
\]
\[
\begin{align*}
\frac{dI_h(t)}{dt} &= \lambda S_h(t)I_h(t) - (d_1 + v)I_h(t), \quad t \neq kT, \\
I_h(t^*) &= (1 - \phi)I_h(t), \quad t = kT,
\end{align*}
\]
then it holds that
\[
I_h(t) = \frac{\lambda S_h(0)I_h(0)}{d_1 + v} \left[ I_h^* - \frac{\lambda S_h(0)I_h(0)}{d_1 + v} e^{-\left(\frac{d_1+v}{d_1}(t-T)\right)} \right] > 0 \quad \text{for all } t > 0 \text{ from Lemma 3.2, where } I_h^* = \frac{\lambda S_h(0)I_h(0)}{d_1 + v} \left( 1 - \omega \right) \left( 1 - e^{-\left(\frac{d_1+v}{d_1}t\right)} \right).
\]

In the similar method, for the case \( I_h^* > 0, I_h^0 = 0 \), then we have \( I_h(t) > 0 \) and \( S_h(t) > 0 \) for all \( t > 0 \). This implies that \( (S_h, I_h, S_v, I_v) \notin M_o \) for \( t > 0 \) sufficiently small. It follows that \( M_o \subseteq \{(S_h, 0, S_v, 0) | S_h \geq 0, S_v \geq 0 \} \). Thus, \( M_o = \{(S_h, 0, S_v, 0) | S_h \geq 0, S_v \geq 0 \} \). It is clear that \( E_0 = (S_h^0, 0, S_v^0, 0) \) is a unique fixed point of \( \tilde{P} \) in \( M_o \).

In the following, we need to prove \( W^+(E_0) \cap X_0 = \varnothing \).

We write \( x^0 = (S_h^0, I_h^0, S_v^0, I_v^0) \in X_0 \). By the continuity of the solutions with respect to the initial conditions, \( \forall \epsilon > 0 \), there exist \( \delta_0 > 0 \), such that for all \( x^0 \in X_0 \) with \( x^0 - E_0 \leq \delta_0 \), it holds that
\[
\left\| u_t(x^0) - u(t, E_0) \right\| \leq \epsilon, \quad \forall t \in [0, T].
\]

Now, we show that
\[
\limsup_{t \to \infty} d(\tilde{P}^m(x^0), E_0) \geq \delta_0.
\]
Suppose not hold, then \( \limsup_{t \to \infty} d(\tilde{P}^m(x^0), E_0) < \delta_0 \) for some \( x^0 \in X_0 \). Without loss of the generality, we can assume that \( d(\tilde{P}^m(x^0), E_0) < \delta_0, \forall m \geq 0 \). Thus, we obtain that
\[
\left\| u_t(t, \tilde{P}^m(x^0)) - u(t, E_0) \right\| \leq \epsilon, \quad \forall t \in [0, T] \quad \text{and} \quad \forall m \geq 0.
\]
For any \( \tilde{t}_i \geq 0 \), let \( \tilde{t}_i = m\omega + t' \), where \( t' \in [0, \omega] \) and \( m = \left\lfloor \frac{\tilde{t}_i}{\omega} \right\rfloor \). \( \left\lfloor \frac{\tilde{t}_i}{\omega} \right\rfloor \) is the greatest integer less than or equal to \( \tilde{t}_i/\omega \). So, we have that
\[
\left\| u_t(t, x^0) - u(t, E_0) \right\| \leq \epsilon, \quad \forall \tilde{t}_i \geq 0.
\]
It follows that
\[
0 \leq I_h(t) \leq \epsilon, \quad 0 \leq I_v(t) \leq \epsilon, \quad \forall \tilde{t}_i > \tilde{t}_i.
\]
Then, by the first, third, fifth and seventh equations of system (2.1), we have
\[
\begin{align*}
\frac{dS_h(t)}{dt} &\geq -\left( \lambda \epsilon + d_1 \right) S_h(t), \quad t \neq kT, \\
\frac{dS_v(t)}{dt} &\geq \lambda - \beta \epsilon S_h(t) - d_2 S_v(t), \\
S_h(t^*) &= S_h(t) + \delta, \\
S_v(t^*) &= S_v(t), \quad t = kT,
\end{align*}
\] (5.2)
Consider an auxiliary system...
Using the same method as aforementioned, we have that (5.3) admits a positive periodic solution \((\tilde{S}_h(t), \tilde{S}_v(t))\). Since \(r(Q^t\Phi_{M_1}(t)) < 1\) holds. Then, there exists a small enough \(\varepsilon\) such that \(r(Q^t\Phi_{M_1+M_2}(t)) < 1\), and \(r(Q^t\Phi_{M_1+M_2}(t))\) is continuous for small \(\varepsilon\), where

\[
M_{\varepsilon} = \begin{pmatrix}
\varepsilon & 0 \\
0 & \varepsilon
\end{pmatrix}.
\]

As before, we have that \((\tilde{S}_h(t), \tilde{S}_v(t))\) is globally asymptotically stable, and meanwhile, \(\lim_{\varepsilon \to 0} \tilde{S}_h(t) = S_h(t)\), \(\lim_{\varepsilon \to 0} \tilde{S}_v(t) = S_v(t)\), thus there exist \(\varepsilon_i\) small enough and a constant \(\xi > 0\), such that

\[
\begin{cases}
\tilde{S}_h(t) \geq S_h(t) - \xi, \\
\tilde{S}_v(t) \geq S_v(t) - \xi,
\end{cases}
\]

for \(\xi < \varepsilon_i\).

On the other hand, the standard comparison theorem implies that there exist \(\tilde{t}_z \geq \tilde{t}_1\) and \(\varepsilon_i\) such that

\[
\begin{cases}
S_h(t) \geq \tilde{S}_h(t) - \xi, \\
S_v(t) \geq \tilde{S}_v(t) - \xi,
\end{cases}
\]

for all \(\tilde{t}_z \geq \tilde{t}_2\). Then, for all \(\tilde{t}_z \geq \tilde{t}_2\), we have

\[
\begin{cases}
S_h(t) \geq S_h(t) - \tilde{\xi} + S_h(t) - \gamma, \\
S_v(t) \geq S_v(t) - \tilde{\xi} + S_v(t) - \gamma,
\end{cases}
\]

for all \(t \geq \tilde{t}_2\), where \(\gamma = \xi + \varepsilon_i\).

By the second, fourth, sixth and eighth equations of system (2.1), we have

\[
\begin{align*}
\frac{dI_h(t)}{dt} & \geq \lambda(S_h(t) - \gamma)I_h(t) - (d_i + \nu)I_h(t), \\
\frac{dI_v(t)}{dt} & \geq \beta(S_v(t) - \gamma)I_v(t) - d_\gamma I_v(t), \\
I_h(t') & = (1 - \phi)I_h(t), \\
I_v(t') & = I_v(t),
\end{align*}
\]

for \(t \neq kT\), and

\[
\begin{align*}
I_h(t') = (1 - \phi)I_h(t), \\
I_v(t') = I_v(t),
\end{align*}
\]

for \(t = kT\) (5.4)

Set \(M_{\gamma}\) be the \(2 \times 2\) matrix function such that

\[
M_{\gamma} = \begin{pmatrix}
0 & \gamma \\
\gamma & 0
\end{pmatrix},
\]

where \(\gamma\) is small enough.
By Lemma 3.1 and the standard comparison principle, it follows that there exists a positive $T$-periodic function $W(t) = (W_1(t), W_2(t))$ such that $J_2(t) = \exp\left(p_2 t\right)W(t)$ is a solution of system (5.4), where 
\[ p_2 = \frac{1}{T} \ln r\left(\Phi_{F - Y - M_p}(T)\right). \]
Since $r\left(\Phi_{F - Y - M_p}(T)\right) > 1$, and $r\left(\Phi_{F - Y - M_p}(T)\right)$ is continuous for small $\gamma$. So we can choose $\gamma$ small enough, such that $r\left(\Phi_{F - Y - M_p}(T)\right) > 1$. It follows that $p_2 > 0$, we can choose $\bar{I}_\epsilon > \bar{I}_0$ such that 
\[ I_\epsilon(\bar{I}_\epsilon) \geq W_1(0), I_\epsilon(\bar{I}_\epsilon) \geq W_2(0). \]

By the comparison principle we have
\[ I_\epsilon(t) \geq \exp^{p_2(t-\bar{I}_\epsilon)} W_1(t-\bar{I}_\epsilon), I_\epsilon(t) \geq \exp^{p_2(t-\bar{I}_\epsilon)} W_2(t-\bar{I}_\epsilon) \]
for all $t \geq \bar{I}_\epsilon$. Then, we obtain that $\lim_{t\to+\infty} I_\epsilon(t) = +\infty$ and $\lim_{t\to+\infty} I_\epsilon(t) = +\infty$, which contradicts to the boundedness of $0 \leq I_\epsilon(t) \leq \bar{I}_0$. Thus we have proved $W'(E_0) \cap X_0 = \emptyset$, which implies each orbit in $M_\epsilon$ converges to $E_0$, and hence $E_0$ is acyclic in $M_\epsilon$.

Therefore, the Lemma 5.1 is satisfied for system (2.1). Furthermore, we obtain that the disease is permanence, when $R_0 > 1$.

6. Conclusion

In this paper, a vector-borne epidemic model for Huanglongbing with impulsive control is established. Under the reasonable assumptions (H1)-(H8), one studied the threshold dynamics behavior of the model. Based on comparison theorem of impulsive differential equation and method of enlarging and reducing, we proved that if the $R_0 < 1$, the disease-free equilibrium is global stability, and Huanglongbing is uniformly persistent if $R_0 > 1$. We only consider replanting susceptible and rouging infective in model, spraying insecticides to kill psyllid is not. It’s a lot of room for us to improve.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

References


Computing the Enclosures Eigenvalues Using the Quadratic Method

Shurouq Mohammad Abusamra

Department of Mathematics, Faculty of Science, King Abdul-Aziz University, Jeddah, KSA
Email: ashrouq@yahoo.com

Abstract
In this article, We compute the enclosures eigenvalues (upper and lower bounds) using the quadratic method. The Schrodinger operator (A) (harmonic and anharmonic oscillator model) has used as an example. We study a new technique to get more accurate bounds. We compare our results with Boulton and Strauss method.

Keywords
Quadratic Method, Enclosures Eigenvalues, Boulton and Strauss Method

1. Introduction
Galerkin method is one of the best methods for determining upper bounds for the eigenvalues of semi-definite operators, unfortunately this method cannot find enclosures eigenvalue. This paper shows how to compute enclosures of the eigenvalues of self-adjoint operators by the Quadratic method. At first, we study the second-order relative spectrum (The Quadratic method) in [1] [2], and Boulton & Strauss method in [3] [4]. These methods have used for computing eigenvalue enclosures (upper and lower bounds) of the eigenvalues of self-adjoint operators. The quadratic method, which relies on calculation of the second-order which is providing, certified a priori intervals of spectral enclosure. Then we study our new technique which gives more accurate results, we also follow the results that have been published by Boulton & Hobiny in [5]. The method will be examined by harmonic and anharmonic oscillator models.

Second-order relative spectra were first considered by Davies (1998) in the context of resonances for general self-adjoint operators in [6] [7]. It was then suggested by Shargrodsky and subsequently by Levitin and Shargorodsky (2000) in [8] that the second order relative spectra can also be employed for the pollu-
tion-free computation of eigenvalues in gaps of the essential spectrum. Various implementations, including on models from elasticity, solid state, physics, relativistic quantum mechanics and magneto hydrodynamics confirm that the Quadratic method is a reliable tool for eigenvalue approximation in the spectral pollution regime. Properties of second order relative spectra have been studied recently by Bolton & Leviton in [9] and then by Bolton & Strauss (2007, 2011) in [3] [4]. Bolton and Strauss extended this method to normal operators and optimal convergence rates for eigenvalues and estimated that by an order of magnitude for the harmonic & anharmonic oscillator models, by cut the interval into sub-interval around the eigenvalues in the Spectrum, and the approximation enclosure eigenvalues results of these models are more accurate than the Quadratic method around \( \lambda \).

Our improvement depends on domain expansion around the first five eigenvalues in the spectrum, we will take a value for \( a \) less than \( \lambda_1 \) and a value for \( b \) greater than \( \lambda_5 \) and calculate the conjugate pairs of eigenvalues \( (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5) \) (i.e.; around \( \lambda_1 \) we will choose \( a_1 < \lambda_1 < \lambda_2 < \lambda_3 < \lambda_4 < \lambda_5 < b \)). In this paper we studied two models (harmonic and anharmonic oscillator) which are:

1): \( \mathcal{H}^{har}(x) = -u''(x) + x^2 u(x) \).
2): \( \mathcal{H}^{anh}(x) = -u''(x) + x^4 u(x) \).

**Notation**

- Below \( \mathcal{H} \) denotes a generic separable Hilbert space with inner product \( \langle \cdot, \cdot \rangle \) and norm \( \| \cdot \| \).
- Let the operator \( (\mathcal{A} : D(\mathcal{A}) \rightarrow \mathcal{H}) \) be self-adjoint. We will write \( \text{Spec}(\mathcal{A}) \) to denote the spectrum of \( (\mathcal{A}) \).
- \( \mathcal{H} \) is real inner product space means that \( \mathcal{H} \) is real vector space on which there is an inner product \( \langle x, y \rangle \) associating a real number to each pair of elements \( x, y \). The norm is real function such that \( x = \langle x, x \rangle^{1/2} \), and the distance is: \( d(x, y) = x - y = (x - y, x - y)^{1/2} \). And: \( d(x, z) \leq d(x, y) + d(y, z) \), \( x, y, z \in \mathcal{H} \), which is called triangular inequality.
- For any pair of elements \( x, y \) of \( \mathcal{H} \) satisfies the following properties:
  \[
  \langle x, x \rangle \geq 0 \\
  \langle y, x \rangle = \langle x, y \rangle \\
  \langle y, x_1 + bx_2 \rangle = \langle x_1 y + bx_2 y \rangle \\
  \langle x, ay_1 + by_2 \rangle = a \langle x, y_1 \rangle + b \langle x, y_2 \rangle \\
  \]
- Given a subspaces \( \mathcal{L} \subset D(\mathcal{A}) \) of dimension \( n \) such that:
  \[
  \mathcal{L} = \text{Span} \{ b_j \}_{j=1}^n \\
  \]
  we will write
  \[
  \mathcal{A}_i = [a^i(h_j, h_k)]_{j,k=1}^n \in \mathbb{C}^{n \times n} \\
  \]
- The discrete spectrum: is the set of eigenvalues of finite multiplicity.

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• The essential spectrum of \( \mathbb{A} \): is the remaining part of \( \text{spec} (\mathbb{A}) \).

• \( \text{Spec}_{\text{ess}} (\mathbb{A}) = \frac{\text{spec} (\mathbb{A})}{\text{Spec}_{\text{disc}} (\mathbb{A})} \).

• Theorem: Let \( \mathbb{A} \) be a self-adjoint operator:
- \( \lambda \in \text{spec} (\mathbb{A}) \), there exists a sequence \( u_j \in \text{Dom}(\mathbb{A}) \) such that:
- \[ \| \mathbb{A} u_j , \mathbb{A} u_j \| \rightarrow 0 \quad \text{as} \quad j \rightarrow \infty \] [This sequence is called Weyls sequence].
- \( \lambda \in \text{Spec}_{\text{ess}} (\mathbb{A}) \leftrightarrow (\mathbb{A} - \lambda \mathbb{I}) \) is not Fredholm.

• The min-max theorem tells us if \( \mathcal{R} \) is bounded above and below by the mini-mum and maximum eigenvalues respectively \( (\lambda_{\text{min}}, \lambda_{\text{max}}) \), and we have:
  \[ \lambda_{\text{min}} \text{ (if any)} \leq \mathcal{R} (x) \leq \lambda_{\text{max}} \text{ (if any)} \], where \( \mathcal{R} (x) \) denoted by Rayleigh-Ritz quotient on a self-adjoint operator \( \mathbb{A} \) as:
  \[ \mathcal{R}_u (x) = \frac{\langle \mathbb{A} x, x \rangle}{\langle x, x \rangle}, x \in \text{Dom} (\mathbb{A}). \]

2. The Quadratic Method

We begin by describing the basic framework of the Quadratic method associated to a self-adjoint operator. First considered by Davis (1998) in the context of resonances for general self-adjoint operator in [6] [7], it was then suggested by Shargrodsky and subsequently by Levitin and Shargorodsky (2000) in [8], Properties of second order relative spectra have been studied recently by Boulton & Levitin in [9] and then by Boluton & Strauss (2007, 2011) in [3] [4]. For the benefit of the reader, we include here some definitions.

Let \( \mathbb{A} \) be self-adjoint operator on Hilbert space \( \mathcal{H} \). Let \( \mathcal{L} \subset \text{Dom} (\mathbb{A}) \), a number \( z \in \mathbb{C} \) belongs to second order spectrum \( \text{Spec}_2 (\mathbb{A}, \mathcal{L}) \) if there exists \( u, v \in \mathcal{L} \) such that:

\[ \{(\mathbb{A} - z\mathbb{I})u, (\mathbb{A} - z\mathbb{I})v\} = 0. \tag{2.1} \]

For all \( u \) is nonzero. Means that the Second Order Spectrum usually contain complex numbers, but it turns out that if \( z \in \text{Spec}_2 (\mathbb{A}, \mathcal{L}) \) then:

\[ \text{Spec} (\mathbb{A}) \left[ (\text{Re} (z) - \text{Im} (z)), (\text{Re} (z) + \text{Im} (z)) \right] \neq \varnothing. \]

Consider: \( \mathcal{E} = \{ e_1, e_2, e_3 \} \).

Be a basis of \( \mathcal{L} \) [ \( \mathcal{L} \) is finite dimensional subspace of \( \text{Dom} (\mathbb{A}) \)], then the Quadratic matrix polynomial is:

\[ Q(z) = \mathbb{A}_2 - 2z\mathbb{A}_1 + z^2\mathbb{A}_0. \tag{2.2} \]

where the mass, stiffness and bending matrices are:

\[ [\mathbb{A}_0]_{ij} = (e_j, e_i), [\mathbb{A}_1]_{ij} = (\mathbb{A} e_j, e_i), [\mathbb{A}_2]_{ij} = (\mathbb{A} e_j, \mathbb{A} e_i) \]

We define the spectrum of \( Q(z) \) as the set of \( z \in \mathbb{C} \) such that

\[ Q(z) = 0 \quad \text{for some} \quad u \in \mathcal{L} / \{0\}. \tag{2.3} \]

The standard way of finding \( \text{Spec}(Q) \) is to reach to the linear pencil eigenvalue problem:
\[ Au = \lambda u \quad \text{for some} \quad u \in \mathcal{L} \setminus \{0\}. \quad (2.4) \]

In this method we need to construct companion matrices which depend on the quadratic matrix polynomial to find the enclosure eigenvalues.

We have two possible companion matrices form which are given by

\[
S = \begin{pmatrix} 0 & I \\ -A_2 & 2A_1 \end{pmatrix}, \quad T = \begin{pmatrix} I & 0 \\ 0 & A_0 \end{pmatrix}
\]

And

\[
S = \begin{pmatrix} 0 & I \\ -A_2 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} I & 0 \\ 2A_1 & A_0 \end{pmatrix}
\]

The eigenvalues of the matrix polynomial can be determined from one of this companion matrices form

**LEMMA 2.1:** Let \( Q(z) \) is defined and be singular then:

\[
\det(Q(z)) = 0 \iff \det(S - zT) = 0. \quad (2.5)
\]

For \( S = \begin{pmatrix} 0 & I \\ -A_2 & 2A_1 \end{pmatrix}, \quad T = \begin{pmatrix} I & 0 \\ 0 & A_0 \end{pmatrix} \)

\[
\det(S - zT) = \begin{vmatrix} 0 - zI & I - 0 \\ -A_2 - 0 & 2A_1 - zA_1 \end{vmatrix} = -zI(2A_1 - zA_1) + A_2 I = \det(Q(z))
\]

Indeed, the assertion that \( Q(z) \) is singular is equivalent to the existence of \( u \neq 0 \) such that

\[
A_2 u - 2zA_1 u + z^2 A_0 u = 0. \quad (2.6)
\]

Denoting \( v = zu \), this can be rewritten as

\[
A_2 u - 2A_1 v + zA_0 v = 0. \quad (2.7)
\]

In turn, the latter is equivalent to:

\[
\begin{pmatrix} 0 & I \\ -A_2 & 2A_1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = z \begin{pmatrix} I & 0 \\ 0 & A_0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}
\] (2.8)

As needed for the verification of (2.3).

**LEMMA 2.2:** Let \( \mathcal{L} = \text{Span}\{b_j\}_{j=1}^{\infty} \subset \text{Dom}(A) \) and \( P \) is the orthogonal projection onto \( \mathcal{L} \), then:

\[
\text{Spec}_2(A; \mathcal{L}) = \left\{ z \in \mathbb{C} : \exists u \in \mathcal{L} \left( (A - zI)\begin{pmatrix} u \\ v \end{pmatrix} = 0, \forall v \in \mathcal{L} \right) \right\} \quad (2.9)
\]

If \( \mathcal{L} \subset \text{Dom}(A) \), then:

\[
\text{Spec}_2(A; \mathcal{L}) = \left\{ z \in \mathbb{C} : \exists u \in \mathcal{L} \left( (A - z)^2 u, v = 0, \forall v \in \mathcal{L} \right) \right\}
\]

(2.10)

Typically \( \text{Spec}_2(A; \mathcal{L}) \) contains non-real points. From the lemma it is easy to see that \( z \in \text{Spec}_2(A; \mathcal{L}) \) if and only if \( \exists \in \text{Spec}_2(A; \mathcal{L}) \).
We now discuss a strategy suggested by Davis and Plum for computing the spectrum of self-adjoint operator.

Let \( \mathcal{A} = \mathcal{A}^* \) and \( \mathcal{L} \subset \text{Dom}(\mathcal{A}) \) for \( z \in \mathbb{C} \) to \( \text{Spec}(\mathcal{A}) \), consider the function \( \mathbb{F} : \mathbb{C} \to [0, \infty) \) be given:

\[
\mathbb{F}(z) = \min_{v \in \mathcal{L}} \frac{\| (z - \mathcal{A}) v \|}{\| v \|} \tag{2.11}
\]

Then \( \mathbb{F}(z) \) is an upper bound for the distance from \( z \) to the spectrum of \( \mathcal{A} \), that means:

\[
\text{Dist}[z, \text{Spec}(\mathcal{A})] = \min \{ (z - \lambda) : \lambda \in \text{spec}(\mathcal{A}) \}.
\]

So \( \mathbb{F}(z) \geq \text{dist}[z, \text{Spec}(\mathcal{A})] \). \hspace{1cm} (2.12)

Assume that \( z \) not in \( \text{Spec}(\mathcal{A}) \), since \( \mathcal{L} \subset \text{Dom}(\mathcal{A}) \) then:

\[
\mathbb{F}(z) = \min_{\| u \| < 1} \frac{\| (z - \mathcal{A})^{-1} u \|}{\| u \|} \\
\geq \sup_{\| u \| < 1} \frac{\| (z - \mathcal{A})^{-1} u \|}{\| u \|} \\
= \| (z - \mathcal{A})^{-1} \|^{-1} = \text{dist}[z, \text{Spec}(\mathcal{A})] \text{ so } z \in \mathbb{C}
\]

Therefore \( \mathbb{F}(x) \) can be small only when \( x \in \mathbb{R} \) is close to \( \text{Spec}(\mathcal{A}) \), so we can make a connection between \( \text{Spec}(\mathcal{A}) \) and \( \mathbb{F}(x) \).

Suppose \([a, b]\) be an interval of the spectrum of \( \mathcal{A} \) which lies in it, and 
\( \mathcal{L} \subset \text{Dom}(\mathcal{A}^2) = \{ u \in \text{Dom}(\mathcal{A}) : Lu \in \text{Dom}(\mathcal{A}) \} \).

let \( [a = z_0, z_1, z_2, \ldots, z_n = b], \forall z \in \mathbb{R} \),

\[
\mathbb{F}(z^2) = \min_{\| v \| < 1} \frac{\langle (z - \mathcal{A})^2 v, v \rangle}{\| v \|} \\
= \min_{\| v \| < 1} \frac{\| (z - \mathcal{A}) v \|}{\| v \|} \\
= \min_{\| v \| < 1} \frac{\| Q(v) \|}{\| v \|} \\
= \| G(z) \|, \text{ v is nonzero}
\]

Clearly, \( \text{Spec}(\mathcal{L}) = \{ z \in \mathbb{C} : G(z) = 0 \} \). For \( z \in \mathbb{C} \), let \( Q(z) \) be as in (2.1), \( v \neq 0 \):

\[
G(z) = \min_{v \in \mathcal{L}} \frac{\| Q(v) \|}{\| v \|} \tag{2.14}
\]

Thus the zeros of \( G(x) \) appear in conjugate pair.

**THEOREM 2.1:** (Shargorodsky):

For \( \lambda \in \text{Spec}(\mathcal{L}) \), and \( G(\lambda) = 0 \), let 
\( \lambda_{\text{app}} = \text{Re}(\lambda) + |\text{Im}(\lambda)|, \lambda_{\text{low}} = \text{Re}(\lambda) - |\text{Im}(\lambda)| \)
Then:
\[
\left[ \lambda_{\text{min}}, \lambda_{\text{app}} \right] \cap \text{Spec} \left( \mathcal{A} \right) \neq \emptyset.
\] (2.15)

Let \( \lambda = a + ib \) for \( \alpha, \beta \in \mathbb{R} \), then:
\[
\left( \lambda - \mathcal{A} \right) u, \left( \lambda - \mathcal{A} \right) v = \left( (\alpha - \mathcal{A}) u, (\beta - \mathcal{A}) v \right) + 2i\beta \left( (\alpha - \mathcal{A}) u, v \right) - \beta^2 u, v
\]
Then either \( \beta = 0 \), \( u \in \mathcal{L} \), \( u = v \) we get:
\[
\left\| (\alpha - \mathcal{A}) u \right\|^2 - \beta^2 \left\| u \right\|^2 + 2i\beta \left( (\alpha - \mathcal{A}) u, u \right) = 0
\]
So:
\[
\beta^2 = \frac{\left\| (\alpha - \mathcal{A}) u \right\|^2}{\left\| u \right\|^2} \quad \text{and} \quad \left( (\alpha - \mathcal{A}) u, u \right) = 0
\] (2.16)

So:
\[
\left| \beta \right| = \frac{\left\| (\alpha - \mathcal{A}) u \right\|}{\left\| u \right\|} \geq \text{dist} \left[ \alpha, \text{Spec} \left( \mathcal{A} \right) \right]
\] (2.17)

3. The Boulton and Strauss Method

The method of second order relative spectra has been shown to reliably approximate the discrete spectrum for a self-adjoint operator in [2]. Boulton and Strauss extended this method to normal operators and find optimal convergence rates for eigenvalues and estimated that by an order of magnitude in [3] [4]. The spectrum of \( \mathcal{A} \), \( \text{Spec} \left( \mathcal{A} \right) \), may be expressed as the union of the discrete spectrum consisting of all isolated eigenvalues of finite multiplicity, \( \text{Spec}_{\text{dis}} \left( \mathcal{A} \right) \), and the essential spectrum, where.

\[
\text{Spec}_{\text{ess}} \left( \mathcal{A} \right) = \frac{\text{spec} \left( \mathcal{A} \right)}{\text{Spec}_{\text{dis}} \left( \mathcal{A} \right)}
\] (3.1)

In most standard situations the essential spectrum can be found analytically, but points in \( \text{Spec}_{\text{dis}} \left( \mathcal{A} \right) \) are usually estimated by numerical procedures. The standard numerical techniques, such as the Quadratic method, aim at solving Galerkin approximate problems posed in weak form: find \( u \in \mathcal{L} - \{0\} \) and \( \lambda \in \mathbb{R} \) such that \( \left\langle \lambda u, v \right\rangle = \lambda \left\langle u, v \right\rangle \) \( \forall v \in \mathcal{L} \) (\( \mathcal{L} \) is finite dimensional). Boulton and Strauss method depends on sub-interval around the eigenvalues in \( \text{Spec} \left( \mathcal{A} \right) \) to find more accurate approximation enclosure eigenvalues than the Quadratic method around \( \lambda \). Let:
\[
\lambda_{\text{min}}(\mathcal{A}) = \inf \left[ \text{Spec} \left( \mathcal{A} \right) \right] \quad \text{and} \quad \lambda_{\text{max}}(\mathcal{A}) = \sup \left[ \text{Spec} \left( \mathcal{A} \right) \right].
\] (3.2)

\[\forall \mathcal{L} \subset \text{Dom} \left( \mathcal{A} \right); \text{Spec}_{\lambda} \left( \mathcal{A}, \mathcal{L} \right) \subset \left[ \lambda_{\text{min}}(\mathcal{A}), \lambda_{\text{max}}(\mathcal{A}) \right].\]

\[\lim_{n \to \infty} \left[ \text{Spec}_{\lambda} \left( \mathcal{A}, \mathcal{L}_n \right) \right] \subseteq \left[ \lambda_{\text{min}}(\mathcal{A}), \lambda_{\text{max}}(\mathcal{A}) \right] \quad \text{for any sequence \( \mathcal{L}_n \).}\]

By corresponding limit sets which has been stimulated by the following property: if \( (a, b) \cap \text{Spec} \left( \mathcal{A} \right) = \emptyset \), then:
\[
\text{Spec}_{\lambda} \left( \mathcal{A}, \mathcal{L} \right) \cap (a, b) = \emptyset, \quad \forall \mathcal{L} \subset \text{Dom} \left( \mathcal{A} \right).
\] (3.3)

whenever \( z \in \text{Spec}_{\lambda} \left( \mathcal{A}, \mathcal{L} \right) \).

Thus inclusions of points in the spectrum of \( \mathcal{A} \) are achieved from \( \text{Re} \left( z \right) \) with...
a two-sided explicit residual given by $|\text{Im}(z)|$. Indeed, if $(a, b) \cap \text{Spec}(\mathcal{A}) = \lambda$ and $z \in \text{Spec}_2(\mathcal{A}, \mathcal{L})$ with $z \in D(a, b)$, then:

$$
\left[ \text{Re}(z) - \frac{|\text{Im}(z)|^2}{b - \text{Re}(z)}, \text{Re}(z) + \frac{|\text{Im}(z)|^2}{\text{Re}(z) - a} \right] \cap \text{Spec}(\mathcal{A}) = \{ \lambda \} \quad (3.4)
$$

### 4. New Technique to Get More Accurate Results

we described a new technique to get more accurate bounds for the eigenvalues in $\text{Spec}(\mathcal{A})$. Hobiny was discussed the Quadratic method to estimate enclosure eigenvalues for self-adjoint operator and illustrated the results on the harmonic and anharmonic oscillator models to know the accuracy and efficiency for this method in [5]. We discussed also the bounds of the size of the enclosure eigenvalues and studied them in the context of one dimensional Schrödinger operators, and illustrated that the conjugate pairs of the eigenvalues in $\text{Spec}_2(\mathcal{A}, \mathcal{L})$ are closed to the real line, so the conjugate pairs will give small intervals $(a, b)$s enclosing points in $\text{Spec}_2(\mathcal{A}, \mathcal{L})$.

After that Boulton in [9] [10] [11], was modified this studying and gave a value for each $a$, $b$ within a specific domain for the first five eigenvalues in $\text{Spec}(\mathcal{A})$, (i.e.: $a_1 < \lambda_1 < b_1$, $a_2 < \lambda_2 < b_2$, $a_3 < \lambda_3 < b_3$, $a_4 < \lambda_4 < b_4$, $a_5 < \lambda_5 < b_5$).

Our improvement depends on domain expansion around the first five eigenvalues in $\text{Spec}(\mathcal{A})$, we will take a value for $a$ less than $\lambda_1$ and a value for $b$ greater than $\lambda_5$ and calculate the conjugate pairs of eigenvalues $(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)$ (i.e; around $\lambda_1$ we will choose $a_1 < \lambda_1 < \lambda_2 < \lambda_3 < \lambda_4 < \lambda_5 < b_5$ where:

$$
(\lambda_1)_{\text{low}} = \text{Re}(z) - \frac{|\text{Im}(z)|^2}{b_1 - \text{Re}(z)} \quad (4.1)
$$

$$
(\lambda_1)_{\text{app}} = \text{Re}(z) + \frac{|\text{Im}(z)|^2}{\text{Re}(z) - a_1} \quad (4.2)
$$

And the same technique for other eigenvalues. Then we will apply $a$, $b$ in our programs to find the first five enclosure eigenvalues and produce our comparison by numerical experiments on the harmonic and anharmonic oscillator models.

### 5. Schrödinger Operator

Consider the trial subspace $\mathcal{L}$ constructed via the finite element method on finite segment.

Let $\mathcal{L} > 0$ and set $\mathcal{A} = \mathcal{H}$ then, the general Schrödinger equation is:

$$
\mathcal{H}_\mathcal{L} u(x) = -u^*(x) + V(x) u(x); \quad x \in (-\mathcal{L}, \mathcal{L}) \quad (5.1)
$$

where $V(x)$ is called the potential function, and it must be bounded below $u(\mathcal{L}) = u(-\mathcal{L}) = 0$; which is called the Dirichlet Boundary Conditions.

If we take $u(x)$ as a common factor then:

$$
\left[ -\frac{d^2}{dx^2} + V(x) \right] u(x) = \lambda u(x) \quad (5.2)
$$
Take:

\[ \mathcal{H} = -\frac{d^2}{dx^2} + V(x) \quad \text{then} \quad \mathcal{H}u(x) = \lambda u(x) \tag{5.3} \]

so the operator \( \mathcal{H} \) is a differential equation.

If \( V(x) = x^2 \)

then the equation is related to harmonic oscillator model.

If \( V(x) = x^4 \)

then the equation is related to anharmonic oscillator model.

We compare between the Quadratic method, Boulton & Strauss and our development on these models to calculate the enclosure eigenvalues to know which one is the best in this field.

6. Harmonic Oscillator Model

The Harmonic Oscillator is one of the most important models of quantum theory.

Let \( \mathcal{H}_{\text{har}} = \mathcal{A} \), for \( V(x) = x^2 \), then the exact eigenvalue is \( \lambda_{j+1} = 2j + 1; \quad j \in \mathbb{N} \).

By Schrodinger equation we have:

\[ \mathcal{H}_{\text{har}}(x) = -u''(x) + x^2u(x) \tag{6.1} \]

And \( \mathcal{H}_{\text{har}}(x) = \lambda u(x) \tag{6.2} \)

This equation can be solved explicitly and we can find the approximation eigenvalues using Matlab program by matrices \( M, N, R \) where,

\[
M = \left[ \begin{array}{c} \langle \Psi', \Psi' \rangle \\ \langle x^2\Psi', \Psi \rangle \\ \langle \Psi, \Psi \rangle \end{array} \right]
\]

\[
N = \left[ \begin{array}{c} \langle \Psi', \Psi' \rangle \\ \langle x^2\Psi', \Psi \rangle \\ \langle \Psi, \Psi \rangle \end{array} \right]
\]

\[
R = \left[ \begin{array}{c} \langle \Psi', \Psi' \rangle \\ \langle x^2\Psi', \Psi \rangle \\ \langle \Psi, \Psi \rangle \end{array} \right]
\]

And \( \lambda_{\text{app}} = \frac{M + N}{R} \).

Now we compute \( \text{Spec}_2\left( \mathcal{H}_{\text{har}}, L_{\text{har}} \right) \) as described before and calculate the eigenvalues enclosure. All the coefficients of the matrices were found analytically.

\textbf{NOTE:} In our improvement method we choose \( n = 200, L = 6 \) to compare our results with Boulton and Strauss method, which approximate the first five eigenvalues of harmonic and anharmonic models with \( n = 200, L = 6 \).

\textbf{Figure 1} and \textbf{Figure 2} show the conjugate pair for each eigenvalue with upper and lower bounds of eigenvalues in \( \text{Spec}_2\left( \mathcal{H}_{\text{har}} \right) \).

\textbf{Trial 1:} In this trial we use the Quadratic method, Boulton & Strauss method, and our improvement with: \( n = 200, L = 6 \), to find the first five approximation enclosure eigenvalues (upper and lower eigenvalues) of \( \mathcal{H}_{\text{har}} \), where \( h = \frac{2L}{n} \) (see Table 1, Figure 3).
We calculate the Error between the exact and the lower bound for the first five approximation enclosure eigenvalues of $\mathcal{H}_n^{\text{har}}$. error $= \text{exact}_j - \lambda_j$. The slope of the graphs is close to the value (6) in all cases as $n$ increases (Figure 3).

**Table 1.** Approximating enclosures for the first five eigenvalues of $\mathcal{H}_n^{\text{har}}$ with $n = 200$, $\lambda_{\text{low}}$ is the lower bound of the enclosing $\lambda$ and $\lambda_{\text{upp}}$ is the upper bound.

<table>
<thead>
<tr>
<th>$\lambda_j$</th>
<th>Quadratic method</th>
<th>Boulton and Straus method</th>
<th>Our improvement method</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda_1$</td>
<td>Upp 0.999999763730525</td>
<td>1.000000118025628</td>
<td>1.0000000118025628</td>
</tr>
<tr>
<td></td>
<td>low 0.999999763730525</td>
<td>1.000000118025628</td>
<td>1.0000000118025628</td>
</tr>
<tr>
<td>$\lambda_2$</td>
<td>Upp 3.0000212522912</td>
<td>3.000000000000000</td>
<td>3.000000000000000</td>
</tr>
<tr>
<td></td>
<td>low 2.9999997875754936</td>
<td>2.9999997875754936</td>
<td>2.9999997875754936</td>
</tr>
<tr>
<td>$\lambda_3$</td>
<td>Upp 3.00000212522912</td>
<td>3.000000000000000</td>
<td>3.000000000000000</td>
</tr>
<tr>
<td></td>
<td>low 2.9999997875754936</td>
<td>2.9999997875754936</td>
<td>2.9999997875754936</td>
</tr>
<tr>
<td>$\lambda_4$</td>
<td>Upp 5.000000000000000</td>
<td>5.000000000000000</td>
<td>5.000000000000000</td>
</tr>
<tr>
<td></td>
<td>low 4.999999463031893</td>
<td>4.999999463031893</td>
<td>4.999999463031893</td>
</tr>
<tr>
<td>$\lambda_5$</td>
<td>Upp 6.000000000000000</td>
<td>6.000000000000000</td>
<td>6.000000000000000</td>
</tr>
<tr>
<td></td>
<td>low 5.999999463031893</td>
<td>5.999999463031893</td>
<td>5.999999463031893</td>
</tr>
<tr>
<td>$\lambda_6$</td>
<td>Upp 7.000000000000000</td>
<td>7.000000000000000</td>
<td>7.000000000000000</td>
</tr>
<tr>
<td></td>
<td>low 6.999999463031893</td>
<td>6.999999463031893</td>
<td>6.999999463031893</td>
</tr>
<tr>
<td>$\lambda_7$</td>
<td>Upp 8.000000000000000</td>
<td>8.000000000000000</td>
<td>8.000000000000000</td>
</tr>
<tr>
<td></td>
<td>low 7.999999463031893</td>
<td>7.999999463031893</td>
<td>7.999999463031893</td>
</tr>
<tr>
<td>$\lambda_8$</td>
<td>Upp 9.000000000000000</td>
<td>9.000000000000000</td>
<td>9.000000000000000</td>
</tr>
<tr>
<td></td>
<td>low 8.999999463031893</td>
<td>8.999999463031893</td>
<td>8.999999463031893</td>
</tr>
</tbody>
</table>

**Figure 1.** Second order spectra relative to $L^{\text{har}}$. The horizontal axis is the real part of the points in $\text{Spec}_z(\mathcal{H}_n^{\text{har}}, L^{\text{har}})$ and the vertical axis is imaginary part.

**Figure 2.** $\text{Spec}_z(\mathcal{H}_n^{\text{har}}, L^{\text{har}})$ and illustration on the end points of the segment given in Theorem 2.1, $\text{Re}(z) - |\text{Im}(z)|$ and $\text{Re}(z) + |\text{Im}(z)|$. 

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Figure 3. Approximating enclosures for the first five eigenvalues of $\mathcal{H}_{\text{har}}$ by the: Quadratic, Boulton & Strauss, and Our improvement methods with $n = 200$, $\lambda_{\text{low}}$ is the lower bound of the enclosing $\lambda$ and $\lambda_{\text{upp}}$ is the upper bound. (a) This sub-figure shows the error between the exact and lower bound for the first five approximation eigenvalues of $\mathcal{H}_{\text{har}}$ using the quadratic method; (b) This sub-figure shows the error between the exact and lower bound for the first five approximation eigenvalues of $\mathcal{H}_{\text{har}}$ using the Boulton & Strauss method; (c) This sub-figure shows the error between the exact and lower bound for the first five approximation eigenvalues of $\mathcal{H}_{\text{har}}$ using our improvement method.
7. Anharmonic Oscillator Model

The anharmonic oscillator is another model of Schrodinger equation in one dimension. This model is one of the most important problems of quantum mechanics.

We choose this model to show that the technique for both methods can be applied to operator where the exact spectrum is not known.

Let $\mathcal{H}^{anh} = A$, for $V(x) = x^4$ then; by Schrodinger equation we have

$$\mathcal{H}^{anh}(x) = -u''(x) + x^4 u(x)$$  \hspace{1cm} (7.1)

And $\mathcal{H}^{anh}(x) = \lambda u(x)$ \hspace{1cm} (7.2)

The exact eigenvalue is unknown.

This equation can be solved explicitly and we can find the approximation eigenvalues using Matlab program by matrices $M, N, R$ where:

$$M = \left[\left[\Psi, \Psi'\right]\right]$$

$$N = \left[\left[x^4 \Psi, \Psi\right]\right]$$

$$R = \left[\left[\Psi, \Psi\right]\right]$$

We compute $\text{Spec}_{\mathcal{L}}(\mathcal{H}^{anh}, L^{anh})$, as in Figure 4.

**Trial 2:** In this trial we use the Quadratic method, Boulton & Strauss method, and Our improvement technique with: $n = 200, L = 6$, to find the first five approximation enclosure eigenvalues (upper and lower eigenvalues) of $\mathcal{H}^{anh}$, where

$$h = \frac{2L}{n}$$

(see Table 2, Figure 5).

**Table 2.** Approximating enclosures for the first five eigenvalues of $\mathcal{H}^{anh}$ with $n = 200$, $\lambda_{low}$ is the lower bound of the enclosing $\lambda$ and $\lambda_{upp}$ is the upper bound.

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>Quadratic method</th>
<th>Boulton and Strauss method</th>
<th>Our improvement method</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda_1$</td>
<td>Upp 1.060362557981696</td>
<td>1.060362252449401</td>
<td>1.061066062252193</td>
</tr>
<tr>
<td></td>
<td>low 1.060361563013190</td>
<td>1.060362061273195</td>
<td>1.059658118801031</td>
</tr>
<tr>
<td>$\lambda_2$</td>
<td>Upp 3.799676880351021</td>
<td>3.799673918715551</td>
<td>3.797040801641944</td>
</tr>
<tr>
<td></td>
<td>low 3.799638443628093</td>
<td>3.799672602711578</td>
<td>3.797040801641944</td>
</tr>
<tr>
<td>$\lambda_3$</td>
<td>Upp 7.455732063205368</td>
<td>7.455701633603200</td>
<td>7.462745619537849</td>
</tr>
<tr>
<td></td>
<td>low 7.455666688132128</td>
<td>7.455694527185979</td>
<td>7.448650264962926</td>
</tr>
<tr>
<td>$\lambda_4$</td>
<td>Upp 11.644110613912629</td>
<td>11.644257012619969</td>
<td>11.659764086317372</td>
</tr>
<tr>
<td></td>
<td>low 11.64469763222517</td>
<td>11.644727274867936</td>
<td>11.629726975328792</td>
</tr>
<tr>
<td>$\lambda_5$</td>
<td>Upp 16.262159917276765</td>
<td>16.261854836128617</td>
<td>16.289304652770664</td>
</tr>
<tr>
<td></td>
<td>low 16.261391679238212</td>
<td>16.26470991625957</td>
<td>16.23437516123545</td>
</tr>
</tbody>
</table>
Figure 4. Second order spectra relative to $L^\text{anh}_n$. The horizontal axis is the real part of the points in $\text{Spec}_2 \left( \mathcal{H}^\text{anh}_n, L^\text{anh}_n \right)$ and the vertical axis is imaginary part.

Figure 5. Approximating enclosures for the first five eigenvalues of $\mathcal{H}^\text{anh}_n$ by the Quadratic, Boulton & Strauss, and Our improvement methods with $n = 200$, $\lambda_{\text{low}}$ is the lower bound of the enclosing $\lambda$ and $\lambda_{\text{upp}}$ is the upper bound.
8. Conclusions

We compute \( \text{Spec}_2(\mathcal{H}^{\text{har}}_6, L_6^{\text{har}}) \) and \( \text{Spec}_2(\mathcal{H}^{\text{anh}}_6, L_6^{\text{anh}}) \), as we described before and calculate the eigenvalues enclosure. All the coefficient of the matrices \( A_0 \), \( A_1 \) and \( A_2 \) were found analytically.

Figure 1 and Figure 4 show \( \text{Spec}_2(\mathcal{H}^{\text{har}}_6, L_6^{\text{har}}) \) and \( \text{Spec}_2(\mathcal{H}^{\text{anh}}_6, L_6^{\text{anh}}) \) for value \( n = 200 \), clearly in both the second-order relative spectra is not the same.

In Table 1 and Table 2 we show the approximation of the first five eigenvalues enclosures of \( \mathcal{H}^{\text{har}}_6, \mathcal{H}^{\text{anh}}_6 \) respectively, with \( n = 200 \) by three different methods (Quadratic method, Boulton & Strauss method, and Our improvement) to compare between these methods and identify which one is more accurate for computing eigenvalues.

For the Harmonic oscillator model, the error between the approximation results and the first five exact eigenvalues (1, 3, 5, 7, 9) in our technique is less than the error between the exact and the approximation eigenvalues by the Boulton and Strauss method, and the same thing between of the Quadratic method results and the exact eigenvalues, so the approximation enclosure eigenvalues by our technique is more accurate and effective than the Boulton & Strauss method and the Quadratic method, also Figure 3 shows that clearly.

For the anharmonic oscillator model, the exact eigenvalues are unknown but by results and Figure 5, we can confirm the previous result.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

References


Fourier-Series Representation of Discontinuous Functions and Its Physical Applications

Sami M. Al-Jaber, Iyad Saadeddin

Department of Physics, An-Najah National University, Nablus, Palestine
Email: Jaber@najah.edu, iyads@najah.edu

Abstract

In this work, Fourier-series representation of a discontinuous function is used to highlight and clarify the controversial problem of finding the value of the function at a point of discontinuity. Several physical situations are presented to examine the consequences of this kind of representation and its impact on some widely well-known problems whose results are not clearly understood or justified.

Keywords

Electric Field, Charged Conductors, Electrostatics, Fourier Series

1. Introduction

The problem of a conducting sphere of radius $R$ and uniform charge $Q$ on its surface is discussed in introductory physics textbooks [1] [2]. For such a sphere, Gauss’s law gives that the electric field, at a point $r$, is zero inside ($r < R$) and $kQ/r^2$ outside ($r > R$). Usually, the question of finding the electric field on points on the surface of the sphere is not raised due to the discontinuity of the field at such points. The limit of the electric field as $r \to R$ from inside gives zero and from outside gives $kQ/R^2$ and therefore the electric field is ill-defined on the surface. Most textbooks as in [1] and [2], authors calculate, by direct use of Gauss law, the electric field just outside a conducting surface and obtain the correct value $kQ/R^2$. This value is the same as the outside limiting value of the electric field of the conducting sphere. This causes a misleading and confusion among students for the electric field on the surface and may claim that the electric field there is $kQ/R^2$. This is not a correct conclusion and one should admit the ambiguity of the electric field on the surface and care must be taken when
dealing with such a situation. In some cases one has to assign a value for the electric field at a very small patch on the surface for the purpose of calculating the electrostatic pressure (force per unit area) on the surface. This has been done by Griffiths in his Electrodynamics book [3]. Griffiths assigned the average value between the inside and outside limits and got the value $kQ/2R^2$ for the electric field on the surface of the conductor, and he arrived at the correct well-known value of the electrostatic pressure on the surface [4], namely, $\sigma^2/2\varepsilon_0$, where $\sigma$ is the surface charge density and $\varepsilon_0$ is the electric permittivity of free space.

Other researchers [5] [6] [7] [8] [9] considered the electric field which suffers a jump at the surface and concluded that the appropriate value of the electric field at the surface is the average value between the inside and the outside limits.

In the light of the above debate and due to the ambiguity of this problem, the aim of the present paper is to give a proof that the value of the electric field at the discontinuity point on the surface is in fact the average value between the inside and the outside limits of the electric field as $r \to R$. Fourier series expansion will be used to obtain our result.

### 2. Fourier Series Representation of a Discontinuous Function

Expansion of a function by a Fourier series has been of a great advantage in physics and engineering [10]-[18] because it allows one to more easily manipulate functions that are discontinuous or difficult to represent analytically. Some investigators have been working on fast and accurate simulations for Fourier representation of discontinuous function that represent scientific problems [19]-[25]. The value of the expanded function $f(x)$ at its point of discontinuity is the average of the upper and lower limits of the function [26] [27]. This means that the Fourier series converges to half-way between these two limits. Mathematically, at a point of discontinuity, $x_0$, the Fourier series converges to the value $f(x_0)$ given by

$$f(x_0) = \frac{1}{2} \lim_{\varepsilon \to 0} [f(x_0 + \varepsilon) + f(x_0 - \varepsilon)]$$

The Fourier series expansion of the function $f(x)$, with period $2L$ is conventionally written as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[ a_n \cos \left( \frac{n\pi x}{L} \right) + b_n \sin \left( \frac{n\pi x}{L} \right) \right]$$

where $a_0, a_n, b_n$ are constants called Fourier coefficients. Due to the orthogonality of the sine and cosine functions, the Fourier coefficients are readily obtained with the result

$$a_0 = \frac{1}{L} \int_{-L}^{L} f(x) \, dx$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \left( \frac{n\pi x}{L} \right) \, dx$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \left( \frac{n\pi x}{L} \right) \, dx$$
The factor $\frac{1}{2}$ which appears in the $a_0$ term in Equation (2) is included so that Equation (4) may be applied for $n = 0$ as well as $n > 0$.

In order to demonstrate Equation (1), we consider a specific function $f(x)$, which is given by

$$f(x) = \begin{cases} 0 & x \in (-L, 0) \\ 1 & x \in (0, L) \end{cases}$$

Applying Equations (3)-(5), one immediately gets $a_0 = 1$, $a_n = 0$ and $b_n = 2/n\pi$ for $n$ = odd only. Therefore, Equation (2) yields

$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1}{n} \sin \left( \frac{n\pi x}{L} \right)$$

Obviously, at the point of discontinuity ($x = 0$) the series in Equation (7) converges to 1/2, which is the average value between the two limits of $f(x)$ below and above the jump.

### 3. A Uniformly Charged Conducting Sphere

In order to discuss the problem of the value of the electric field on the surface of a conducting sphere, we consider a conducting sphere of radius $R$ which is uniformly charged by a charge $Q$. Direct application of Gauss’s law gives the electric field at a point $r$ from the sphere’s center, with the result

$$E = \begin{cases} 0 & r < R \\ \frac{kQ}{r^2} & r > R \end{cases}$$

with $k = 1/4\pi\varepsilon_0$. Obviously, the electric field suffers a discontinuity at points on the surface since its limit from inside gives zero while from outside gives $kQ/R^2$.

In order to find the electric field at points on the surface, we proceed as follows:

We consider a conducting object which carries a surface charge density $\sigma$ on its surface. The widely well-known quantity which is usually invoked is the electric field just outside the conductor. The usual treatment to derive this is by considering a cylindrical Gaussian surface whose half of its length inside the conductor and the other half is outside as in Figure 2. Application of Gauss’s law yields $E = \sigma/\varepsilon_0$, which is found in most standard introductory physics textbooks [2]. Therefore, the electric field is zero inside the conductor and a constant ($\sigma/\varepsilon_0$) just outside. The result shows that the electric field suffers a finite jump on the surface, so that the electric field is written as

$$E = \begin{cases} 0 & -\varepsilon \leq x < 0 \\ \frac{\sigma}{\varepsilon_0} & 0 < x \leq \varepsilon \end{cases}$$
where \( \varepsilon \) is a small quantity and \( x = 0 \), the discontinuity point, represents a point on the surface. The point now is to expand the electric field in Fourier series over the interval \([-\varepsilon, \varepsilon]\). Comparing Equation (9) with Equation (6) and noting that \( L \rightarrow \varepsilon \), the Fourier coefficients, given in Equations (3)-(5) are easily calculated with the result \( a_0 = \sigma/\varepsilon_0 \), \( a_n = 0 \) and \( b_n = 2\sigma/n\varepsilon_0 \). Therefore, Equation (2) gives the Fourier expansion of the electric field which is given by

\[
E = \frac{\sigma}{2\varepsilon_0} + \frac{2\sigma}{\pi\varepsilon_0} \sum_{n=odd}^{\infty} \frac{1}{n} \sin \left( \frac{n\pi x}{\varepsilon} \right)
\]  

(10)

It should be clear that the above series converges to \( \sigma/2\varepsilon_0 \) at the discontinuity point \( (x = 0) \) which is the average value of the electric field between its two limits from inside and outside. Therefore, by using the relation \( \sigma = q/4\pi R^2 \) for the conducting sphere problem, we get our required result for the electric field on the surface,

\[
E = \frac{1}{2} \left[ \lim_{r \to x^-} E + \lim_{r \to x^+} E \right] = \frac{1}{2} \frac{Q}{4\pi\varepsilon_0 R^2} = \frac{kQ}{2R^2}
\]  

(11)

Our result in Equation (11) removes the ambiguity of the electric field at points on the surface of a conducting sphere and assigns a value of this field at its discontinuity point. This interesting result must be explained at the undergraduate level for physics and engineering students, since it has been avoided in almost all undergraduate physics textbooks. It should be emphasized that our result, beside its mathematical interest, it also has applications in physical situations in which the value of a function at its point of discontinuity is necessary in order to derive some relevant physical quantities. In the next section, three physical situations will be presented to demonstrate the use of our result and to show that the value of a discontinuous function at its point of discontinuity is the average value of the function at that point, which is the value where the Fourier series of the function converges.

### 4. Some Physical Problems Involve Discontinuous Functions

Here, we consider three physical situations that involve functions that suffer a jump at a point on a boundary. Therefore, assigning a value of the function at that point is necessary in order to achieve a relevant physical quantity.

#### 4.1. The First Problem: Electrostatic Pressure on Surface of a Conductor

The first problem deals with the calculation of electrostatic pressure on the surface of a conductor which contains a surface charge density \( \sigma \). This problem has been discussed by Griffiths [28] in his famous textbook on electrodynamics. There, Griffiths considered a patch on the surface of the conductor in attempt to find the electrostatic pressure on the surface of the conductor. He argued (but not rigorously) that the electric field on the patch is the average value between the value of electric field just outside the conductor and
\( \sigma/\epsilon_0 \) and its zero value inside. Hence he arrived at the value \( \sigma/2\epsilon_0 \) at the patch, and therefore the pressure (force per unit area) is just \( \sigma^2/2\epsilon_0 \). However, this argument seems a bit dodgy because the patch is not a point so that part of the patch creates a field that affects the other part of the patch. So our result can now be applied to determine the value of the electric field at a point on the surface. For that purpose, we construct a small cylindrical surface as shown in Figure 1. We assume \( \epsilon \) to be very small, so that, \( \lim_{\epsilon \to 0} E = 0 \) and \( \lim_{\epsilon \to 0} E = \sigma/\epsilon_0 \). Therefore, the average of these two limits gives the correct value of the electric field at a point on the surface of the conducting sphere, namely, \( E = \sigma/2\epsilon_0 \) and hence the well-known value of the electrostatic pressure on the surface of the sphere.

**4.2. The Second Problem: Energy Aspects in Charging a Capacitor**

The well-known two capacitor problem has been of great interest since long time ago, and variety of approaches have been considered to explain the energy loss in this problem [29] [30] [31] [32]. The essence of the problem amounts to the problem of charging a capacitor of capacitance \( C \) by a power supply of electromotive force \( V_0 \) with a series resistor of resistance \( R \). At the end of charging, the energy stored in the capacitor will be \( \frac{1}{2}CV_0^2 \) and exactly the same amount will be dissipated regardless of the value of \( R \). In ref. [30], the authors used superconducting wires and used the flux of energy carried by the Poynting vector to calculate the energy stored in the capacitor and the energy loss. In such situation, the charge on the capacitor involves a step function, namely, \( Q(t) = Q\delta(t) \). In their derivation of their final result for the stored energy, \( U \), they encountered the integral

\[
U = -\frac{Q^2}{C} \int \theta(t)\delta(t) dt = \frac{Q^2}{C} \theta(0)
\]

and a similar one for the energy loss. The value of the step function at the discontinuity point \( t = 0 \) was used by taking the average value between the limits from below and above \( t = 0 \) as

\[
\theta(0) = \frac{1}{2}[\theta(0^-) + \theta(0^+)] = \frac{1}{2}
\]

Therefore, the average value at the discontinuity point has a crucial role in deriving the energy loss in this process.

**4.3. The Fermi Distribution Function**

In this subsection, we present our third physical situation for the use of the average value at the jump of a discontinuous function. The average number of fermions, in a single-particle state with energy \( \epsilon \) is given by the Fermi-Dirac distribution function [33],

\[
\langle n(\epsilon) \rangle = \frac{1}{e^{(\epsilon - \mu)/k_B T} + 1}
\]

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where \( \mu \) is the chemical potential, \( k_B \) is the Boltzmann constant and \( T \) is the temperature in Kelvin, as shown in Figure 2. We note that the Fermi distribution function behaves like a step function

\[
\lim_{T \to 0} n(\varepsilon) = \begin{cases} 
0 & \varepsilon > \mu \\
1 & \varepsilon < \mu 
\end{cases} = \theta(\varepsilon - \mu) 
\tag{15}
\]

Therefore, the average population number of fermions at zero temperature is \( 1/2 \), which is at the midpoint of the jump. One can also observe from Equation (14) that for any temperature \( T > 0 \), the average population number is \( 1/2 \) when the energy is equal to the chemical potential.

5. Conclusion

In this paper, the authors examined the behavior of a discontinuous function and its application in some physical systems. Fourier series representation of such function has been studied, and it has been pointed out that, at the point of discontinuity, this series converges to the average value between the two limits of the function about the jump point. So for a step function, this convergence occurs at the exact value of one half. The obtained result clarifies and solves a controversial problem about the value of the electrostatic field at points on the surface of a conducting sphere, which is usually avoided in introductory physics books. As an application of our result, three physical systems have been discussed and the average value of the function at the discontinuity point has been used in such systems: The first deals with the calculation of the electrostatic pressure on the sur-
face of a conductor, the second concerns the calculation of the energy loss in charging a capacitor using superconducting wires and the third is the behavior of the population function of Fermi gas at zero temperature.

**Conflicts of Interest**

The authors declare no conflicts of interest regarding the publication of this paper.

**References**


A Modern Method for Constructing the S-Box of Advanced Encryption Standard

W. Eltayeb Ahmed

1Mathematics and Statistics Department, Faculty of Science, Imam Mohammad Ibn Saud Islamic University, Riyadh, KSA
2Department of Basics and Engineering Sciences, Faculty of Engineering, University of Khartoum, Khartoum, Sudan
Email: waahmed@imamu.edu.sa

Abstract
The substitution table (S-Box) of Advanced Encryption Standard (AES) and its properties are key elements in cryptanalysis ciphering. We aim here to propose a straightforward method for the non-linear transformation of AES S-Box construction. The method reduces the steps needed to compute the multiplicative inverse, and computes the matrices multiplication used in this transformation, without a need to use the characteristic matrix, and the result is a modern method constructing the S-Box.

Keywords
Advanced Encryption Standard, S-Box, Extended Euclidean Algorithm, Greatest Common Divisor, XOR Operation

1. Introduction
The S-Box table of AES is taken as a lookup table to substitute an input byte by another, this table is constructed using a non-linear transformation depends on the usual method taking more calculation steps to give the corresponding byte.

The S-Box plays a fundamental role in encryption and decryption processes, as byte substitution appears in many steps. At the first round of the encryption process, we add the plaintext matrix to the key matrix, then we substitute each byte by another byte according to S-Box, for example, to substitute the byte $xy$(say), we take the byte in the cell that has $x$ as the column index and $y$ as the row index, we do this substitute byte step in all rounds of the encryption process, and in all round of the decryption process, we do the inverse substitute byte step, to substitute the byte $xy$(say), we take the index of the column, and the index of the row of the cell that contains $xy$, as the left and the right character of the result byte, respectively. The S-Box (Table 1), involves substitution bytes for all
bytes from \{00\} to \{FF\} in hexadecimal presentation.

The S-Box is constructed using the following operations [1]:

1) Finding the multiplicative inverse of an input byte in the finite field \( GF(2^8) \) based on the irreducible polynomial \( P(x) = x^8 + x^4 + x^3 + x + 1 \).

2) Multiplying this multiplicative inverse by a specific matrix (matrix \( M \)).

3) Adding the multiplication result to a specific vector \( \{63\} = 01100011 \).

We convert the hexadecimal presentation of the input byte into binary presentation as \( \{76543210\} \) and write it as a polynomial \( A(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 \), let its multiplicative inverse be \( T(x) = t_n x^n + t_{n-1} x^{n-1} + \ldots + t_1 x + t_0 \), we multiply \( T(x) \) by the following characteristic matrix:

\[
M = \begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\
0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 
\end{bmatrix}
\]

Then, we add the result to \( \{01100011\} \).

We note that, for the input \{00\} the output is \{63\}.

Table 1. The AES S-Box.

<table>
<thead>
<tr>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>63</td>
<td>7C</td>
<td>77</td>
<td>7B</td>
<td>F2</td>
<td>6B</td>
<td>6F</td>
<td>C5</td>
<td>30</td>
<td>01</td>
<td>67</td>
<td>2B</td>
<td>FE</td>
<td>D7</td>
<td>AB</td>
</tr>
<tr>
<td>1</td>
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<td>82</td>
<td>C9</td>
<td>7D</td>
<td>FA</td>
<td>59</td>
<td>4F</td>
<td>00</td>
<td>AD</td>
<td>D4</td>
<td>A2</td>
<td>AF</td>
<td>9C</td>
<td>A4</td>
<td>72</td>
</tr>
<tr>
<td>2</td>
<td>B7</td>
<td>FD</td>
<td>93</td>
<td>26</td>
<td>36</td>
<td>3F</td>
<td>F7</td>
<td>CC</td>
<td>34</td>
<td>A5</td>
<td>E5</td>
<td>F1</td>
<td>71</td>
<td>D8</td>
<td>31</td>
</tr>
<tr>
<td>3</td>
<td>04</td>
<td>C7</td>
<td>23</td>
<td>C3</td>
<td>18</td>
<td>96</td>
<td>05</td>
<td>9A</td>
<td>07</td>
<td>12</td>
<td>80</td>
<td>E2</td>
<td>EB</td>
<td>27</td>
<td>B2</td>
</tr>
<tr>
<td>4</td>
<td>09</td>
<td>83</td>
<td>2C</td>
<td>1A</td>
<td>1B</td>
<td>6E</td>
<td>5A</td>
<td>A0</td>
<td>52</td>
<td>3B</td>
<td>D6</td>
<td>B3</td>
<td>29</td>
<td>E3</td>
<td>2F</td>
</tr>
<tr>
<td>5</td>
<td>53</td>
<td>D1</td>
<td>00</td>
<td>ED</td>
<td>20</td>
<td>FC</td>
<td>B1</td>
<td>5B</td>
<td>6A</td>
<td>CB</td>
<td>BE</td>
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<td>4A</td>
<td>4C</td>
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<td>D0</td>
<td>EE</td>
<td>AA</td>
<td>FB</td>
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<td>4D</td>
<td>33</td>
<td>85</td>
<td>45</td>
<td>F9</td>
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<td>9F</td>
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<td>F5</td>
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<td>B6</td>
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<td>0C</td>
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<td>EC</td>
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<td>97</td>
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<td>DE</td>
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<td>0B</td>
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<td>0A</td>
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<td>C8</td>
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<td>76</td>
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<td>BA</td>
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<td>25</td>
<td>2E</td>
<td>1C</td>
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<td>B4</td>
<td>C6</td>
<td>E8</td>
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<td>74</td>
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<td>4B</td>
<td>BD</td>
<td>8B</td>
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<td>D9</td>
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<td>87</td>
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<td>CE</td>
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<td>28</td>
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<tr>
<td>F</td>
<td>8C</td>
<td>A1</td>
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<td>2D</td>
<td>0F</td>
<td>B0</td>
<td>5A</td>
<td>BB</td>
</tr>
</tbody>
</table>
1.1. Problem Statement

We search for an easier and straightforward method for constructing the AES S-Box.

1.2. Proposed Solution

The multiplicative inverse of an input byte can be computed in clear steps using an iterated formula.

Multiplying the multiplicative inverse matrix by the characteristic matrix can be determined directly from this multiplicative inverse using simple XOR operations, without a need to use the characteristic matrix.

2. Traditional Way

In cryptography, the extended Euclidean algorithm has wide uses especially for finding a multiplicative inverse (modular inverse).

Euclidean algorithm is used to find the greatest common divisor of two integers \( a \) and \( b \), (denoted by \( \gcd(a,b) \)).

When \( b > a \), and

\[
b - r = aq
\]

for some integers \( r \) and \( q \), we say

\[
r = b \pmod{a}
\]

and if \( b \equiv 0 \pmod{a} \) then

\[
\gcd(a,b) = a
\]

With the polynomials \( A(x) \) and \( P(x) \), we write \( \gcd(A(x),P(x)) \) [2].

The algorithm below gives \( \gcd(A(x),P(x)) \), where \( A(x) < P(x) \)

---

Algorithm (1): Euclidean algorithm [3]

Input: Polynomials \( A(x), P(x) \).

Output: \( \gcd(A(x),P(x)) \).

1) While \( A(x) \neq 0 \) do

   a) \( r(x) = P(x) \mod A(x), \) \( P(x) = A(x), \) \( A(x) = r(x) \).

2) Return \( P(x) \).

The step (1.(a)) of the algorithm (1) involves the division algorithm:

\[
P(x) = A(x)q(x) + r(x)
\]

where \( 0 \leq r(x) = P(x) \mod A(x) < A(x) \).

It implies that [4]

\[
\gcd(A(x),P(x)) = \gcd(A(x),r(x))
\]

If \( r(x) \neq 0 \), the step will be repeated, let us write the repeated application of the division algorithm as:

\[
P(x) = q_i(x)A(x) + r_i(x), \quad 0 \leq r_i(x) < A(x)
\]
When $r_i(x) = 0$, and since

$$\gcd(A(x), P(x)) = \gcd(r_{i-1}(x), r_i(x))$$

we get

$$\gcd(A(x), P(x)) = r_{i-1}(x)$$

The extended form of the Euclidean algorithm is called Extended Euclidean algorithm, it gives (besides $\gcd(A(x), P(x))$, $X(x)$ and $Y(x)$ such that

$$\gcd(A(x), P(x)) = A(x)X(x) + P(x)Y(x)$$

Rewrite the equations of the system (7) as:

$$r_1(x) = P(x) - q_1(x)A(x)$$
$$r_2(x) = A(x) - q_2(x)r_1(x)$$
$$r_3(x) = r_1(x) - q_3(x)r_2(x)$$

$$r_{i-2}(x) = r_{i-3}(x) - q_{i-2}(x)r_{i-3}(x)$$
$$r_{i-1}(x) = r_{i-2}(x) - q_{i-1}(x)r_{i-2}(x)$$

Then, in the last equation of system (11), $r_{i-1}(x) = r_{i-3}(x) - q_{i-1}(x)r_{i-3}(x)$, replace $r_{i-2}(x)$ with its value from the above equation (it involves $r_{i-3}(x)$), then replace $r_{i-3}(x)$ with its value from the above equation, continue doing this replacement, we obtain

$$r_{i-1}(x) = A(x)X(x) + P(x)Y(x)$$

In our problem $1 \leq i < 8$, and since the multiplicative inverse only exists when the $\gcd$ is 1 [5].

$$r_{i-1}(x) = 1$$

The multiplicative inverse [2] of $A(x)$ modulo $P(x)$ is $A^{-1}(x)$ such that

$$A(x)A^{-1}(x) = 1 \pmod{P(x)}$$

When $\gcd(A(x), P(x)) = 1$,

$$1 = A(x)X(x) + P(x)Y(x)$$
\[ 1 \pmod{P(x)} = (A(x)X(x) + P(x)Y(x)) \pmod{P(x)} \]  
(17)

and since

\[ P(x)Y(x) = 0 \pmod{P(x)} \]  
(18)

we get

\[ X(x) = A^{-1}(x) \]  
(19)

So, the procedure of the extended Euclidean algorithm finds the greatest common divisor, also it finds the multiplicative inverse.

Below an algorithm to find \( A^{-1}(x) \), we will denote \( A^{-1}(x) \) by \( T(x) \).

Algorithm (2): Extended Euclidean algorithm [3]

Input: Polynomials \( A(x), P(x) \).
Output: The multiplicative inverse of \( A(x) \).

1) Set \( y_1(x) = 0 \), \( y_2(x) = 1 \).
2) While \( A(x) \neq 1 \) do
   a) \( q(x) = P(x) \div A(x) \), \( r(x) = P(x) + q(x)A(x) \).
   b) \( y_1(x) = y_1(x) + y_2(x)q(x) \).
   c) \( y_2(x) = y_1(x), y_1(x) = y_2(x) \).
   d) \( P(x) = A(x), A(x) = r(x) \).
3) Return \( y_1(x) \).

Now, we have \( T(x) = (t_2t_4t_3t_5t_6t_0) \), we multiply it (from the left) by matrix \( M \)

\[
M(T(x)) = \begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{bmatrix}
= \begin{bmatrix}
t_0 + t_4 + t_5 + t_6 + t_7 \\
t_0 + t_4 + t_5 + t_6 + t_7 \\
t_0 + t_4 + t_5 + t_6 + t_7 \\
t_0 + t_4 + t_5 + t_6 + t_7 \\
t_0 + t_4 + t_5 + t_6 + t_7 \\
t_0 + t_4 + t_5 + t_6 + t_7 \\
t_0 + t_4 + t_5 + t_6 + t_7 \\
t_0 + t_4 + t_5 + t_6 + t_7 \\
\end{bmatrix}
\]  
(20)

Then, we add the result to \( \{63\} = 01100011 \) to obtain the output of the input \( A(x) = (a_7a_6a_5a_4a_3a_2a_1a_0) \)

\[
\begin{bmatrix}
t_0 + t_4 + t_5 + t_6 + t_7 \\
t_0 + t_4 + t_5 + t_6 + t_7 \\
t_0 + t_4 + t_5 + t_6 + t_7 \\
t_0 + t_4 + t_5 + t_6 + t_7 \\
t_0 + t_4 + t_5 + t_6 + t_7 \\
t_0 + t_4 + t_5 + t_6 + t_7 \\
t_0 + t_4 + t_5 + t_6 + t_7 \\
t_0 + t_4 + t_5 + t_6 + t_7 \\
\end{bmatrix}
+ \begin{bmatrix}
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
\end{bmatrix}
= \begin{bmatrix}
t_0 + t_4 + t_5 + t_6 + t_7 + 1 \\
t_0 + t_4 + t_5 + t_6 + t_7 + 1 \\
t_0 + t_4 + t_5 + t_6 + t_7 + 1 \\
t_0 + t_4 + t_5 + t_6 + t_7 + 1 \\
t_0 + t_4 + t_5 + t_6 + t_7 + 1 \\
t_0 + t_4 + t_5 + t_6 + t_7 + 1 \\
t_0 + t_4 + t_5 + t_6 + t_7 + 1 \\
t_0 + t_4 + t_5 + t_6 + t_7 + 1 \\
\end{bmatrix}
\]  
(21)
Example

Using the traditional way, we want to find the output byte that corresponds to the input byte \( 53 \) (Table 2).

\( \{53\} = 01010011, \quad A(x) = x^6 + x^4 + x + 1, \quad P(x) = x^8 + x^4 + x^3 + x + 1. \)

**Iteration 1**

\( y_2(x) = 0, \quad y_1(x) = 1, \)

\( q(x) = x^2 + 1, \quad r(x) = x^2, \)

\( y(x) = x^2 + 1, \quad y_2(x) = 1, \quad y_1(x) = x^2 + 1, \)

\( P(x) = x^6 + x^4 + x + 1, \quad A(x) = x^2. \)

**Iteration 2**

\( q(x) = x^4 + x^2, \quad r(x) = x + 1, \)

\( y(x) = x^6 + x^2 + 1, \quad y_2(x) = x^2 + 1, \quad y_1(x) = x^6 + x^2 + 1, \)

\( P(x) = x^7, \quad A(x) = x + 1. \)

**Iteration 3**

\( q(x) = x + 1, \quad r(x) = 1, \)

\( y(x) = x^6 + x^3 + x, \quad y_2(x) = x^6 + x^2 + 1, \quad y_1(x) = x^7 + x^6 + x^3 + x, \)

\( P(x) = x + 1, \quad A(x) = 1. \)

\[ T(x) = y_1(x) = x^7 + x^6 + x^3 + x = 11001010. \]

The output is \( 11101101 = ED. \)
Table 2. To find the output of [53].

<table>
<thead>
<tr>
<th></th>
<th>...</th>
<th>3</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>5</td>
<td>...</td>
<td>??</td>
<td>...</td>
</tr>
<tr>
<td>1</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>

3. Modern Way

We use the formula [6] below to find the multiplicative inverse.

3.1. The Iterated Formula

The iterated formula

\[
T_i(x) = (q_i(x))(T_{i-1}(x)) + T_i(x), \quad 2 \leq i < 8
\]  

(24)

where \( T_0(x) = 1, T_1(x) = q_i(x) \), gives the multiplicative inverse \( T = T_i \) when \( r_i = 1 \).

To show that, we use the system (11).

When \( i = 1 \), \( r_1(x) = 1 \),

\[
r_1(x) = P(x) - (q_1(x))(A(x))
\]  

(25)

\[
1 = P(x) - (q_1(x))(A(x))
\]  

(26)

We obtain \( T(x) = q_1(x) = T_1(x) \). (Equation (24), takes this as given).

When \( i = 2 \), \( r_2(x) = 1, \ r_1(x) \neq 0 \),

\[
r_2(x) = A(x) - (q_2(x))(r_1(x))
\]  

(27)

\[
1 = A(x) - (q_2(x))(P(x) - (q_1(x))(A(x)))
\]  

(28)

\[
= (1 - (q_2(x))(q_1(x))(A(x)) - (q_2(x))(P(x))
\]  

(29)

We obtain \( T(x) = (q_2(x))(q_1(x)) + 1 \). From Equation (24)

\[
T(x) = T_2(x) = q_2(x)T_1(x) + T_0(x) = (q_2(x))(q_1(x)) + 1
\]  

(30)

When \( i = 3 \), \( r_3(x) = 1, \ r_1(x) \neq 0, \ r_2(x) \neq 0 \),

\[
r_3(x) = r_1(x) - q_3(x)r_2(x)
\]  

(31)

\[
1 = P(x) - q_3(x)A(x) - q_1(x)((1 - q_2(x)q_1(x))A(x) - q_2(x)P(x))
\]  

(32)

We obtain

\[
T(x) = (q_3(x))(1 - (q_2(x))(q_1(x))) - q_1(x) = q_3(x)T_2(x) + T_1(x)
\]  

(33)

and from Equation (24)

\[
T(x) = T_3(x) = q_3(x)T_2(x) + T_1(x)
\]  

(34)

By this way, we can show that Equation (24) gives \( T(x) \) for \( 2 \leq i < 8 \), when \( r_i = 1 \).
Below an algorithm to find $T(x)$ using the modern way.

Algorithm (3): Modern way to find a multiplicative inverse

Input: Polynomials $A(x), P(x)$.

Output: $T(x)$, the multiplicative inverse of $A(x)$.

1) Set $T_0(x) = 0$, $T_1(x) = 1$.

2) $q(x) = P(x) \text{div} A(x)$, $r(x) = P(x) + q(x) A(x)$.

3) $T(x) = q(x) T_0(x) + T_1(x)$

4) If $r(x) = 1$ then return $T(x)$, stop.

5) Else $P(x) = A(x)$, $A(x) = r(x)$.

6) Go to 2

Now, we want to multiply $T(x)$ by the matrix $M$.

First, write $M$ as

$$
M = \begin{bmatrix}
1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
$$

Let

$$
M_1 = \begin{bmatrix}
1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 \\
1 & 1 & 1 & 1 \\
\end{bmatrix}, \quad M_2 = \begin{bmatrix}
1 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
$$

And write $T(x)$ as

$$
T = \begin{bmatrix}
T_0 \\
T_1 \\
T_2 \\
T_3 \\
T_4 \\
T_5 \\
T_6 \\
T_7 \\
\end{bmatrix}
$$

Let

$$
T_1 = \begin{bmatrix}
T_0 \\
T_1 \\
T_2 \\
T_3 \\
\end{bmatrix}, \quad T_2 = \begin{bmatrix}
T_4 \\
T_5 \\
T_6 \\
T_7 \\
\end{bmatrix}
$$
Then

\[
M_1 T_1 = \begin{bmatrix}
I_0 \\
I_0 + t_2 \\
I_0 + t_1 + t_2 \\
I_0 + t_1 + t_2 + t_3
\end{bmatrix}
\]  \hspace{1cm} (38)

\[
M_2 T_2 = \begin{bmatrix}
I_4 \\
I_4 + t_6 \\
I_4 + t_5 + t_6 \\
I_4 + t_5 + t_6 + t_7
\end{bmatrix}
\]  \hspace{1cm} (39)

\[
M_1 T_1 = \begin{bmatrix}
I_5 + t_2 + t_3 + t_4 \\
I_5 + t_2 + t_3 \\
I_5 + t_2 \\
I_5
\end{bmatrix}
\]  \hspace{1cm} (40)

\[
M_2 T_2 = \begin{bmatrix}
I_7 + t_6 + t_8 + t_4 \\
I_7 + t_6 + t_5 \\
I_7 + t_6 \\
I_7
\end{bmatrix}
\]  \hspace{1cm} (41)

So, the multiplication of \( M \) and \( T(x) \) gives

\[
\begin{bmatrix}
M_1 T_1 + M_2 T_2 \\
M_1 T_1 + M_2 T_2
\end{bmatrix}
\]

From Equation (38) and Equation (39), we note that the results of these multiplications give the form

\[
\begin{bmatrix}
\text{first element} \\
\text{first + second} \\
\text{first + second + third} \\
\text{first + second + third + fourth}
\end{bmatrix}
\]

of the second matrix, and similarly, Equation (40) and Equation (41), show that the results give the form

\[
\begin{bmatrix}
\text{fourth + third + second + first} \\
\text{fourth + third + second} \\
\text{fourth + third} \\
\text{fourth}
\end{bmatrix}
\]

of the second matrix, so we don’t need to use matrix \( M \) as the traditional method.

In the last step, we add \( M(t(x)) \) to \( \{63\} = 01100011 \).

3.2. Example

Using the modern way, we want to find the output of \( \{53\} \)

\[
\{53\} = 01010011, \quad A(x) = x^6 + x^4 + x + 1, \quad P(x) = x^8 + x^4 + x^3 + x + 1.
\]

First, finding the multiplicative inverse (Table 3).
Table 3. Finding multiplicative inverse of $[53]$.

<table>
<thead>
<tr>
<th>$i$</th>
<th>$A(x)$</th>
<th>$q(x)$</th>
<th>$r(x)$</th>
<th>$P(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$x^2 + x^4 + x + 1$</td>
<td>$x^2 + 1$</td>
<td>$x^2 + 1$</td>
<td>$x^2 + x^4 + x + 1$</td>
</tr>
<tr>
<td>2</td>
<td>$x^2 + 1$</td>
<td>$x^2 + x^3$</td>
<td>$x + 1$</td>
<td>$x^6 + x^4 + x + 1$</td>
</tr>
<tr>
<td>3</td>
<td>$x + 1$</td>
<td>$x + 1$</td>
<td>$1$</td>
<td>$x^2 + 1$</td>
</tr>
</tbody>
</table>

Since $r_1(x) = 1$,

$$T(x) = T_1(x) = (q_3(x))T_2(x) + T_1(x) = (q_3(x))(q_2(x)(q_1(x)) + 1) + q_1(x) = (x + 1)[(x^4 + x^2)(x^2 + 1) + 1] + x^2 + 1 = x^7 + x^6 + x^3 + x = 11001010$$

Then, computing the matrices multiplication:

$$\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} = 1$$

(42)

Last, adding (01100011)

$$\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

(43)

So, the output is $11101101 = ED$.

4. Conclusions

In this paper, a straightforward method for obtaining the Advanced Encryption Standard S-Box look-up table without the traditional use of the characteristic Matrix $M$ is proposed. We have demonstrated that the two methods are equivalent. In addition, the multiplicative inverse of $A(x)$ has been found more elegantly.

In future work, we will investigate the properties and the impact of this technique on cipher complexity analysis.
Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

References


Modelling the Effects of Vertical Transmission in Mosquito and the Use of Imperfect Vaccine on Chikungunya Virus Transmission Dynamics

Martins O. Onuorah¹*, Emmanuel I. Obi², Bala G. Babangida³

¹Department of Physical Sciences, Kampala International University, Kampala, Uganda
²Department of Science Laboratory Technology, Federal Polytechnic Nasarawa, Nasarawa, Nigeria
³Department of Mathematics and Statistics, Islamic University in Uganda, Mbale, Uganda

Email: *martins.onuorah@kiu.ac.ug, eobi39@yahoo.com, rumaya2011@gmail.com


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Abstract

In this paper, a deterministic mathematical model for Chikungunya virus (Chikv) transmission and control is developed and analyzed to underscore the effect of vaccinating a proportion of the susceptible human, and vertical transmission in mosquito population. The disease free, and endemic equilibrium states were obtained and the conditions for the local and global stability or otherwise were given. Sensitivity analysis of the effective reproductive number, \( R_e \) (the number of secondary infections resulting from the introduction of a single infected individual into a population where a proportion is fairly protected) shows that the recruitment rate of susceptible mosquito \( (\Lambda_M) \) and the proportion of infectious new births from infected mosquito \( (\beta) \) are the most sensitive parameters. Bifurcation analysis of the model using center manifold theory reveals that the model undergoes backward bifurcation (coexistence of disease free and endemic equilibrium when \( R_e < 1 \)). Numerical simulation of the model shows that vaccination of susceptible human population with imperfect vaccine will have a positive impact and that vertical transmission in mosquito population has a negligible effect. To the best of our knowledge, our model is the first to incorporate vaccinated human compartment and vertical transmission in (Chikv) model.

Keywords

Chikungunya Virus, Stability, Equilibrium, Vaccination, Endemic

1. Introduction

Chikungunya is a mosquito-borne viral disease that was first observed in Tanzania.
nia in 1952 [1]. In 1964, there was epidemic of Chikungunya in Vellore, Calcutta and Maharashtra state/provinces of India [2]. Ibadan, South Western Nigeria witnessed an epidemic of Chikungunya virus in 1969 when the virus was isolated from 49 patients [3]. The disease has been identified in over 60 countries in Asia, Africa, Europe and America, and the name describes the stooping appearance of the sufferers [4]. It is an RNA virus that belongs to the alphavirus genus and the family [5]. The symptoms include abrupt onset of fever accompanied by joint pain, muscle pain headache, nausea and rash [6]. Occasionally the infection may go unrecognized or be misdiagnosed and could be acute, sub-acute and chronic.

In recent years, the virus has risen from relatively obscurity to become a global public health menace affecting millions of persons throughout the tropical and subtropical regions of the world and as such has also become a frequent cause of travel associated febrile illness [7]. The virus is transmitted through the bite of female *Aedes aegypti* and *Aedes albopictus* mosquitoes. *Aedes aegypti* breeds in the ubiquitous small pools of water found around human habitation [8]. Unlike *Aedes aegypti* which exists in tropical and subtropical area, *Aedes albopictus* can also thrive in temperate regions, thus potentially introducing Chikungunya to new ecological niche [9]. These species of mosquitoes are found biting throughout the daylight hours. Mother to child transmission of Chikungunya virus has been reported [10].

Diagnosis is by confirming the presence of anti-Chikungunya antibody in the patient. At the moment, there is no vaccine or treatment for the disease. Protection is by covering of exposed skin with long pants and long sleeved shirts, insect repellents and insecticide treated mosquito nets. Since the beginning of the 19th century, mathematical model has become a veritable tool in the study of vector-borne diseases [11] [12] [13]. For (Chikv), we cite the following work, Dumont and Domerg [14], propose a model, including human and mosquito compartments that are associated with the time course of the first epidemic of Chikungunya in Reunion Island. Using entomological results, they investigated the links between the episode of 2005 and the outbreak of 2006. Moulay, Azziz and Cadivel [15], developed a Chikungunya transmission model for the spread of the epidemic in both humans and mosquitoes, the model involves a temporal dynamics of vector (*Aedes albopictus*), depending on climatic factors. In the study, they provided estimates of the transmission potential of the virus and assessed the efficacy of the measures undertaken by public health authorities to control the epidemic spread in Italy. Ruiz et al. [16], analyzed the potential risk of Chikungunya introduction into the US, their study combines a climate-based mosquito population dynamics stochastic model with an epidemiological model to identify temporal windows that have epidemic risk.

Pongsumpun and Sangsawang [17], model studied theoretically an age-structured model for Chikungunya involving juvenile and adult human populations, giving conditions for the disease-free and endemic states respectively. They also sug-
gested alternative way for controlling the disease. Yakob and Clements [18],
analysed a simple, deterministic mathematical model for the transmission of the
virus between humans and mosquitoes. They fitted the model to the large Reu-
nion epidemic data and estimated the type reproduction number for Chikungunya, their model provided a close approximation of both the peak incidence of
the outbreak and the final epidemic size.

In this work, we proposed a deterministic mathematical model for the spread,
and control of Chikungunya. Our model attempt to bridge identified gaps in the works
cited above. Specifically, our model incorporated an imperfect vaccinated human
compartment and vertical transmission in the mosquito population.

2. Model Formulation

The chic model is represented by nine non-linear ordinary differential equation
consisting of human-sub population and mosquito sub-population. The human
sub-population is divided into; susceptible human $S_H$, vaccinated human $V_H$,
exposed human $E_H$, infected symptomatic human $I_1$, infected asymptomatic
human $I_2$, recovered Human $R$, such that the total human population,
$N_H = S_H + V_H + E_H + I_1 + I_2 + R$. While the mosquito sub-population is divided
into; susceptible mosquito $S_M$, exposed mosquito $E_M$, and infected mosquito
$I_3$, such that the total mosquito population, $N_M = S_M + E_M + I_3$.

The parameters of the model and their values are given in Table 1, while Figure 1 is the schematic diagram of the transmission dynamics.

The susceptible human sub-population is generated at a constant rate $\Lambda_H$, which includes birth and immigration. The vaccinated population is generated as members of the susceptible population receive vaccination at the rate $\nu$, a proportion of the vaccinated with time lose their immunity at the rate $\psi$ as their vaccine wanes and move back to the susceptible population. Member of the susceptible and vaccinated populations acquire infection at the rate $\alpha_H b_M I_M / N_H$ and $\alpha_H b_M I_M (1 - \varepsilon) / N_H$ respectively and move to the exposed population, where
$\alpha$ is the probability of infection, $b_M$ biting rate of mosquito and $\varepsilon$ (where
$0 < \varepsilon < 1$) is the efficacy of the imperfect vaccine. Members of the exposed population move to either symptomatic infectious population at the rate $\sigma_I$ or to asymptomatic infectious population at the rate $(1 - \sigma_I)$. The recovered population is generated as both symptomatic and asymptomatic infected populations recover with lifelong immunity at the rate $\gamma$. All human population are decreased by natural death at the rate $\mu$, except the two infected populations that are decreased by disease induced death at the rate $\delta$.

The susceptible mosquito population is generated by $\Lambda_M$, this population is decreased by birth from infected mosquito (vertical transmission) at the rate $\beta M I_M$; and as its members take a blood meal from either symptomatic or asymptomatic infected human (horizontal transmission) at the rate $\alpha_M$. The
exposed mosquito population progresses to infected mosquito population at the rate $\sigma_2$. It is assumed that births from infected mosquito do not pass through the exposed class. All sub-populations of mosquito die naturally at the rate $\mu_2$.

**Table 1.** Parameters of the model Equations (1) to (9).

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Meaning</th>
<th>Value</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Lambda_s$</td>
<td>Recruitment rate of susceptible human</td>
<td>0.073</td>
<td>[19]</td>
</tr>
<tr>
<td>$\alpha_s$</td>
<td>Contact rate of susceptible human when bitten by Aides Mosquitoes</td>
<td>0.24</td>
<td>[19]</td>
</tr>
<tr>
<td>$\mu_h$</td>
<td>Natural death of human</td>
<td>0.000039</td>
<td>[20]</td>
</tr>
<tr>
<td>$\sigma_e$</td>
<td>Progression rate of exposed human to Symptomatic and Asymptomatic</td>
<td>0.33</td>
<td>[21]</td>
</tr>
<tr>
<td>$\delta$</td>
<td>Death rate of human due to virus infection</td>
<td>0.02</td>
<td>Assumed</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>Recovery rate of infectious human</td>
<td>0.68</td>
<td>[15]</td>
</tr>
<tr>
<td>$\Lambda_w$</td>
<td>Birth rate of Susceptible Aides Mosquitoes</td>
<td>83.75</td>
<td>[20] [22]</td>
</tr>
<tr>
<td>$\beta$</td>
<td>Proportion of infectious new birth from infected Aides Mosquitoes</td>
<td>0.00005</td>
<td>Assumed</td>
</tr>
<tr>
<td>$\alpha_a$</td>
<td>The rate at which susceptible Aides become infectious</td>
<td>0.24</td>
<td>[19]</td>
</tr>
<tr>
<td>$\sigma_a$</td>
<td>Progression rate of exposed Aedes</td>
<td>0.285</td>
<td>Assumed</td>
</tr>
<tr>
<td>$\mu_a$</td>
<td>Natural death rate of Aides</td>
<td>0.0714</td>
<td>[21] [23]</td>
</tr>
<tr>
<td>$\nu$</td>
<td>The rate at which susceptible human receive vaccine</td>
<td>Variable</td>
<td></td>
</tr>
<tr>
<td>$\psi$</td>
<td>The rate at which vaccine wane</td>
<td>Variable</td>
<td></td>
</tr>
<tr>
<td>$\varepsilon$</td>
<td>Vaccine efficacy where $0 &lt; \varepsilon &lt; 1$</td>
<td>Variable</td>
<td></td>
</tr>
<tr>
<td>$b_a$</td>
<td>Biting rate of mosquito</td>
<td>0.25</td>
<td>[19] [24]</td>
</tr>
</tbody>
</table>

**Figure 1.** Schematic diagram of Chikungunya virus transmission dynamics, Equations (1) to (9).
2.1. The Model Equation

From the model formulation, and schematic diagram Figure 1, we hereby present the model equations.

\[ \frac{dS_H}{dt} = \Lambda_H + \psi V_H - \frac{\alpha H S_H I_M}{N_H} (\nu + \mu_H) S_H, \]  \hspace{1cm} (1)

\[ \frac{dV_H}{dt} = \nu S_H - \frac{\alpha H (1 - \varepsilon) V_H I_M}{N_H} (\psi + \mu_H) V_H, \]  \hspace{1cm} (2)

\[ \frac{dE_H}{dt} = \frac{\alpha H b_M I_M}{N_H} \left(S_H + (1 - \varepsilon) V_H\right) - (\sigma_H + \mu_H) E_H, \]  \hspace{1cm} (3)

\[ \frac{dI_1}{dt} = \sigma_H E_H - (\gamma + \mu_H + \delta) I_1, \]  \hspace{1cm} (4)

\[ \frac{dI_2}{dt} = (1 - \sigma_H) E_H - (\gamma + \mu_H + \delta) I_2, \]  \hspace{1cm} (5)

\[ \frac{dR}{dt} = \gamma I_1 + \gamma I_2 - \mu_R, \]  \hspace{1cm} (6)

\[ \frac{dS_M}{dt} = \Lambda_M - \frac{\alpha M b_M S_M (I_1 + I_2)}{N_H} - \beta M M I_M - \mu_S M, \]  \hspace{1cm} (7)

\[ \frac{dE_M}{dt} = \frac{\alpha M b_M I_M (I_1 + I_2)}{N_H} + \beta M M I_M - (\sigma_M + \mu_M) E_M, \]  \hspace{1cm} (8)

\[ \frac{dI_M}{dt} = \sigma_M E_M - \mu_M I_M. \]  \hspace{1cm} (9)

Adding (1) to (6) gives

\[ \frac{dN_H}{dt} = \Lambda_H - \delta (I_1 + I_2) - \mu_H N_H. \]  \hspace{1cm} (10)

Also adding (7) to (9), gives

\[ \frac{dN_M}{dt} = \Lambda_M - \mu_M N_M. \]  \hspace{1cm} (11)

where

\[ N_H (t) = S_H (t) + V_H (t) + E_H (t) + I_1 (t) + I_2 (t) + R (t), \]  \hspace{1cm} (12)

\[ N_M (t) = S_M (t) + E_M (t) + I_M (t). \]  \hspace{1cm} (13)

(12) and (13) are the total human population and Aides mosquito population respectively.

2.2. Basic Properties

For the Chikungunya model (1) to (9) to be epidemiological meaningful, it is necessary to prove that all its state variables are non-negative for all time. This means that the solution of the model Equations (1) to (9) with non-negative initial data will remain non-negative for all time \( t > 0 \).

\textbf{Lemma 1.}
The closed set
\[
D = \left\{ \left( S_H, V_H, E_H, I_1, I_2, R, S_M, E_M, I_M \right) \in \mathbb{R}^9 : \right. \\
\left. S_H, V_H, E_H, I_1, I_2, R \leq \frac{\Lambda_H}{\mu_1} ; \quad S_M, E_M, I_M \leq \frac{\Lambda_M}{\mu_2} \right\} .
\] (14)

is positively-invariant and attracting with respect to the basic model Equations (1) to (9).

**Proof**

From Equations (10) and (11);
\[
\frac{dN_H}{dt} \leq \Lambda_H - \mu_1 N_H , \quad \frac{dN_M}{dt} \leq \Lambda_A - \mu_2 N_M .
\]

It follows that \( \frac{dN_H}{dt} < 0 \) and \( \frac{dN_M}{dt} < 0 \) if \( N_H (t) > \frac{\Lambda_H}{\mu_1} \) and \( N_A (t) > \frac{\Lambda_M}{\mu_2} \) respectively. Thus a standard comparison theorem as in Lakshmikantham and Martynyuk, [25] can be used to show that
\[
N_H (t) \leq N_H (0) e^{-\mu_1 t} + \frac{\Lambda_M}{\mu_1} \left( 1 - e^{-\mu_1 t} \right) \quad \text{and}
\]
\[
N_M (t) \leq N_M (0) e^{-\mu_2 t} + \frac{\Lambda_M}{\mu_2} \left( 1 - e^{-\mu_2 t} \right) .
\]

In particular \( N_H (t) \leq \frac{\Lambda_H}{\mu_1} \) and \( N_A (t) \leq \frac{\Lambda_A}{\mu_2} \) respectively. Thus \( D \) is positively-invariant. Further, if \( N_H (0) > \frac{\Lambda_H}{\mu_1} \), and \( N_M (0) > \frac{\Lambda_M}{\mu_2} \), then either the solution enters \( D \) in finite time or \( N_H (t) \) approaches \( \frac{\Lambda_H}{\mu_1} \), and \( N_M (t) \) approaches \( \frac{\Lambda_M}{\mu_2} \), and the infected variables \( E_H, I_1, I_2, E_A, I_3 \) approaches 0.

Hence \( D \) is attracting, that is all solutions in \( \mathbb{R}^9 \) eventually enters \( D \). Thus in \( D \), the basic model Equations (1) to (9) is well posed epidemiologically and mathematically according to [26]. Hence it is sufficient to study the dynamics of the basic model Equations (1) to (9).

**Lemma 2.** Let the initial data \( F(0) \geq 0 \), where
\[
F(t) = \left( S_H, V_H, E_H, I_1, I_2, R, S_M, E_M, I_M \right).
\]

Then the solution \( F(t) \) of the Chikungunya virus model (1) to (9) are non-negative for all \( t \geq 0 \). Furthermore form (10) and (11),
\[
\limsup_{t \to \infty} N_H (t) = \frac{\Lambda_H}{\mu_1 + \delta} \quad \text{and} \quad \limsup_{t \to \infty} N_M (t) = \frac{\Lambda_M}{\mu_2} .
\]

**Proof**

\( t_1 = \sup \{ t > 0 : F(t) > 0 \in [0, t] \} \). Thus \( t_1 > 0 \). It follows from (1) that
\[
\frac{d}{dt} \left[ S_H (t) \exp \left[ \alpha_i b_m \int_0^\xi \frac{I_M}{N_H} (\xi) d\xi + (\nu + \mu_i) t \right] \right] = \left( \Lambda_H + \psi V_H \right) \exp \left[ \alpha_i b_m \int_0^\xi \frac{I_M}{N_H} (\xi) d\xi + (\nu + \mu_i) t \right],
\]

So that,
\[
\frac{d}{dt} S_H (t_i) \exp \left[ \alpha_i b_m \int_0^\xi \frac{I_M}{N_H} (\xi) d\xi + (\nu + \mu_i) t_i \right] - S_H (0)
\]
\[
= \int_0^\xi \left( \Lambda_H + \psi V_H \right) \exp \left[ \alpha_i b_m \int_0^\xi \frac{I_M}{N_H} (\xi) d\xi + (\nu + \mu_i) t \right] dp > 0.
\]

Hence,
\[
S_H (t_i) = S_H (0) \exp \left[ -\alpha_i b_m \int_0^\xi \frac{I_M}{N_H} (\xi) d\xi + (\nu + \mu_i) t_i \right]
\]
\[
+ \exp \left[ -\alpha_i b_m \int_0^\xi \frac{I_M}{N_H} (\xi) d\xi + (\nu + \mu_i) t_i \right]
\]
\[
\int_0^\xi \left( \Lambda_H + \psi V_H \right) \exp \left[ \alpha_i b_m \int_0^\xi \frac{I_M}{N_H} (\xi) d\xi + (\nu + \mu_i) t \right] dp > 0.
\]

Similarly, it can be shown that \( F > 0 \), for all \( t > 0 \).

For the second part of the proof, note that,
\[
0 < V_H (t) \leq N_H (t), 0 < E_H (t) \leq N_H (t), 0 < I_t (t) \leq N_H (t),
\]
\[
0 < I_t (t) \leq N_H (t), 0 < R (t) \leq N_H (t), 0 < S_M (t) \leq N_M (t),
\]
\[
0 < E_M (t) \leq N_M (t), 0 < I_M (t) \leq N_M (t)
\]

From Equations (10) and (11),
\[
\frac{\Lambda_H}{\mu_1 + \delta} \leq \liminf_{t \to \infty} N_H (t) \leq \limsup_{t \to \infty} N_H (t) = \frac{\Lambda_H}{\mu_1 + \delta},
\]

and
\[
\frac{\Lambda_M}{\mu_2} \leq \liminf_{t \to \infty} N_M (t) \leq \limsup_{t \to \infty} N_M (t) = \frac{\Lambda_M}{\mu_2}.
\]

as required.

3. Results

3.1. Local Stability of Disease Free Equilibrium (DFE)

The basic model (1) to (9) has a DFE, \( E_0 \), obtained by setting the right-hand sides of the model equations to zero, which gives:
\[
E_0 = \left( S_H^*, V_H^*, E_H^*, I_H^*, S_M^*, E_M^*, I_M^* \right)
\]
\[
= \left( \frac{\Lambda_H (\psi + \mu_1)}{(\psi + \mu_1 + \nu) \mu_1}, \frac{\nu \Lambda_H}{(\psi + \mu_1 + \nu) \mu_1}, 0, 0, 0, 0, 0, \frac{\Lambda_M}{\mu_2}, 0, 0 \right).
\]

The linear stability of \( E_0 \) can be established using the next generation Matrix operator method on the system (1) to (9). Using the notation in [23], the matric-
es $F$ and $V$, for the new infection terms and the remaining transfer terms, are, respectively, given by,

$$F = \begin{bmatrix}
0 & 0 & 0 & 0 & \frac{\alpha h_m S^*_H (1 - \varepsilon) V^*_H}{N^*_H} \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & \frac{\alpha h_m S^*_H}{N^*_H} & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}, \quad (21)$$

and,

$$V = \begin{bmatrix}
K_1 & 0 & 0 & 0 & 0 \\
-\sigma_1 & K_4 & 0 & 0 & 0 \\
-(1 - \sigma_1) & 0 & K_4 & 0 & 0 \\
0 & 0 & 0 & K_5 & 0 \\
0 & 0 & 0 & -\sigma_2 & K_6
\end{bmatrix}, \quad (22)$$

where,

$$K_1 = \gamma + \mu, K_2 = \psi + \mu, K_3 = \sigma_1 + \mu,$$

$$K_4 = \gamma + \mu + \delta, K_5 = \sigma_2 + \mu, K_6 = \mu - \beta M$$

$$R_s = \frac{1}{2} \left( \frac{M_1 + \sqrt{M_2 + M_3}}{M_4} \right), \quad (24)$$

$$M_1 = \Lambda M \beta K_1 K_2 K_3 N^*_H,$$  \quad (25)

$$M_2 = \Lambda M \beta K_1 K_3 K_4 N^*_H,$$  \quad (26)

$$M_3 = 4 K_2 K_3 K_4 \sigma_2 \alpha h_m S^*_H \left( \alpha h_m S^*_H + (1 - \varepsilon) \right) N^*_H,$$  \quad (27)

$$M_4 = \beta K_1 K_3 K_5 K_6 N^*_H.$$  \quad (28)

Hence using theorem 2 of [23] the following results are established.

**Theorem 1** The disease free equilibrium, $E_0$, of the model (2.1) to (2.9) is locally asymptotically stable (LAS) if $R_s < 1$, and unstable if $R_s > 1$.

### 3.2. Global Stability of Disease Free Equilibrium

Consider the feasible region:

$$D_1 = \{ X \in D_1: S_H \leq S^*_H, V_H \leq V^*_H, R \leq R^*, S_M \leq S^*_M \},$$ \quad (29)

$$X = \{ S_H, V_H, E_H, I_1, I_2, R, S_M, E_M, I_M \}.$$ \quad (30)

**Lemma 3.** The region $D_1$ is positively invariant for the Chikungunya model
Proof

From Equations (1) to (9) and (20), we have that, the only non-zero compartments at disease free equilibrium are:

\[
\begin{align*}
\frac{dS_H}{dt} &= \Lambda_H + \psi V_H - \frac{\alpha b_H S_H I_M}{N_H} - (v + \mu_{1}) S_H, \\
\frac{dV_H}{dt} &= v S_H - \frac{\alpha b_H (1 - \varepsilon)V_H I_M}{N_H} - (\psi + \mu_{1}) V_H, \\
\frac{dS_M}{dt} &= \Lambda_M - \frac{\alpha b_M S_M (I_1 + I_2)}{N_H} - \beta \Lambda_M I_M - \mu_{2} S_M
\end{align*}
\]

(31)

Such that,

\[
\begin{align*}
\frac{dS_H}{dt} &= \Lambda_H + \psi V_H - \frac{\alpha b_H S_H I_M}{N_H} - (v + \mu_{1}) S_H, \\
&\leq \Lambda_H + \psi V_H - (v + \mu_{1}) S_H \\
&\leq (v + \mu_{1}) \left[ \frac{\Lambda_H (\psi + \mu_{1})}{(\psi + \mu_{1} + v) \mu_{1}} + \psi \frac{\nu \Lambda_H}{(\psi + \mu_{1} + v) \mu_{1}} - S_H \right] \\
&= (v + \mu_{1}) (S_H^* + \psi V_H^* - S_H)
\end{align*}
\]

(32)

Hence,

\[
S_H(t) \leq S_H^* + \psi V_H^* - \left[ S_H^* - \psi V_H^* - S_H(0) \right] e^{(v+\mu_{1})t}.
\]

(33)

Thus if \( N_H^* = \frac{\Lambda_H}{\mu_{1}} \) and \( S_H(0) < S_H^* + \psi V_H^* \) for all \( t \geq 0 \), then \( S_H(t) \leq S_H^* + \psi V_H^* \) for all \( t \geq 0 \).

Similarly, it follows from Equation (7) of our model and (20) where \( S_M^* = \frac{\Lambda_M}{\mu_{2}} \).

We have that,

\[
\begin{align*}
\frac{dS_M}{dt} &= \Lambda_M - \frac{\alpha b_M S_M (I_1 + I_2)}{N_H} - \beta \Lambda_M I_M - \mu_{2} S_M \\
&\leq \Lambda_M - \mu_{2} S_M \leq \mu_{2} \left[ \frac{\Lambda_M}{\mu_{2}} - S_M \right] = \mu_{2} (S_M^* - S_M)
\end{align*}
\]

(34)

Hence,

\[
S_M(t) \leq S_M^* - \left[ S_M^* - S_M(0) \right] e^{-\mu_{2}t}.
\]

(35)

Thus if \( N_M^* = \frac{\Lambda_M}{\mu_{2}} \) and \( S_M(0) < S_M^* \) for all \( t \geq 0 \), then \( S_M(t) \leq S_M^* \) for all \( t \geq 0 \).

In summary, we have shown that \( D_1 \) is positively invariant and attracting with respect to the solutions of our model Equations (1) to (9).

Theorem 2

The DFE of the basic model (1) to (9) is Global Asymptotical Stability (GAS) in \( D_1 \), whenever \( R_c \leq 1 \).
Proof

To prove the GAS of the DFE we adopt the approach in [27]. Let \( X = (S_H, V_H, R, S_M) \) and \( Z = (E_H, I_1, I_2, E_M, I_M) \) and group our model Equations (1) to (8) into:

\[
\frac{dX}{dt} = F(X, 0), \\
\frac{dZ}{dt} = G(X, Z).
\]

where \( F(X, 0) \) is the right hand side of \( \dot{S}_H, \dot{V}_H, \dot{R}, \dot{S}_M \) with \( E_H = I_1 = I_2 = E_M = I_M = 0 \) and \( G(X, Z) \), the right hand side of \( E_H, \dot{I}_1, \dot{I}_2, \dot{E}_M, \dot{I}_M \). Next we consider the reduced system:

\[
\frac{dX}{dt} = F(X, 0) \text{ given as,}
\]

\[
\frac{dS_H}{dt} = \Lambda - \mu_i S_H, \\
\frac{dV_H}{dt} = \nu S_H - (\psi + \mu_i) V_H, \\
\frac{dR}{dt} = -\mu_i R, \\
\frac{dS_M}{dt} = \Lambda_M - \mu_i S_M.
\]

Let \( X^* = (S_H^*, V_H^*, R^*, S_M^*) \) be an equilibrium of (37) we show that \( X^* \) is a global stable equilibrium in \( D_1 \).

To do this, we solve the Equations (37), which gives

\[
S_H(t) = \frac{\Lambda}{\nu + \mu_i} + V_H^* - \frac{\Lambda}{\nu + \mu_i} + (\psi + \mu_i) V_H^* e^{-(\psi + \mu_i) \mu_i t},
\]

\[
S_H(t) \to \frac{\Lambda}{\nu + \mu_i} + V_H^*,
\]

as \( t \to \infty \).

\[
V_H(t) = \frac{\nu \Lambda}{\nu + \mu_i} - \frac{\nu \Lambda}{\nu + \mu_i} e^{-(\psi + \mu_i) \mu_i t} + V_H(0) e^{-(\psi + \mu_i) \mu_i t},
\]

\[
V_H(t) \to \frac{\nu \Lambda}{\nu + \mu_i},
\]

as \( t \to \infty \).

\[
R(t) = R(0) e^{-\rho t}, R(t) \to 0,
\]

as \( t \to \infty \).

\[
S_M(t) = \frac{\Lambda}{\mu_i} - \frac{\Lambda}{\mu_i} e^{-\rho t} + S_M(0) e^{-\rho t}, S_M(t) \to \frac{\Lambda_M}{\mu_i},
\]

as \( t \to \infty \).
This asymptotic dynamics is independent of initial conditions in $D$. Hence the solution of $xxx$ converges globally in $D_1$.

Next we are required to show that $G(X,Z)$ satisfies the following two conditions in [19] pp246 namely;

$$G(X,0) = 0,$$

$$G(X,Z) = D_2\hat{G}(X^*,0)Z - \hat{G}(X,Z), \hat{G}(X,Z) \geq 0,$$  (43)

where,

$$(X^*,0) = \left(\frac{\Lambda_H (\psi + \mu_1)}{(\psi + \mu_1 + \nu) \mu_i}, \frac{\nu \Lambda_H}{(\psi + \mu_1 + \nu) \mu_i}, 0, \frac{\Lambda_M}{\mu_2}\right).$$  (44)

and $D_2G(X^*,0)$ is the Jacobian of $G(X,Z)$ taken with respect to $(E_H, I_1, I_2, E_M, I_M)$ and evaluated at $(X^*,0)$, which is an M-Matrix (the off diagonal elements are non-negative).

Thus,

$$D_2G(X^*,0) = \begin{bmatrix} -k_5 & 0 & 0 & 0 & Q_1 \\ \sigma_1 & -k_4 & 0 & 0 & 0 \\ 1-\sigma_1 & 0 & -k_4 & 0 & 0 \\ 0 & \frac{\alpha_2 b_m S_M^*}{N_H^*} & \frac{\alpha_2 b_m S_M^*}{N_H^*} & -k_5 & 0 \\ 0 & 0 & 0 & \sigma_2 & -k_6 \end{bmatrix},$$  (45)

$$\hat{G}(X,Z) = \begin{bmatrix} 0 & 0 & 0 & 0 & Q_2 I_M \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\beta \Lambda_M} \end{bmatrix},$$  (46)

where,

$$Q_1 = \frac{\alpha_1 b_m S_H^* + (1-\varepsilon) + V_H^*}{N_H^*},$$

$$Q_2 = \frac{N_H^*}{S_H^* + (1-\varepsilon) + V_H^*} \left(1 - \frac{N_H^*}{S_H^* + (1-\varepsilon) + V_H^*} \frac{S_H^* + (1-\varepsilon) + V_H^*}{N_H^*}\right),$$

$$Q_3 = \frac{1 - N_H^*}{S_M^* N_H^*} I_1,$$

$$Q_4 = \frac{1 - N_H^*}{S_M^* N_H^*} I_2.$$

Further $S_H \leq S_H^*$, $V_H \leq V_H^*$ and $S_M \leq S_M^*$ in $D_1$. Thus, it follows that
\( \left(1 - \frac{S_H}{S_W}\right) > 0, \left(1 - \frac{V_H}{V_W}\right) > 0 \) and \( \left(1 - \frac{S_M}{S_M}\right) > 0 \). Hence \( \hat{G}(X,Z) \geq 0 \).

Therefore, by the theorem 2 in [28], the disease-free equilibrium is globally asymptotically stable since in the absence of disease induced mortality the human population is constant.

### 3.3. Sensitivity Analysis

Here we present the sensitivity index of the parameters of the effective reproductive number \( (R_c) \). Sensitivity tells us how important each parameter is to disease transmission. Such information, is crucial not only to experimental design, but also to data assimilation and reduction of complex nonlinear model [29]. Sensitivity Analysis is commonly used to determine the robustness of model prediction to parameter values, since there are usually errors in data collection and presumed parameter values. It is used to determine parameters that have high impact on the \( (R_c) \) and should be targeted by intervention strategies. Sensitivity indexes allows us to measure the relative changes in a variable when a parameter changes. The normalized forward sensitivity index of a variable with respect to a parameter is the ratio of relative changes in the parameter when the variable is a differentiable function of the parameter. The sensitivity index may be alternatively defined using partial derivatives. The sensitivity index of our model is given in Table 2.

**Table 2.** Sensitivity analysis index for the effective basic reproductive number.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Sensitivity index</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha_i )</td>
<td>0.06</td>
</tr>
<tr>
<td>( \alpha_i )</td>
<td>0.37</td>
</tr>
<tr>
<td>( \mu_i )</td>
<td>0.63</td>
</tr>
<tr>
<td>( \mu_i )</td>
<td>0.87</td>
</tr>
<tr>
<td>( \sigma_i )</td>
<td>0.02</td>
</tr>
<tr>
<td>( \sigma_i )</td>
<td>0.87</td>
</tr>
<tr>
<td>( \beta )</td>
<td>1.25</td>
</tr>
<tr>
<td>( \psi )</td>
<td>-0.12</td>
</tr>
<tr>
<td>( \varepsilon )</td>
<td>-0.86</td>
</tr>
<tr>
<td>( \delta )</td>
<td>0.13</td>
</tr>
<tr>
<td>( \gamma )</td>
<td>0.5</td>
</tr>
<tr>
<td>( \Lambda_x )</td>
<td>0.62</td>
</tr>
<tr>
<td>( \Lambda_x )</td>
<td>1.6</td>
</tr>
<tr>
<td>( \nu )</td>
<td>0.46</td>
</tr>
</tbody>
</table>
From Table 2, the most sensitive parameter of \( R_c \) is the recruitment rate of susceptible mosquito (\( \Lambda_M \)) followed by the proportion of infectious new birth from infected mosquito (\( \beta \)) while the natural birth rate of mosquito (\( \mu \)) and the rate at which exposed mosquito become infectious (\( \sigma_2 \)) are equally sensitive to the \( R_c \) according to the model. This means that any policy or practice capable of reducing these parameters will go a long way in reducing the menace of Chikungunya and at the long run, result to eradication.

**Endemic Equilibrium**

Let \( E_i = \left( S_i^*, V_i^*, E_i^*, R_i^*, I_1^*, I_2^*, R^*, S_M^*, E_M^*, I_M^* \right) \),

represents any arbitrary endemic equilibrium of the model (1) to (9). Further, let

\[
\lambda_i^* = \frac{\alpha_i b_i I_i^*}{N_i^*}, \lambda_M^* = \frac{\alpha_i b_i (I_i^* + I_M^*)}{N_i^*}.
\]

be the forces of infection of humans and vectors at steady state, respectively. Solving (1) to (9) in terms of \( \lambda_i^* \) and \( \lambda_M^* \), we have;

\[
S_i^* = \frac{\Lambda_i (\lambda_i^* + k_i)}{(\lambda_i^* + k_i) \left( \lambda_i^* + k_4 \right) + \psi \nu}, \quad V_i^* = \frac{\nu \lambda_i}{(\lambda_i^* + k_2) \left( \lambda_i^* + k_4 \right) + \psi \nu},
\]

\[
E_i^* = \frac{\lambda_i^* (\lambda_i^* + k_i) + (1-\epsilon) \psi \nu}{(\lambda_i^* + k_2) \left( \lambda_i^* + k_4 \right) + \psi \nu} k_5, \quad I_i^* = \frac{\psi \lambda_i \lambda_i^* \left( \lambda_i^* + k_2 \right) \left( \lambda_i^* + k_4 \right) + (1-\epsilon) \psi \nu}{(\lambda_i^* + k_2) \left( \lambda_i^* + k_4 \right) + \psi \nu} k_4 k_5,
\]

\[
I_2^* = \frac{(1-\sigma_i) \Lambda_i \lambda_i^* \left( \lambda_i^* + k_2 \right) \left( \lambda_i^* + k_4 \right) + (1-\epsilon) \psi \nu}{(\lambda_i^* + k_2) \left( \lambda_i^* + k_4 \right) + \psi \nu} k_4 k_5, \quad R^* = \frac{\psi \lambda_i \lambda_i^* \left( \lambda_i^* + k_2 \right) \left( \lambda_i^* + k_4 \right) + (1-\epsilon) \psi \nu}{(\lambda_i^* + k_2) \left( \lambda_i^* + k_4 \right) + \psi \nu} k_4 k_5 \mu_i.
\]

\[
S_M^* = \frac{k_i k_s \Lambda_i}{\lambda_M^* (k_i k_s + \beta \Lambda_i \sigma_2) + \mu_i k_i k_s}, \quad E_M^* = \frac{\psi \lambda_i \lambda_M^*}{k_i k_s \Lambda_i}, \quad R^* = \frac{\psi \lambda_i \lambda_M^*}{k_i k_s \Lambda_i} + \frac{\mu_i k_i k_s}{k_i k_s \Lambda_i}.
\]

Substituting (20) into (19) we have;

\[
\lambda_i^* = \frac{\alpha_i b_i \Lambda_i \lambda_i^* \left( \lambda_i^* + k_2 \right) + (1-\epsilon) \psi \nu}{(\lambda_i^* + k_2) \left( \lambda_i^* + k_4 \right) + \psi \nu} k_4 k_5,
\]

\[
\lambda_i^* = A \left( \lambda_i^* \right)^2 + B \left( \lambda_i^* \right)^3 + C \left( \lambda_i^* \right)^4 + D \left( \lambda_i^* \right)^5 - E.
\]

where,

\[
A = \left( \alpha_i b_i \Lambda_i k_i k_s + \beta \Lambda_i \sigma_2 \right) \left( \mu_i k_i k_s \right),
\]

\[
B = \left( T_i k_i k_s \sigma_2 \right) \left( k_i k_s \left( \mu_i k_i k_s + \psi \nu \right) \right) - \left( \alpha_i \lambda_i^* \left( b_i \right)^2 \Lambda_i \sigma_2 \right) k_i k_s \mu_i,
\]

\[
C = \left[ \alpha_i b_i \Lambda_i \sigma_2 \left( k_i (1-\epsilon) \nu \right) k_i k_s \\
+ \beta \Lambda_i \sigma_2 \left( k_i k_s \mu_i k_i k_s + k_i k_s \mu_i + k_i k_s + \psi \nu \right) \left( 1-\nu \right) \nu \right] \left( \alpha_i \lambda_i^* \left( b_i \right)^2 \Lambda_i \sigma_2 \right) k_i k_s \mu_i
\]

\[
+ \left( \psi \lambda_i \lambda_i^* \right) \Lambda_i \sigma_2 \left( k_i (1-\epsilon) \nu \right) k_i k_s \mu_i.
\]
$D = \left( \alpha_2 b_m \Lambda_2 \sigma_2 \left( k_2 + (1-\varepsilon)\nu \right) \left( \alpha_2 b_m \Lambda_2 k_2 k_6 + \beta \Lambda_2 \sigma_2 \left( k_2 \left( \mu_k + \nu \right) \right) \right) \right)\left( k_2 \left( \mu_k + \nu \right) \right)$

$$- \left( \alpha_2 \sigma_2 \left( b_m \right) \Lambda_2 \sigma_2 \left( k_2 + (1-\varepsilon)\nu \right) \left( k_2 + k_2 \right) k_2 \mu_k, \right)$$

$$+ \left( k_2 k_2 k_4 - \nu \nu k_2 k_4 \mu_k \right) \left( \alpha_2 \sigma_2 \left( b_m \right) \Lambda_2 \sigma_2 \right),$$

$$E = \alpha_2 \sigma_2 \left( b_m \right) \Lambda_2 \sigma_2 \left( k_2 + (1-\varepsilon)\nu \right) k_2 k_2 k_4 - \nu \nu k_2 k_4 \mu_k.$$ (56)

**Theorem 3.6.** The Chikungunya basic model (1) to (9) undergoes backward bifurcation whenever the coefficient $a$ in equation is positive.

**Proof.** To prove this theorem, we use the Centre Manifold theory as in Castillo-Chavez and songs [30] [31] see the theorem in **Appendix A**.

Let $S_{H} = x_1$, $V_{H} = x_2$, $E_{H} = x_3$, $I_{1} = x_4$, $I_{2} = x_5$, $R = x_6$, $S_{M} = x_7$, $E_{M} = x_8$, $I_{M} = x_9$ so that $N_{H} = x_1 + x_2 + x_3 + x_4 + x_5 + x_6$ and $N_{M} = x_7 + x_8 + x_9$. Further by using vector notation $X = (x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9)^T$. Equations (1) to (9) can be written as $\frac{dX}{dt} = \left( f_1 + f_2 + f_3 + f_4 + f_5 + f_6 + f_7 + f_8 + f_9 \right)^T$ as follow:

$$\frac{dx_1}{dt} = \Lambda_2 V_2 - \frac{\alpha_2 b_m x_2}{x_1 + x_2 + x_3 + x_4 + x_5 + x_6} - k_1 x_1,$$

$$\frac{dx_2}{dt} = \nu x_2 - \frac{\alpha_2 b_m \left( 1-\varepsilon \right) x_2}{x_1 + x_2 + x_3 + x_4 + x_5 + x_6} - k_2 x_2,$$

$$\frac{dx_3}{dt} = \frac{\alpha_2 b_m x_2}{x_1 + x_2 + x_3 + x_4 + x_5 + x_6} \left( x_1 + \left( 1-\varepsilon \right) x_2 \right) - k_3 x_3,$$

$$\frac{dx_4}{dt} = \sigma_4 x_3 - k_4 x_4,$$

$$\frac{dx_5}{dt} = \left( 1-\sigma_4 \right) x_5 - (\gamma + \mu_1 + \delta) x_5,$$

$$\frac{dx_6}{dt} = \gamma x_4 + \gamma x_5 - \mu_1 x_6,$$

$$\frac{dx_7}{dt} = \frac{\alpha_2 b_m x_5}{x_1 + x_2 + x_3 + x_4 + x_5 + x_6} - \beta \Lambda_2 x_9 - \mu_2 x_7,$$

$$\frac{dx_8}{dt} = \frac{\alpha_2 b_m x_5}{x_1 + x_2 + x_3 + x_4 + x_5 + x_6} + \beta \Lambda_2 x_9 - k_3 x_8,$$

$$\frac{dx_9}{dt} = \sigma_3 x_8 - k_6 x_9.$$  (57)

Because it’s not always convenient to use $R_0 = 1$ as bifurcation parameter, we choose $P = P^*$ where $P^* = \alpha_2 b_m$ as the bifurcation parameter such that,

$$P^* = \frac{1}{2} \left( \frac{M_4}{M_1 + \sqrt{M_2 + M_5}} \right),$$ (58)

where

$$M_5 = 4k_2 k_2 k_4 \sigma_2 x_1^* \left( \alpha_2 b_m x_1^* + (1-\varepsilon) x_2^* \right).$$ (59)
The Jacobian of (57) evaluated at $E_0$ with $\alpha \beta = P$, denoted by $J^*$ is given
\[
J^* = \begin{bmatrix}
-k_i & 0 & 0 & 0 & 0 & 0 & 0 & -Q_3 \\
\nu & -k_2 & 0 & 0 & 0 & 0 & 0 & -Q_6 \\
0 & 0 & -k_3 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \sigma_1 & -k_4 & 0 & 0 & 0 & 0 \\
0 & 0 & 1-\sigma_1 & 0 & -k_4 & 0 & 0 & 0 \\
0 & 0 & 0 & \gamma & \gamma & -\mu_2 & 0 & 0 \\
0 & 0 & 0 & Q_7 & 0 & \mu_2 & 0 & -\beta \Lambda_M \\
0 & 0 & 0 & Q_7 & 0 & 0 & -k_5 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \sigma_2 & -k_6
\end{bmatrix}
\] (60)

where,
\[
Q_3 = \frac{\alpha \beta x_i^* x_i}{x_1 + x_2 + x_3 + x_4 + x_5 + x_6},
\]
\[
Q_6 = \frac{\alpha \beta (1-\epsilon) x_i^*}{x_1 + x_2 + x_3 + x_4 + x_5 + x_6},
\]
\[
Q_7 = \frac{\alpha \beta x_i^* x_j}{x_1 + x_2 + x_3 + x_4 + x_5 + x_6}.
\] (61)

It follows that (60) has a right eigenvector denoted by
\[
v = v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10},\]
where
\[
v_1 = \frac{-Q_i (k_i k_2 + \nu \psi) + \psi (Q_i k_i + \nu Q_i) v_9}{(k_i k_2 - \nu \psi) k_i},
\]
\[
v_2 = \frac{-(Q_i k_i + \nu Q_i) v_9}{k_i k_2 - \nu \psi},
\]
\[
v_3 = \frac{Q_1 v_9}{k_3},
\]
\[
v_4 = \frac{\sigma Q_1 v_9}{k_3 k_4},
\]
\[
v_5 = \frac{(1-\sigma_1) Q_1 v_9}{k_3 k_4},
\]
\[
v_6 = \frac{\gamma Q_i k_i (Q_i + (1-\sigma_1)) v_9}{k_3 k_4},
\]
\[
v_7 = \frac{(\beta \Lambda_M k_2^2 k_3 + Q_i Q_i k_i) v_9}{k_2 k_2 \mu_2},
\]
\[
v_8 = \frac{Q_i k_i (\sigma_1 + (1-\sigma_1)) v_9}{k_3 k_4 k_5},
\]
\[
v_9 = v_9.
\] (62)

And a left eigenvector given by
\[
w = w_1, w_2, w_3, w_4, w_5, w_6, w_7, w_8, w_9, w_{10},\]
where
\[ w_1 = \frac{\nu w_4}{k_1}, \]
\[ w_2 = w_2, \]
\[ w_3 = \frac{Q_1 w_2 + \gamma w_6}{k_j k_4}, \]
\[ w_5 = \frac{\gamma w_6 - Q_3 w_7}{k_j k_4}, \]
\[ w_6 = w_6, \]
\[ w_7 = w_7, \]
\[ w_8 = w_8 = 0. \]  
(63)

Computation of \( a \)

\[ \frac{\partial^2 f_1}{\partial x_1 \partial x_2} = \frac{\alpha_1 b_m x_1^*}{x_1 + x_2}, \]
\[ \frac{\partial^2 f_1}{\partial x_1 \partial x_3} = \frac{\partial^2 f_1}{\partial x_2 \partial x_3} = \frac{\partial^2 f_1}{\partial x_4 \partial x_5} = \frac{\partial^2 f_1}{\partial x_6 \partial x_7} = \frac{\partial^2 f_1}{\partial x_1 \partial x_2} = \frac{\alpha_1 b_m x_1^*}{x_1 + x_2}, \]
\[ \frac{\partial^2 f_1}{\partial x_4 \partial P} = \frac{\partial^2 f_1}{\partial x_5 \partial P} = \frac{\partial^2 f_1}{\partial x_6 \partial P} = \frac{\partial^2 f_1}{\partial x_7 \partial P} = \frac{\partial^2 f_1}{\partial x_1 \partial x_2} = \frac{\alpha_1 b_m x_1^*}{x_1 + x_2}, \]
\[ \frac{\partial^2 f_1}{\partial x_4 \partial x_5} = \frac{\partial^2 f_1}{\partial x_4 \partial x_6} = \frac{\partial^2 f_1}{\partial x_4 \partial x_7} = \frac{\partial^2 f_1}{\partial x_5 \partial x_6} = \frac{\partial^2 f_1}{\partial x_5 \partial x_7} = \frac{\partial^2 f_1}{\partial x_6 \partial x_7} = \frac{\partial^2 f_1}{\partial x_1 \partial x_2} = \frac{\alpha_1 b_m x_1^*}{x_1 + x_2}. \]

\[ a = \sum_{k, l, j=1}^{n} v_k w_j w_i \frac{\partial^2 f_i}{\partial x_i \partial x_j} (0, 0) \]
\[ = v_k w_j \frac{\alpha_1 b_m x_1^*}{x_1 + x_2} (w_1 + w_2 + w_3 + w_4 + w_5 + w_6) \]
\[ - v_k w_j \frac{\alpha_1 b_m x_1^*}{x_1 + x_2} \left( k_1 + \psi x_2^* + \frac{2 \psi x_1^* + k_3 x_1^*}{x_1^* + x_2^*} \right) \]
\[ - v_k w_j \left( w_4 + w_5 \right) \left( \frac{\mu_2}{x_1^* + x_2^*} \right) \]
\[ - v_k w_j \left( w_4 + w_5 \right) \left( \frac{\mu_2}{x_1^* + x_2^*} \right) \]
\[ = \sum_{k, l, j=1}^{n} v_k w_j \frac{\partial^2 f_i}{\partial x_i \partial P} = v_7 \left( w_4 + w_5 + w_7 \right) \left( \frac{x_7^*}{x_1^* + x_2^*} \right). \]

\[ b = \sum_{k, l, j=1}^{n} v_k w_j \frac{\partial^2 f_i}{\partial x_i \partial P} = v_7 \left( w_4 + w_5 + w_7 \right) \left( \frac{x_7^*}{x_1^* + x_2^*} \right). \]

\[ \text{3.4. Vaccine Impact Analysis} \]

Vaccine was believed to confer life-long immunity until 1990s. This was the norm as it was approximately correct for most available vaccine for infectious children diseases. But most vaccines used for combating adult infectious diseases today are defective and thus immunity conferred on the recipients wane
with time. It is expected that the future Chikv vaccine will also be defective and hence the need to assess its effectiveness in $R_C$ a community. In this paper, the vaccine impact analysis is done by differentiating effective reproductive number with respect to the proportion $p$ of susceptible individuals vaccinated at equilibrium, according to [32], 
\[ p = \frac{V_p}{N_H} \]  
\[ \frac{\partial R_C}{\partial p} = \frac{R_C e}{1(1 - p e)}, \]  
i.e. since $0 < e < 1$ we have that $\frac{\partial R_C}{\partial p} < 0$, hence $R_C$ is a decreasing function of $p$. This means that a vaccination program with $p > 0$ and $e > 0$ at equilibrium, the future vaccine will have a positive impact. Besides, there exist a $p_C$ such that $R_C(p_C) = 1$ given by $\frac{1}{e}\left(1 - \frac{1}{R_C}\right)$ and for vaccination of proportion of susceptible $p > p_C$ the number of new-cases reduces to zero faster than when $p < p_C$.

4. Numerical Simulation

To further verify the analytical results in the model, the ode 45 code embedded in matlab was used to simulate some parameters of the model. Table 1 provided values of the parameters while initial values of the state variables were chosen arbitrarily. Figures 2(A)-(D) and Figures 3(A)-(D) are simulation of the various model compartments with time. Figure 4 is the simulation of some compartments

![Figure 2](image_url)

**Figure 2.** Plot of the various populations with parameters as in Table 1. (A) is the simulation of susceptible human against time, the plot shows that the susceptible human decreases with time due to the proportion that gets infected but slows down after some days, perhaps due to the vaccination and other control measures. (B) is the simulation of the vaccinated compartment. The plot shows a steady increase initially, but began to slope down after few days, this could be due to the fact that a proportion of the class are infectious as the vaccine is imperfect. (C) is the simulation of the exposed compartment with time, the plot shows a steady decline as members become infectious and progress to either the symptomatic or asymptomatic compartment. Finally (D) is the simulation of the symptomatic compartment with time. The plot shows a steady decline and tends to zero after about 20 days. This could be attributed to recovery from the infection.
Figure 3. Plot of the various populations with parameters as in Table 1. (A) is the simulation of the asymptomatic infected compartment with time, it shows a sharp increase at the onset of the epidemic, followed by a decline. (B) is the simulation of the recovered compartment with time, it shows a steady increase at the initial time, got to a peak and then remains a constant as time progresses. (C) is the simulation of susceptible mosquito compartment with time. It maintains a steady increase until perhaps due to short life cycle. (D) is the simulation of exposed mosquito compartment with time. The plot shows a steady decline with time as proportion progresses to infected compartment.

Figure 4. Simulation of the Human populations with varying values of $\beta$. (A) is the effect of the vertical transmission ($\beta$) on the susceptible compartment, while (B), (C) and (D) is the effect on same on the vaccinated, exposed, and symptomatic infected human compartment respectively. It is obvious from the plots that $\beta$ has negligible effect in all the compartments and hence on the transmission of Chikungunya virus according to the model analysis and simulation.
with various values of the vertical transmission rate \( \beta \). Figure 5 is a contour plot of the effective basic reproduction number as a function of recruitment rate of susceptible mosquito \( \Lambda_M \) and vertical transmission rate \( \beta \) while Figure 6 is the contour plot of effective basic reproductive number with varying values of vaccine efficacy \( \varepsilon \) and vaccinated proportion. Finally, Figure 7 is a simulation of the new cases of Chikungunya with different values of vaccine efficacy \( \varepsilon \) and vaccination rate \( \nu \). The figures and detailed caption are presented below.

![Simulation of the chikv model displaying a contour graph of \( R_C \) as a function of recruitment rate of susceptible mosquito; and recruitment rate of infected mosquito \( \beta \) with parameter values as listed in Table 1.](image)

**Figure 5.** Simulation of the chikv model displaying a contour graph of \( R_C \) as a function of recruitment rate of susceptible mosquito; and recruitment rate of infected mosquito \( \beta \) with parameter values as listed in Table 1.

![Simulation of the chikv model displaying a contour graph of \( R_C \) as a function of vaccinated human population and vaccine efficacy \( \varepsilon \); with parameter values as listed in Table 1.](image)

**Figure 6.** Simulation of the chikv model displaying a contour graph of \( R_C \) as a function of vaccinated human population and vaccine efficacy \( \varepsilon \); with parameter values as listed in Table 1.
5. Conclusion

A deterministic mathematical model for Chikungunya virus dynamics was developed using the standard incidence approach. The model assumed that the offspring of infected mosquito is infected at birth (vertical transmission) and also through blood meal from symptomatically and as-symptomatically infected human (horizontal transmission). For the subhuman population, only horizontal transmission was considered and the virus infection in human is assumed fatal, though with a very low rate. The disease free and endemic equilibrium was obtained and analyzed for both local and global asymptotically stability. The analysis shows that the model undergoes backward bifurcation when the effective basic reproductive number $R_e \leq 1$. Numerical simulation of the model shows that the effect of vertical transmission of the mosquito sub-population in the dynamics of the virus is negligible, even when the rate is high as shown in Figures 4(A)-(D). Further, the contour plot of the effective basic reproductive number $R_e$ with respect to the vaccine efficacy $\varepsilon$ and the proportion of susceptible vaccinated (Figure 6) gave the rates at which the $R_e$ is above, below and equal to unity, this confirms that the use of imperfect vaccine will be effective. Figure 6 also reveals a linear relationship between the effective basic reproductive number and the two parameters in question unlike Figure 5. Also the graph of Chikungunya new case (Figure 7) shows a decrease in new cases with high vaccine efficacy $\varepsilon$ and proportion of vaccinated susceptible $\nu$. Hence buttressing the point made in Figure 6.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.
References


search, 3, 302–312.


Appendix A

**Castilo-Chaevz and Song [3]**

Consider the following general system of ordinary differential equations with a parameter $\phi$.

$$\frac{dx}{dt} = f(x, \phi): R^n \times R \rightarrow R^n \quad \text{and} \quad f \in C^2 \left( R^n \times R \right)$$

where $0$ is an equilibrium point of the system (that is, $f(0, \phi) = 0$ for all $\phi$) and

(A1) $A = D_x f(0, 0) = \left( \frac{\partial f_i}{\partial x_j} (0, 0) \right)$ is the linearization matrix of the system 2.10 around the equilibrium 0 with $\phi$ evaluated at 0;

(A2) Zero is a simple eigenvalues of $A$ and other eigenvalues of $A$ have negative real parts;

(A3) Matrix $A$ has a right eigenvector $w$ and left eigenvector $v$ (each corresponding to zero eigenvalues).

Let $f_k$ be the $k$th component of $f$ and

To do this we need the values of $a$ and $b$ given below:

$$a = \sum_{k,i=1}^{n} v_i w_j \frac{\partial^2 f_k}{\partial x_i \partial x_j} (0, 0)$$

$$b = \sum_{k,i=1}^{n} v_i w_j \frac{\partial^2 f_k}{\partial x_i \partial x_j} (0, 0)$$

then, the local dynamics of the system around equilibrium point 0 is totally determined by the signs of $a$ and $b$, particularly,

1) $a > 0, b > 0$, when $\phi < 0$ with $|\phi| < 1$, 0 is locally asymptotically stable and there exists a positive unstable equilibrium; when $0 \phi < 1$, 0 is unstable and there exists a negative, locally asymptotically stable equilibrium;

2) $a < 0, b < 0$, when $\phi < 0$ with $|\phi| < 1$, 0 is unstable; when $0 < \phi < 1$, 0 is locally asymptotically stable equilibrium and there exists a positive unstable equilibrium;

3) $a < 0, b > 0$, when $\phi$ changes from negative to positive, 0 changes its stability from stable to unstable. Correspondingly a negative unstable equilibrium becomes positive and locally asymptotically stable.
New Analytical Study of the Effects Thermo-Diffusion, Diffusion-Thermo and Chemical Reaction of Viscous Fluid on Magneto Hydrodynamics Flow in Divergent and Convergent Channels

Abdul-Sattar J. A. Al-Saif¹, Abeer Majeed Jasim²

¹Department of Mathematics, College of Education for Pure Science, University of Basrah, Basrah, Iraq
²Department of Mathematics, College of Science, University of Basrah, Basrah, Iraq

Abstract

In this paper, the magneto hydrodynamic (MHD) flow of viscous fluid in a channel with non-parallel plates is studied. The governing partial differential equation was transformed into a system of dimensionless non-similar coupled ordinary differential equation. The transformed conservations equations were solved by using new algorithm. Basically, this new algorithm depends mainly on the Taylor expansion application with the coefficients of power series resulting from integrating the order differential equation. Results obtained from new algorithm are compared with the results of numerical Range-Kutta fourth-order algorithm with help of the shooting algorithm. The comparison revealed that the resulting solutions were excellent agreement. Thermo-diffusion and diffusion-thermo effects were investigated to analyze the behavior of temperature and concentration profile. Also the influences of the first order chemical reaction and the rate of mass and heat transfer were studied. The computed analytical solution result for the velocity, temperature and concentration distribution with the effect of various important dimensionless parameters was analyzed and discussed graphically.

Keywords

Thermo-Diffusion, Diffusion-Thermo, Chemical Reaction, Analytical Approximate Solution, Mass and Heat Transfer, Magneto Hydrodynamics
1. Introduction

The importance of thermal-diffusion and diffusion-thermo effects for various fluid flows has been studied by Eckert and Drake [1]. Olajuwon [2] examined convection heat and mass transfer in a hydromagnetic flow of a second grade fluid past a semi-infinite stretching sheet in the presence of thermal diffusion and thermal radiation. Kumar et al. [3] have investigated thermal diffusion and radiation effects on unsteady magneto hydrodynamics (MHD) flow through porous medium with variable temperature and mass diffusion in the presence of heat source or sink. Magnetohydrodynamics is the study of the interaction between magnetic fields and moving, conducting fluids [4] and the behavior of an electrically conducting fluid in the presence of a magnetic field. In this case, a force is produced inside the fluid which is proportional to fluid velocity and this force always opposes the flow. Another way to produce a force inside a flowing fluid, not known widely, is the application of an externally applied magnetic as well as an externally applied electric field. This force is called Lorentz force and can be generated by a strip wise arrangement of flush mounted electrodes and permanent magnets of alternating polarity and magnetization. The Lorentz force which acts parallel to the plate can either assist or oppose the flow. The idea of using a Lorentz force to stabilize a boundary layer flow over a flat plate belongs probably to Gailitis and Lielausis [5] [6]. It is a known fact that the temperature and concentration gradients present mass and energy fluxes, respectively. Concentration gradients result in Dufuor effect (diffusion-thermo) but Soret effect (thermal-diffusion) is due to temperature gradients. The heat and mass transfer with chemical reaction plays an important role in designing of chemical processing equipment, damage of crops due to frost, formulation and dispersion of fog. The mass transfer can be defined as a phenomenon when there is an escape of vapors into the atmosphere while heat transfer happens when there is heating or cooling of a liquid or fluid. That is, both of these phenomena play an important role in the industry. Because nonlinearity of the equations for these problem exact solutions is known, so many analytical techniques have been studied. Homotopy analysis method [7] [8] and Adomian’s decomposition method [9] [10] [11] [12] are also analytical techniques used to solve the nonlinear equations. In this article the governing equations of the problem contain a system of partial differential equations which are transformed by usual transformation into a non-dimensional system of partial coupled non-linear differential equations. The purpose of this article is to investigate the diffusion-thermo and thermal diffusion effects on converging and diverging channel in the presence of chemical reaction. In present problem we apply a new technique to solve the equation governing the flow of viscous fluid in diverging and converging channels called a new algorithm. This new algorithm that includes the use of several steps, first integration and then we use Taylor expansion in addition to the last step involves extracting the value of derivatives. MHD, Soret, Dufour and chemical reaction effects are taken into account. Influences of physical parameters on
temperature and concentration profiles are discussed for both diverging and converging channels with the help of graphs. The structure of this paper is organized as follows: Section 2 definitions of Mathematical formulation. Section 3 explains Description of the new scheme. Section 4, we apply new algorithm method to solve the magneto hydrodynamic (MHD) flow of viscous fluid in a channel with non-parallel plates show its ability and efficiency in finding new approximate solutions. Section 5 evidences that a new algorithm is converged through new theorems with its application. Section 6 discusses the effect of physical parameters on velocity, temperature, concentration profiles by help tables and graphics. Section 7 introduces conclusions of the present work.

2. Mathematical Formulation

Consider the flow of an incompressible fluid due to source or sink that is located at the intersection of two rigid plane walls angled $2\alpha$ apart. Radial and symmetric nature of the flow is taken into consideration. Induced magnetic field is ignored and an applied magnetic field is considered that is applied across the flow direction. Under the aforesaid assumptions velocity field takes the form $V = [u_r, 0, 0]$, where $u_r$ is a function of both $r$ and $\theta$. Soret and Dufour effects are also considered that are incorporated in energy and concentration equations respectively. The fluid is also assumed to be chemically reacting. Also, the temperature and concentration are also the function of both $r$ and $\theta$. While the results of angle opening $\alpha$ on concentration profile show that the increase in angle gives a decrease in concentration profile. The governing equations for mass, motion, energy, and mass transfer in polar coordinates under imposed assumptions become [13] [14] (Figure 1).

\[
\frac{1}{r}\frac{\partial}{\partial r}(ru_r) = 0, \tag{1}
\]

\[
u \frac{\partial u_r}{\partial r} = -\frac{1}{\rho} \frac{\partial p}{\partial r} + u_r \left[ \frac{\partial^2 u_r}{\partial r^2} + \frac{1}{r} \frac{\partial u_r}{\partial r} + 1 \frac{\partial^2 u_r}{\partial \theta^2} - u_r \right] - \frac{\sigma B_0^2}{\rho} u_r, \tag{2}
\]

\[-\frac{1}{\rho r} \frac{\partial p}{\partial \theta} + 2\nu \frac{\partial u_r}{\partial \theta} = 0, \tag{3}\]

\[
\rho c_p u_r \frac{\partial T}{\partial r} = k \left[ \frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{1}{r^2} \frac{\partial^2 T}{\partial \theta^2} \right] + \mu \left[ 4 \left( \frac{\partial u_r}{\partial r} \right)^2 + \left( \frac{\partial u_r}{\partial \theta} \right)^2 \right] + \frac{DK_{oc} \partial C}{C_i} \left[ \frac{\partial^2 C}{\partial r^2} + \frac{1}{r} \frac{\partial C}{\partial r} + \frac{1}{r^2} \frac{\partial^2 C}{\partial \theta^2} \right] \tag{4}
\]

\[
\frac{\partial C}{\partial r} = D \left[ \frac{\partial^2 C}{\partial r^2} + \frac{1}{r} \frac{\partial C}{\partial r} + \frac{1}{r^2} \frac{\partial^2 C}{\partial \theta^2} \right] + \frac{DK_T}{T_w} \left[ \frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{1}{r^2} \frac{\partial^2 T}{\partial \theta^2} \right] \tag{5}\]

the boundary conditions are,

\[
u = U, \quad \frac{\partial u_r}{\partial \theta} = 0, \quad \frac{\partial T}{\partial \theta} = 0, \quad \frac{\partial C}{\partial \theta} = 0, \text{ at } \theta = \alpha, \]
where $p$ is the fluid pressure, $\nu = \frac{\mu}{\rho}$ is kinematic viscosity, $D$ in order are the specific heat and coefficient of mass diffusivity. $K, K_T, D$ correspondingly are the thermal conductivity, thermal-diffusion ratio and coefficient of mass diffusivity. Further $C_s, T_w, T_m, C_w, K_1$ represent the concentration susceptibility, temperature at wall, mean fluid temperature, concentration at the wall and the chemical reaction constant respectively. From the continuity Equation (1), we can write

$$F(\theta) = ru_\alpha (r, \theta),$$

with the use of dimensionless parameters award \cite{6}

$$f(\eta) = \frac{F(\theta)}{rU}, \quad \eta = \frac{\theta}{\alpha}, \quad \beta(\eta) = \frac{T}{T_w}, \quad \phi(\eta) = \frac{C}{C_w}.$$ (8)

Eliminating $p$ from Equations (1) and (2) using Equations (7) and (8), we get a system of nonlinear ordinary differential equation for the normalized velocity profile $f(\eta)$, temperature profile $\beta(\eta)$ and concentration profile $\phi(\eta)$

$$f''''(\eta) + 2\alpha Re f'(\eta) f''(\eta) + (4 - Ha) \alpha^2 f''(\eta) = 0,$$

$$\beta''(\eta) + E_P \left[ 4\alpha^2 f''(\eta) + (f'(\eta))^2 \right] + D_P f''(\eta) = 0,$$

$$\phi''(\eta) + S_s \phi'' - S_s \gamma \alpha^2 \phi(\eta) = 0,$$

$$f(0) = 1, \quad f'(0) = 0, \quad f(1) = 0,$$

$$\beta'(0) = 0, \quad \beta(1) = 1,$$

$$\phi(0) = 0, \quad \phi(1) = 1,$$ (9)

where,

$$Re = \frac{U \alpha}{\mu} \begin{cases} \text{divergent channel : } \alpha > 0, U > 0, & H_a = \sqrt{\frac{\sigma B_0}{\mu}}, \quad P_r = \frac{\mu c_p}{K} \end{cases}$$

$$\text{convergent channel : } \alpha < 0, U < 0.$$ (10)
\[ E_c = \frac{U^2}{c_f T_w}, \quad D_f = \frac{DK_f C_w}{\nu C_f T_w}, \quad S_c = \frac{\nu}{D}, \quad S_c = \frac{DK_f T_w}{\nu T_w C_w}, \] represent Reynolds, Hartman, Prandtl, Eckert, Dufour, Schmidt and Soret number respectively while \( \gamma = \frac{K_1}{\nu} \) is the first order chemical reaction parameter. The local Nusselt and Sherwood numbers are defined by

\[ \frac{Nu}{K} = \beta' = -\frac{1}{\alpha} (1), \]

\[ \frac{Sh}{DC_w} = \varphi' = -\frac{1}{\alpha} (1). \]

### 3. Description of the New Algorithm

This section describes how to obtain a new scheme to calculate the coefficients of the power series solution resulting from solving nonlinear ordinary differential equations to find analytical-approximate solution. These coefficients are important basis to construct the solution formula, therefore they can be computed recursively by differentiation ways. To illustrate the computation and operations for these coefficients and derivation the new scheme, we summarized the detail a new outlook in the following steps.

**Step (1):** Consider the non-linear differential equation as follows:

\[ H(f(\eta), f'(\eta), f''(\eta), f'''(\eta), \ldots, f^{(n-1)}(\eta), f^{(n)}(\eta)) = 0, \]

integrating Equation (13) with respect to \( \eta \) on \([0, \eta]\) yield

\[ f(\eta) = f(0) + f'(0)\eta + f''(0)\frac{\eta^2}{2!} + \cdots + f^{(n-1)}(0)\frac{\eta^{n-1}}{(n-1)!} + \mathcal{L}^{-1} G[f(\eta)], \]

where,

\[ G[f(\eta)] = H(f(\eta), f'(\eta), f''(\eta), \ldots, f^{(n-1)}(\eta)), L^{-1} = \int_0^\eta \int_0^\nu \cdots \int_0^\nu (d\eta)^n. \]

**Step (2):** We take Taylor series expansion of the function \( G[f(\eta)] \) about \( \eta = \eta_0 \) as follows

\[ G[f(\eta)] = \sum_{n=0}^{\infty} (\Delta \eta)^n \frac{d^n G(f_0(\eta))}{d\eta^n}, \]

rewriting the Equation (16)

\[ G[f(\eta)] = G[f_0(\eta)] + \frac{\Delta \eta}{1!} G'[f_0(\eta)] + \frac{(\Delta \eta)^2}{2!} G''[f_0(\eta)] + \cdots \]

Now, we assume that \( \Delta \eta = \max \{\eta, \eta_0\} \) and substituting Equation (17) in Equation (14), we obtain

\[ f(\eta) = f_0 + f_1 + f_2 + f_3 + f_4 + \cdots, \]
where,
\[ f_0 = f(0) + f'(0)\eta + f''(0)\frac{\eta^2}{2!} + \cdots + f^{(n-1)}(0)\frac{\eta^{(n-1)}}{(n-1)!}, \quad f_i = L^{-1}G[f_0(\eta)]. \]
\[ f_2 = L^{-1}\frac{\max\{\eta, \eta_0\}}{1!}G'[f_0(\eta)], \quad f_3 = L^{-1}\frac{\max\{\eta, \eta_0\}^2}{2!}G''[f_0(\eta)], \]
\[ f_i = L^{-1}\frac{\max\{\eta, \eta_0\}^i}{3!}G^i[f_0(\eta)], \ldots \]

(19)

**Step (3):** We focus on computing the derivatives of \( G \) with respect to \( \eta \) which is the crucial part of the proposed method. Let start calculating
\[ G[f(\eta)], G'[f(\eta)], G''[f(\eta)], \ldots. \]
\[ G[f(\eta)] = H(f(\eta), f'(\eta), f''(\eta), f'''(\eta), \ldots, f^{(n-1)}(\eta)), \]
\[ G'[f(\eta)] = \frac{dG[f(\eta)]}{d\eta} = G_{ff} \cdot f_0 + G_{f}\cdot f_0' + \cdots + G_{f^{(n-1)}} \cdot f_0^{(n-1)}, \]
\[ G''[f(\eta)] = \frac{d^2G[f(\eta)]}{d\eta^2} = G_{ff} \cdot (f_0)^2 + G_{f\cdot f'} \cdot (f_0)^2 + \cdots + G_{f^{(n-1)} \cdot f_0^{(n-1)}}, \]
\[ G'''[f(\eta)] = \frac{d^3G[f(\eta)]}{d\eta^3} = G_{ff} \cdot (f_0)^3 + G_{f\cdot f'\cdot f''} \cdot (f_0)^3 + \cdots + G_{f^{(n-1)} \cdot f_0^{(n-1)}}, \]
\[ G^n[f(\eta)] = \frac{d^nG[f(\eta)]}{d\eta^n} = G_{ff} \cdot (f_0)^n + G_{f\cdot f'\cdot f''\cdots f^{(n-1)}}, \ldots \]

(20)

(21)

(22)
The calculations are more complicated in the second and third derivatives because of the product rules. Consequently, the systematic structure on calculation is extremely important. Fortunately, due to the assumption that the operator $G$ and the solution $f$ are analytic functions, then the mixed derivatives are equivalence.

We note that the derivatives function to $f$ unknown, so we suggest the following hypothesis

\[
f_n = f_1 = L^{-1}G[f_0(\eta)], \quad f_{qq} = f_2 = L^{-1}\frac{\max \{\eta, \eta_0\}}{1!}G'[f_0(\eta)],
\]

\[
f_{qqq} = f_3 = L^{-1}\frac{(\max \{\eta, \eta_0\})^2}{2!}G''[f_0(\eta)],
\]

\[
f_{qqqq} = f_4 = L^{-1}\frac{(\max \{\eta, \eta_0\})^3}{3!}G'''[f_0(\eta)],
\]

\[
f_{qqqqq} = f_5 = L^{-1}\frac{(\max \{\eta, \eta_0\})^4}{4!}G''''[f_0(\eta)],
\]

Therefore Equations (20)-(23) are evaluated by

\[
G[f_0(\eta)] = H\left(f_0(\eta), f_0'(\eta), f_0''(\eta), \ldots, f_0^{(n-1)}(\eta)\right),
\]

\[
G'[f_0(\eta)] = G_{f_0} \cdot f_1 + G_{f_0} \cdot (f_1)' + G_{f_0} \cdot (f_1)'' + \ldots + G_{f_0}^{(n-1)} \cdot (f_1)^{(n-1)} + G_{f_0}^{(n)} \cdot (f_1)^{(n)}.
\]
\[ + \cdots + G_{j_0 (a-1)} \cdot (f_1) (f_1)^{(a-1)} + G_{j_0} \cdot (f_2) + G_{j_0 (a-1)} \cdot (f_1)^{(a-1)} \cdot f_1 \]
\[ + G_{j_0 (a-1)} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} + \cdots + G_{j_0 (a-1) j_0 (a-1)} \cdot (f_1)^{(a-2)} + \cdots + G_{j_0} \cdot (f_2)^{(a-1)}, \]
\[ G^* [f_0 (\eta)] \]
\[ = G_{f_0 f_0 f_0} \cdot (f_1)^{(a-1)} + G_{f_0 f_0 f_0} \cdot (f_1)^{(a-1)} + \cdots + G_{f_0 f_0 f_0 (a-1)} \cdot (f_1)^{2} \cdot (f_1)^{(a-1)} \]
\[ + G_{f_0} \cdot 2 (f_1) + G_{f_0 f_0 f_0} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} + \cdots \]
\[ + G_{f_0 f_0 f_0 (a-1)} \cdot (f_1)^{(a-1)} + \cdots + G_{f_0 f_0 f_0 (a-1) f_0} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} \]
\[ + G_{f_0 f_0 f_0 (a-1)} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} + \cdots + G_{f_0 f_0 f_0 (a-1) f_0} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} \]
\[ + \cdots + G_{f_0 f_0 f_0 (a-1) f_0} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} \cdot f_1 + \cdots \]
\[ + G_{f_0 f_0 f_0 (a-1)} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} \cdot f_1 + \cdots \]
\[ + G_{f_0 f_0 f_0 (a-1) f_0} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} \cdot f_1 + \cdots \]
\[ + G_{f_0 f_0 f_0 (a-1) f_0} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} \cdot f_1 + \cdots \]
\[ + G_{f_0 f_0 f_0 (a-1) f_0} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} \cdot f_1 + \cdots \]
\[ + G_{f_0 f_0 f_0 (a-1) f_0} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} \cdot f_1 + \cdots \]
\[ + G_{f_0 f_0 f_0 (a-1) f_0} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} \cdot f_1 + \cdots \]
\[ + G_{f_0 f_0 f_0 (a-1) f_0} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} \cdot f_1 + \cdots \]
\[ + G_{f_0 f_0 f_0 (a-1) f_0} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} \cdot f_1 + \cdots \]
\[ + G_{f_0 f_0 f_0 (a-1) f_0} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} \cdot f_1 + \cdots \]
\[ + G_{f_0 f_0 f_0 (a-1) f_0} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} \cdot f_1 + \cdots \]
\[ + G_{f_0 f_0 f_0 (a-1) f_0} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} \cdot f_1 + \cdots \]
\[ + G_{f_0 f_0 f_0 (a-1) f_0} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} \cdot f_1 + \cdots \]
\[ + G_{f_0 f_0 f_0 (a-1) f_0} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} \cdot f_1 + \cdots \]
\[ + G_{f_0 f_0 f_0 (a-1) f_0} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} \cdot f_1 + \cdots \]
\[ + G_{f_0 f_0 f_0 (a-1) f_0} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} \cdot f_1 + \cdots \]
\[ + G_{f_0 f_0 f_0 (a-1) f_0} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} \cdot (f_1)^{(a-1)} \cdot f_1 + \cdots \]

**Step (4):** Substituting Equations (25)-(28) in Equation (18) we get the required analytical-approximate solution for the Equation (13).

4. Application of the New Algorithm to the Magneto Hydrodynamic (MHD) Flow of Viscous Fluid in a Channel with Non-Parallel Plates

The new algorithm described in the previous section can be used as a powerful solver to the nonlinear differential Equations (9)-(10) and to find new an analytical-approximate solution. From step (1) we have

\[ f (\eta) = f (0) + f' (0) \eta + f'' (0) \eta^2 + \cdots \]

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\[ \beta(\eta) = \beta(0) + \beta'(0)\eta + L_1; [-E_P\left[ 4\alpha^2 f^2(\eta) - (f'(\eta))^2 \right] - D_P\phi(\eta)], \]
\[ \phi(\eta) = \phi(0) + \phi'(0)\eta + L_1; [-S_S, \beta^* + S_S\alpha^2 \phi(\eta)], \quad (29) \]

rewrite the Equation (29) as follows
\[ f(\eta) = A_1 + A_2\eta + A_3\frac{\eta^2}{2!} + L_1 G_1[f(\eta)], \]
\[ \beta(\eta) = B_1 + B_2\eta + L_1 G_2[\beta(\eta)], \]
\[ \phi(\eta) = C_1 + C_2\eta + L_1 G_3[\phi(\eta)], \quad (30) \]

where,
\[ A_1 = f(0), \quad A_2 = f'(0), \quad A_3 = f''(0), \]
\[ B_1 = \beta(0), \quad B_2 = \beta'(0), \quad C_1 = \phi(0), \quad C_2 = \phi'(0), \]
\[ G_1[f] = -2\alpha R e F(\eta)f'(\eta) - (4 - Ha)\alpha^2 f'(\eta), \]
\[ G_2[\beta] = -E_P\left[ 4\alpha^2 f^2(\eta) - (f'(\eta))^2 \right] - D_P\phi(\eta), \]
\[ G_3[\phi] = -S_S, \beta^* + S_S\alpha^2 \phi(\eta), \]

and \[ L_1^{-1}(\cdot) = \int_0^{\eta_0} (d\eta)^2, \quad L_2^{-1}(\cdot) = \int_0^{\eta_0} (\cdot)^2. \quad (31) \]

From the boundary conditions the Equation (30) becomes
\[ f(\eta) = 1 + A_1\frac{\eta^2}{2!} + L_1 G_1[f(\eta)], \]
\[ \beta(\eta) = B_1 + L_1 G_2[\beta(\eta)], \]
\[ \phi(\eta) = C_1 + L_1 G_3[\phi(\eta)]. \quad (32) \]

From step (2) suppose that \[ \Delta\eta = \max \{1, 0\} = 1, \] yield
\[ f_0 = 1 + A_1\frac{\eta^2}{2!} \quad f_1 = L_1 G_1[f_0(\eta)], \quad f_2 = L_1 G_2[f_1(\eta)], \ldots, \]
\[ \beta_0 = B_1, \quad \beta_1 = L_1 G_2[\beta_0(\eta)], \quad \beta_2 = L_1 G_2[\beta_1(\eta)], \ldots, \]
\[ f_0 = C_1, \quad \phi_1 = L_1 G_3[\phi_0(\eta)], \quad \phi_2 = L_1 G_3[\phi_1(\eta)], \ldots, \quad (33) \]

and the analytical-approximate solution are
\[ f(\eta) = f_0 + f_1 + f_2 + f_3 + \cdots, \]
\[ \beta(\eta) = \beta_0 + \beta_1 + \beta_2 + \beta_3 + \cdots, \]
\[ \phi(\eta) = \phi_0 + \phi_1 + \phi_2 + \phi_3 + \cdots \quad (34) \]

From step (3) yields
\[ G_1[f(\eta)] = -2\alpha R e F(\eta)f'(\eta) - (4 - Ha)\alpha^2 f'(\eta), \]
\[ G_2[\beta(\eta)] = -E_P\left[ 4\alpha^2 f^2(\eta) - (f'(\eta))^2 \right] - D_P\phi(\eta), \]
\begin{align}
G_1[\phi(\eta)] &= -S_y S^* + S_y y^2 \phi(\eta), \\
G_1'[f(\eta)] &= \frac{dG_1[f(\eta)]}{d\eta} = G_{1, f'\eta} + G_{1, f'} (f_\eta)' \\
G_2[\beta(\eta)] &= \frac{dG_2[\beta(\eta)]}{d\eta} = G_{2, f'\eta} + G_{2, f'} (f_\eta)' + G_{2} \phi^* (\phi_\eta)'
\end{align}

\begin{align}
G_1'[f(\eta)] &= \frac{d^2G_1[f(\eta)]}{d\eta^2} \\
 &= G_{yy} (f_\eta)^2 + 2 G_{yf'} f_{\eta} (f_\eta) + G_{f'f'} (f_\eta)^2 + G_{f'\eta} f_{\eta 
\end{align}

\begin{align}
G_2[\beta(\eta)] &= \frac{d^2G_2[\beta(\eta)]}{d\eta^2} \\
 &= G_{yy} (f_\eta)^2 + 2 G_{yf'} f_{\eta} (f_\eta) + G_{f'f'} (f_\eta)^2 + G_{f'\eta} f_{\eta}
\end{align}
We note that the derivatives of $f$ with respect to $\eta$ that are given in (19), can be computing by Equations (35)-(38) as

$$
G_1^* [f_0 (\eta)] = d^3 G_3 \left[ \phi (\eta) \right] \\
= G_{3p_3} \cdot (\beta_{\eta})^3 + 3 \cdot G_{3p_3} \cdot (\beta_{\eta})^2 \cdot \phi_{\eta} + 3 \cdot G_{3p_3} \cdot (\beta_{\eta}) \cdot \phi_{\eta}^* + 3 \cdot G_{3q_3} \cdot (\beta_{\eta})^3 + 3 \cdot G_{3q_3} \cdot (\beta_{\eta})^2 \cdot \phi_{\eta} + 3 \cdot G_{3q_3} \cdot (\beta_{\eta}) \cdot \phi_{\eta}^* + 3 \cdot G_{3q_3} \cdot \phi_{\eta} \cdot \beta_{\eta} + G_{3q_3} \cdot (\phi_{\eta}) + 3 \cdot G_{3p_{3q}} \cdot (\beta_{\eta}) \cdot \phi_{\eta} + G_{3p_{3q}} \cdot (\beta_{\eta})^* \cdot \phi_{\eta} + G_{3p_{3q}} \cdot (\beta_{\eta})^* \cdot \phi_{\eta}^* .
$$

(38)

...
The extraction of the first derivatives of $G$ can be represented as:

$$G_{i,j} = -2\alpha Re f_0'(\eta), \quad G_{i,0} = 0, \quad G_{0,j} = -2\alpha Re,$$

$$G_{1,0} = G_{1,0} = G_{1,0} = G_{1,0} = 0,$$

$$G_{1,j} = -2\alpha Re f_0'(\eta) - (4 - Ha)\alpha^2, \quad G_{1,0} = -2\alpha Re, \quad G_{1,0} = 0,$$

$$G_{2,0} = G_{2,0} = G_{2,0} = G_{2,0} = 0, \quad G_{2,0} = -D_p,$$

$$G_{2,j} = -8E_p\alpha^2 f_0'(\eta), \quad G_{2,0} = -8E_p\alpha^2,$$

$$G_{3,0} = G_{3,0} = G_{3,0} = G_{3,0} = 0, \quad G_{3,0} = 0, \quad G_{3,0} = -8E_p\alpha^2,$$

$$G_{4,0} = G_{4,0} = G_{4,0} = G_{4,0} = 0, \quad G_{4,0} = 0, \quad G_{4,0} = 0, \quad G_{4,0} = 0,$$

from Equation (33) by using Equations (39)-(42), gives the following,

$$f_0 = \frac{1}{2} A_4 \eta^2 + 1,$$

$$\beta_0 = B_1,$$

$$\phi_0 = C_1,$$

$$f_1 = -\frac{1}{120} \alpha Re A_4^2 \eta^6 - \left(\frac{1}{12} \alpha Re A_4 + \frac{1}{6} \alpha^2 A_4 - \frac{1}{24} \alpha^2 Ha A_4\right) \eta^4,$$

$$\beta_1 = -\frac{1}{30} \alpha^2 P_e A_4^2 \eta^6 - \frac{1}{12} \left(4A_4 \alpha^2 + A_4^3\right) P_e \eta^4 - 2\alpha^2 P_e \eta^2,$$

$$\phi_1 = \frac{1}{2} \eta^2 S C_1,$$

$$f_2 = \frac{1}{10800} \alpha^2 Re^2 A_4^2 \eta^{10} + \left(\frac{1}{280} \alpha^2 Re A_4^2 - \frac{1}{1120} \alpha^3 Re Ha A_4^2 + \frac{1}{560} \alpha^2 Re^2 A_4^3\right) \eta^8$$

$$+ \left(\frac{1}{180} \alpha^2 Re A_4 + \frac{1}{45} \alpha^3 Re A_4 - \frac{1}{180} \alpha^3 Re Ha A_4\right) \eta^6$$

$$+ \frac{1}{45} \alpha^4 A_4 + \frac{1}{90} \alpha^4 Re Ha A_4 + \frac{1}{720} \alpha^4 Re Ha^2 A_4\eta^4,$$
\[
\beta_2 = \frac{1}{2700} \alpha^2 \Re^2 \hat{A}_3 P E \eta^{10} + \frac{1}{1680} \alpha \hat{A}_3^2 P E \left( -5 \Re \alpha^3 + 12 \alpha^2 \Re \right) + 20 \alpha^3 + 3 A \Re \right) \eta^4 + 2 S, S, P - r E, \alpha^2 \eta^2,
\]
\[
\phi_2 = \frac{1}{30} \alpha^2 P E, S, S, \hat{A}_3^2 \eta^6 + \left( \frac{1}{12} P E, S, S, \left( 4 A \alpha^2 + \hat{A}_3^2 \right) + \frac{1}{24} S^2 \gamma^2 \alpha^4 C \right) \eta^4,
\]
\[
f_3 = -\frac{1}{1572480} \alpha^4 \Re^2 \hat{A}_3 \eta^{14} - \left( \frac{359}{9979200} \alpha^4 \Re^2 \hat{A}_3^3 \Re \eta^4 + \frac{359}{9979200} \alpha^4 \Re^2 \hat{A}_3 \eta^4 \right) + \frac{29}{226800} \alpha^4 \Re^2 \hat{A}_3 \eta^4 + \frac{29}{113400} \alpha^4 \Re^2 \hat{A}_3 \eta^4
\]
\[
- \frac{29}{56700} \alpha^4 \Re^2 \hat{A}_3 \eta^4 - \frac{1}{10080} \alpha^4 \Re^2 \hat{A}_3 + \frac{1}{1680} \alpha^4 \Re^2 \hat{A}_3
\]
\[
- \frac{1}{6720} \alpha^4 \Re^2 \hat{A}_3 \eta^4 + \frac{1}{840} \alpha^4 \Re^2 \hat{A}_3 + \frac{1}{1680} \alpha^4 \Re^2 \hat{A}_3
\]
\[
+ \frac{1}{13440} \alpha^4 \Re^2 \hat{A}_3 + \frac{1}{1260} \alpha^4 \Re^2 \hat{A}_3 + \frac{1}{1680} \alpha^4 \Re^2 \hat{A}_3
\]
\[
+ \frac{1}{6720} \alpha^4 \Re^2 \hat{A}_3 - \frac{1}{80640} \alpha^4 \Re^2 \hat{A}_3 \eta^4,
\]
\[
\beta_3 = -\frac{1}{393120} P E, \alpha^4 \Re^2 \hat{A}_3 \eta^4 - \frac{1}{9979200} P E, \alpha^4 \Re^2 \hat{A}_3 \eta^4
\]
\[
+ 718 \Re \alpha^2 + 1380 \alpha^3 + 259 A \Re \eta^4 + \frac{1}{453600} P E, \alpha^2 \Re^2 \eta^4 - \left( 49 \alpha^2 \Re^2 \eta^4 + 214 \Re \alpha^2 + 392 \Re \alpha^4 + 120 A \Re \alpha^4 - 232 \Re^2 \alpha^2 - 856 \Re \alpha^4 \right)
\]
\[
- 784 \alpha^4 - 240 A \Re^2 - 480 A \Re \alpha \eta^4 - \frac{1}{20160} P E, \alpha \Re \alpha \eta^4 + 13 A \Re \alpha^2 - 8 \Re \alpha \eta^4 - 16 \Re \alpha^2 - 52 A \Re \alpha^2 - 104 A \Re \alpha^2
\]
\[
+ 8 \Re \alpha^2 + 32 \Re \alpha^3 + 32 \Re \alpha^4 + 52 A \Re \alpha^2 + 208 A \Re \alpha^2 + 208 A, \Re \alpha^2 \eta^4
\]
\[
- \frac{1}{60} S^2 D, S, S, P, E, \alpha^2 \eta^4 - \frac{1}{24} P D, S, S, P, E, \alpha (4 \alpha^2 + A) \eta^4
\]
\[
+ \frac{1}{2} \eta^2 \gamma^2 \alpha^4 \eta^4 - \eta^2 \Re \alpha^2 \eta^4
\]
\[
\phi_3 = -\frac{1}{5400} S, S, P, E, \alpha \Re \eta^4 + \frac{1}{16} S, S, P, E, \alpha \Re \eta^4 - \frac{1}{210} S, S, P, E, \alpha \Re \eta^4
\]
\[
+ 12 \Re \alpha^2 + 20 \alpha^3 + 3 A \Re \eta^4 + a \frac{1}{210} S^2 \gamma^4 S, P, E, \alpha \Re \eta^4
\]
\[
+ \frac{1}{15} \left( \frac{1}{12} S, S, P, E, \alpha A \left( H \alpha^3 + A, H \alpha a - 2 \Re \alpha^2 - 4 \alpha^3 - 2 A, \alpha \right) \right) \eta^4
\]
\[
+ \frac{1}{5} S, \gamma^2 \left( \frac{1}{12} S, S, P, E, \alpha A \left( 4 \alpha^2 + A \right) + \frac{1}{24} S^2 \gamma^2 \alpha^4 C \right) \eta^4
\]
\[
+ \frac{1}{12} S^2 \gamma S - r P, E, \alpha^2 \eta^4 + \frac{1}{4} S, S, P, D, \gamma^2 \Re C \eta^2
\]
From **step (4)** substitution Equations (46)-(42) in Equation (34), the analytical-approximate solution can be resulted as follows:

\[
f(\eta) = 1 + \frac{1}{2} A_2 \eta^2 - \left( \frac{1}{12} \alpha Re A_3 + \frac{1}{6} \alpha^2 A_4 - \frac{1}{24} \alpha^2 Ha A_3 \right) \eta^4 + \left( \frac{1}{180} \alpha^2 Re A_3 \right) \eta^6 + \frac{1}{240} \alpha^4 Re A_3 \eta^8 + \frac{1}{560} \alpha^2 Re A_3 \eta^{10} + \cdots
\]

\[
\beta(\eta) = B_1 + \left( -2 \alpha P E_1 + 2 S S P E_1 \right) \eta^2 + \left( \frac{1}{12} (4 A_2 \alpha^2 + A_2^3) P E_1 \right) \eta^4 - \frac{1}{160} \alpha A_2 P E_1 \eta^6 + \frac{1}{2700} \alpha^2 Re A_3 P E_1 \eta^{10} + \cdots
\]

\[
\phi(\eta) = C_1 + \frac{1}{2} \gamma^2 S C_1 \eta^2 + \left( \frac{1}{12} P E_1 S S \left( 4 A_2 \alpha^2 + A_2^3 \right) \right) \eta^4 + \frac{1}{30} \alpha^2 P E_1 S S C_1 \eta^6 + \cdots
\]

(50)

**5. The Analysis of Convergence**

Here, the analysis of convergence for the analytical-approximate solution (50) that was resulted from the application of new power series algorithm for solving the problem has been extensively studied.

**Definition (1):** Suppose that \( H \) is Banach space, \( R \) is the real numbers and \( G[F,H,P] = \{G_1[F], G_2[H], G_2[P]\} \) is a nonlinear operators defined by \( G[F,H,P]: H^3 \to R^3 \). Then the sequence of the solutions generated from a new algorithm can be written as

\[
F_{n+1} = G_1[F_n], \quad F_n = \sum_{k=0}^{n} f_k, \quad n = 0,1,2,3,\cdots
\]

\[
H_{n+1} = G_2[H_n], \quad H_n = \sum_{k=0}^{n} \beta_k, \quad n = 0,1,2,3,\cdots
\]

(51)

**Definition (2):** Suppose that \( G[F,H,P] \) satisfies Lipschitz condition such that for \( 0 \leq \gamma_1, \gamma_2, \gamma_3 < 1, \quad \gamma_1, \gamma_2, \gamma_3 \in R \), we have

\[
f(\eta) = 1 + \frac{1}{2} A_2 \eta^2 - \left( \frac{1}{12} \alpha Re A_3 + \frac{1}{6} \alpha^2 A_4 - \frac{1}{24} \alpha^2 Ha A_3 \right) \eta^4 + \left( \frac{1}{180} \alpha^2 Re A_3 \right) \eta^6 + \frac{1}{240} \alpha^4 Re A_3 \eta^8 + \frac{1}{560} \alpha^2 Re A_3 \eta^{10} + \cdots
\]
\[ G_1[F_n] - G_1[F_{n-1}] \leq \gamma_1 \|F_n - F_{n-1}\|, \]
\[ G_2[H_n] - G_2[H_{n-1}] \leq \gamma_2 \|H_n - H_{n-1}\|, \]
\[ G_3[P_n] - G_3[P_{n-1}] \leq \gamma_3 \|P_n - P_{n-1}\|. \]  

(52)

Now, we assume that \( G[F_n, H_n, P_n] = G(n) \) for simplify with \( \gamma = \gamma_1 + \gamma_2 + \gamma_3 \), \( 0 \leq \gamma < 1 \) yield,
\[ G(n) - G(n-1) \leq \gamma \|F_n, H_n, P_n\| - (F_{n-1}, H_{n-1}, P_{n-1}) \]  

(53)

The sufficient condition for convergent of the series analytical-approximate solutions \( F_n, H_n, P_n \) is given in the following theorems.

**Theorem (1):**2 The series of the analytical-approximate solution \( \{S_n = (F_n, H_n, P_n)\}_n \) generated from new algorithm converge if the following condition is satisfied:

\[ \|S_n - S_m\| \to 0, \quad \text{as} \quad m \to \infty, \quad \text{for} \quad 0 \leq \gamma < 1, \]  

(54)

**Proof:** From the above definition, the next equation can be written as

\[ \|S_n - S_m\| = \left\| \left( F_n, H_n, P_n \right) - \left( F_m, H_m, P_m \right) \right\| \]

\[ = \left\| \left( f_0 + L^1 \sum_{k=0}^{n} \frac{\Delta^k f \left( \eta \right)}{k!} \frac{d \xi}{d \eta} \right) + \sum_{k=0}^{n} \frac{\Delta^k f \left( \eta \right)}{k!} \right\| \]

\[ = \left\| \left( f_0 + L^1 \sum_{k=0}^{n} \frac{\Delta^k f \left( \eta \right)}{k!} \right) + \sum_{k=0}^{n} \frac{\Delta^k f \left( \eta \right)}{k!} \right\| \]

\[ = \gamma \|S_n - S_m\|, \]

since \( G[F, H, P] \) satisfies Lipschitz condition. Let \( n = m + 1 \), then

\[ \|F_{m+1} - F_m\| \leq \gamma_1 \|F_{m+1} - F_m\|, \]
\[ \|H_{m+1} - H_m\| \leq \gamma_2 \|H_{m+1} - H_m\|, \]
\[ \|P_{m+1} - P_m\| \leq \gamma_3 \|P_{m+1} - P_m\|, \]  

(56)

hence,

\[ \|F_m - F_{m-1}\| \leq \gamma_1 \|F_{m-1} - F_{m-2}\| \leq \cdots \leq \gamma_1^{m-1} \|F_1 - F_0\|, \]
\[ \|H_m - H_{m-1}\| \leq \gamma_2 \|H_{m-1} - H_{m-2}\| \leq \cdots \leq \gamma_2^{m-1} \|H_1 - H_0\|, \]
\[ \|P_m - P_{m-1}\| \leq \gamma_3 \|P_{m-1} - P_{m-2}\| \leq \cdots \leq \gamma_3^{m-1} \|P_1 - P_0\|, \]  

(57)
from Equation (57) we get
\[
\begin{align*}
\|F_2 - F_1\| &\leq \gamma_1 \|F_1 - F_0\| \|H_2 - H_1\| \leq \gamma_2 \|H_1 - H_0\| \|P_2 - P_1\| \leq \gamma_3 \|P_1 - P_0\|, \\
\|F_3 - F_2\| &\leq \gamma_1 \|F_2 - F_0\| \|H_3 - H_2\| \leq \gamma_2 \|H_2 - H_1\| \|P_3 - P_2\| \leq \gamma_3 \|P_2 - P_1\|, \\
\|F_4 - F_3\| &\leq \gamma_1 \|F_3 - F_0\| \|H_4 - H_3\| \leq \gamma_2 \|H_3 - H_2\| \|P_4 - P_3\| \leq \gamma_3 \|P_3 - P_2\|.
\end{align*}
\] (58)

By using triangle inequality, we find that as \( m \to \infty \), we have \( \|S_m - S_n\| \to 0 \), then \( S_n \) is a Cauchy sequence in Banach space \( H^p \).

**Theorem 2:** Let \( G = (G_1, G_2, G_3) \) be a nonlinear operator satisfies Lipschitz condition from \( H^p \) to \( H^p \). If the series analytical-approximate solution \( \{S_n\} \) converges, then it is converged to the solution of the problem (9)-(10).

**Proof.**
\[
\begin{align*}
&\leq \gamma_1 \|F_2 - F_1\| + \gamma_2 \|H_2 - H_1\| + \gamma_3 \|P_2 - P_1\| \\
&\leq (\gamma_1 + \gamma_2 + \gamma_3) \|F_2, H_2, P_2 - (F_1, H_1, P_1)\| \\
&= \gamma \|S_2 - S_1\|
\end{align*}
\]

Therefore, from the Banach fixed-point theorem, there is a unique solution of the problem (9)-(10). We will prove that \( \{S_n\}_{0}^{\infty} \) converges to \( S \).

\[
G[S] = G \left[ \sum_{k=0}^{\infty} S_k \right] = \lim_{n \to \infty} G \left[ \sum_{k=0}^{n} S_k \right] = \lim_{n \to \infty} G[S_n] = \lim_{n \to \infty} S_n = S.
\]

In practice, the theorems (1) and (2) suggest to compute the value of \( \gamma_1, \gamma_2, \gamma_3 \), as described in the following definition.

**Definition 1:** for \( k = 1, 2, 3, \ldots \)
\[
\begin{align*}
\gamma_1^k &= \left\{ \begin{array}{ll}
\frac{\|F_{k+1} - F_k\|}{\|F_k - F_0\|} & \|F_k\| \neq 0, \\
0, & \|F_k\| = 0,
\end{array} \right.
\\
\gamma_2^k &= \left\{ \begin{array}{ll}
\frac{\|H_{k+1} - H_k\|}{\|H_k - H_0\|} & \|H_k\| \neq 0, \\
0, & \|H_k\| = 0,
\end{array} \right.
\\
\gamma_3^k &= \left\{ \begin{array}{ll}
\frac{\|P_{k+1} - P_k\|}{\|P_k - P_0\|} & \|P_k\| \neq 0, \\
0, & \|P_k\| = 0.
\end{array} \right.
\end{align*}
\] (59)
Now, the definition (1) can be applied on the magneto hydrodynamic (MHD) flow of viscous fluid in a channel with non-parallel plates to find convergence, then to obtain for examples as below.

If we choose $\text{Re} = 30$, $\alpha = 3^\circ$, $Ha = 500$, $S_r = S_s = 0.1$, $P_r = 0.1$, $D_f = 0.01$, $E_c = 0.01$, $\gamma = 0.04$ then obtain:

\[ |F_2 - F_0|_1 \leq \gamma_1 |F_1 - F_0|_1 \Rightarrow \gamma_1 = 0.08954317 < 1, \]

\[ |F_3 - F_0|_1 \leq \gamma_1 |F_2 - F_0|_1 \Rightarrow \gamma_1^2 = 0.005099704 < 1, \]

\[ |F_4 - F_0|_1 \leq \gamma_1^2 |F_3 - F_0|_1 \Rightarrow \gamma_1^3 = 0.0001956583630 < 1, \]

\[ |H_2 - H_0|_1 \leq \gamma_2 |H_1 - H_0|_1 \Rightarrow \gamma_2 = 0.2496673982 < 1, \]

\[ |H_3 - H_0|_1 \leq \gamma_2^2 |H_2 - H_0|_1 \Rightarrow \gamma_2^2 = 0.03173895000 < 1, \]

\[ |H_4 - H_0|_1 \leq \gamma_2^3 |H_3 - H_0|_1 \Rightarrow \gamma_2^3 = 0.002169482685 < 1, \]

\[ |P_2 - P_0|_1 \leq \gamma_3 |P_1 - P_0|_1 \Rightarrow \gamma_3 = 0.73232384 < 1, \]

\[ |P_3 - P_0|_1 \leq \gamma_3^2 |P_2 - P_0|_1 \Rightarrow \gamma_3^2 = 0.09141845239 < 1, \]

\[ |P_4 - P_0|_1 \leq \gamma_3^3 |P_3 - P_0|_1 \Rightarrow \gamma_3^3 = 0.003873822310 < 1, \]

\[ |F_1 - F_0|_\infty \leq \gamma |F_0 - F_0|_\infty \Rightarrow \gamma = 0.07399291 < 1, \]

\[ |F_2 - F_0|_\infty \leq \gamma^2 |F_1 - F_0|_\infty \Rightarrow \gamma^2 = 0.004719598 < 1, \]

\[ |F_3 - F_0|_\infty \leq \gamma^3 |F_2 - F_0|_\infty \Rightarrow \gamma^3 = 0.000155752428 < 1, \]

\[ |H_1 - H_0|_\infty \leq \gamma |H_0 - H_0|_\infty \Rightarrow \gamma = 0.2384340096 < 1, \]

\[ |H_2 - H_0|_\infty \leq \gamma^2 |H_1 - H_0|_\infty \Rightarrow \gamma^2 = 0.02467876 < 1, \]

\[ |H_3 - H_0|_\infty \leq \gamma^3 |H_2 - H_0|_\infty \Rightarrow \gamma^3 = 0.001715583134 < 1, \]

\[ |P_1 - P_0|_\infty \leq \gamma |P_0 - P_0|_\infty \Rightarrow \gamma = 0.732255116 < 1, \]

\[ |P_2 - P_0|_\infty \leq \gamma^2 |P_1 - P_0|_\infty \Rightarrow \gamma^2 = 0.087297019 < 1, \]

\[ |P_3 - P_0|_\infty \leq \gamma^3 |P_2 - P_0|_\infty \Rightarrow \gamma^3 = 0.003011824761 < 1, \]

Also, if we get $\alpha = -2^\circ$, $\text{Re} = 10$, $Ha = 110$, $P_r = D_f = 0.2$, $\gamma = 0.4$, $E_c = 0.01$.

\[ |F_2 - F_0|_1 \leq \gamma_1 |F_1 - F_0|_1 \Rightarrow \gamma_1 = 0.03060256592 < 1, \]

\[ |F_3 - F_0|_1 \leq \gamma_1^2 |F_2 - F_0|_1 \Rightarrow \gamma_1^2 = 0.1113608591 < 1, \]

\[ |F_4 - F_0|_1 \leq \gamma_1^3 |F_3 - F_0|_1 \Rightarrow \gamma_1^3 = 0.4872382121 < 1, \]

\[ |H_2 - H_0|_1 \leq \gamma_2 |H_1 - H_0|_1 \Rightarrow \gamma_2 = 0.0004683459276 < 1, \]

\[ |H_3 - H_0|_1 \leq \gamma_2^2 |H_2 - H_0|_1 \Rightarrow \gamma_2^2 = 0.005594660398 < 1, \]

\[ |H_4 - H_0|_1 \leq \gamma_2^3 |H_3 - H_0|_1 \Rightarrow \gamma_2^3 = 0.02712979918 < 1, \]
\[
\left\| P_2 - P_1 \right\|_\gamma \leq \gamma_3 \left\| P_1 - P_0 \right\|_\gamma \Rightarrow \gamma_3 = 0.000005903459683 < 1, \\
\left\| P_3 - P_2 \right\|_\gamma \leq \gamma_5 \left\| P_2 - P_1 \right\|_\gamma \Rightarrow \gamma_5 = 0.001292820508 < 1, \\
\left\| P_4 - P_3 \right\|_\gamma \leq \gamma_7 \left\| P_3 - P_2 \right\|_\gamma \Rightarrow \gamma_7 = 0.0004542885489 < 1, \\
\vdots \\
\left\| F_2 - F_1 \right\|_\gamma \leq \gamma_1 \left\| F_1 - F_0 \right\|_\gamma \Rightarrow \gamma_1 = 0.02756015475 < 1, \\
\left\| F_3 - F_2 \right\|_\gamma \leq \gamma_5 \left\| F_2 - F_1 \right\|_\gamma \Rightarrow \gamma_5 = 0.1104543115 < 1, \\
\left\| F_4 - F_3 \right\|_\gamma \leq \gamma_7 \left\| F_3 - F_2 \right\|_\gamma \Rightarrow \gamma_7 = 0.4872217326 < 1, \\
\vdots \\
\left\| H_2 - H_1 \right\|_\gamma \leq \gamma_2 \left\| H_1 - H_0 \right\|_\gamma \Rightarrow \gamma_2 = 0.00041798914 < 1, \\
\left\| H_3 - H_2 \right\|_\gamma \leq \gamma_5 \left\| H_2 - H_1 \right\|_\gamma \Rightarrow \gamma_5 = 0.005302953669 < 1, \\
\left\| H_4 - H_3 \right\|_\gamma \leq \gamma_7 \left\| H_3 - H_2 \right\|_\gamma \Rightarrow \gamma_7 = 0.02690804181 < 1, \\
\vdots \\
\left\| P_2 - P_1 \right\|_\gamma \leq \gamma_3 \left\| P_1 - P_0 \right\|_\gamma \Rightarrow \gamma_3 = 0.00000534538 < 1, \\
\left\| P_3 - P_2 \right\|_\gamma \leq \gamma_5 \left\| P_2 - P_1 \right\|_\gamma \Rightarrow \gamma_5 = 0.00010743626 < 1, \\
\left\| P_4 - P_3 \right\|_\gamma \leq \gamma_7 \left\| P_3 - P_2 \right\|_\gamma \Rightarrow \gamma_7 = 0.00043058757 < 1, \\
\vdots \\
\sum_{k=0}^{\infty} f_k (\eta), \sum_{k=0}^{\infty} \beta_k (\eta) \text{ and } \sum_{k=0}^{\infty} \phi_k (\eta) \text{ converge to the solutions } f(\eta), \beta(\eta) \text{ and } \phi(\eta) \text{ respectively when } 0 \leq \gamma_1, \gamma_2, \gamma_3 < 1, \quad k = 1, 2, \ldots.
\]

6. Results and Discussions

This section is dedicated to study the influence of various non dimensional physical parameters on velocity field \( f(\eta) \), temperature field \( \beta(\eta) \) and concentration field \( \phi(\eta) \). Also the influence of different parameters on rate of heat transfer and rate of mass transfer are under observation for diverging and converging channels. In Table 1 and Table 2 proof convergence the values \( A_i, B_i \) and \( C_i \) of initial solutions. The stability of these values can be clearly distinguished from the fourth approximation. Table 3 and Table 4 present a comparison of the solutions obtained by new algorithm and numerical algorithm. These tables show that an excellent between the solutions. On the other hand Tables 5-13 are explained to analyze the behavior of Nusselt number and Sherwood number with variation parameters. As for can say that impotent to mention that Nusselt number gives a description of heat transfer rate at the wall, while Sherwood number represents the rate of mass transfer at the wall. In Table 5 and Table 6, Nusselt number with very Reynolds number \( Re \) and channel opening \( \alpha \) are discussed. One can clearly observe that for diverging channel, Nusselt number a rise with increasing in Reynolds number, but for increasing \( Re \) lead to the rate of heat transfer a drop for converging channel. Sherwood number appears to be increasing as the Reynolds number grows for converging channel and decreases for diverging channel. The behavior of Nusselt number in channel
opening is quite opposite for Reynolds number. Table 7 gives description of variation in heat transfer rate and mass transfer rate at the wall with the rising values of Hartmman number. Increase in heat transfer rate and mass transfer rate are observed for increasing Hartmman number $Ha$ in the case of both converging and diverging channels. Table 8 and Table 13 portray that the rate of heat transfer increasing for both diverging and converging channels an increase in Schmidt, Soret, Prandtl, Eckert, Dufour numbers and chemical reaction parameter respectively. Furthermore these tables for both channels clear that the rate of mass transfer at the walls decreases with an increase in Schmidt, Soret, Prandtl, Eckert, Dufour numbers and chemical reaction parameter.

**Table 1.** $Re = 30, Ha = 500, P_r = 0.1, D_f = 0.02, S_r = S_f = 0.1, \gamma = 0.04, \alpha = 3^\circ$.

<table>
<thead>
<tr>
<th>Approximation</th>
<th>$A_i$</th>
<th>$B_i$</th>
<th>$C_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 term</td>
<td>$-2.1999729$</td>
<td>$1.00040723$</td>
<td>$0.99999452$</td>
</tr>
<tr>
<td>2 term</td>
<td>$-2.1998869$</td>
<td>$1.00034130$</td>
<td>$0.99999045$</td>
</tr>
<tr>
<td>3 term</td>
<td>$-2.1993584$</td>
<td>$1.00034466$</td>
<td>$0.99999078$</td>
</tr>
<tr>
<td>4 term</td>
<td>$-2.1993648$</td>
<td>$1.00034465$</td>
<td>$0.99999077$</td>
</tr>
<tr>
<td>5 term</td>
<td>$-2.1993648$</td>
<td>$1.00034465$</td>
<td>$0.99999077$</td>
</tr>
<tr>
<td>6 term</td>
<td>$-2.1993648$</td>
<td>$1.00034465$</td>
<td>$0.99999077$</td>
</tr>
<tr>
<td>7 term</td>
<td>$-2.1993648$</td>
<td>$1.00034465$</td>
<td>$0.99999077$</td>
</tr>
<tr>
<td>8 term</td>
<td>$-2.1993648$</td>
<td>$1.00034465$</td>
<td>$0.99999077$</td>
</tr>
</tbody>
</table>

**Table 2.** $Re = 10, Ha = 110, P_r = 0.2, D_f = 0.2, S_r = S_f = 0.2, \gamma = 0.4, \alpha = -2^\circ$.

<table>
<thead>
<tr>
<th>Approximation</th>
<th>$A_i$</th>
<th>$B_i$</th>
<th>$C_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 term</td>
<td>$-1.89052474$</td>
<td>$1.00059930$</td>
<td>$0.99995131$</td>
</tr>
<tr>
<td>2 term</td>
<td>$-1.88870911$</td>
<td>$1.00065724$</td>
<td>$0.99992738$</td>
</tr>
<tr>
<td>3 term</td>
<td>$-1.88872478$</td>
<td>$1.00065993$</td>
<td>$0.99992621$</td>
</tr>
<tr>
<td>4 term</td>
<td>$-1.88872513$</td>
<td>$1.00065996$</td>
<td>$0.99992618$</td>
</tr>
<tr>
<td>5 term</td>
<td>$-1.88872513$</td>
<td>$1.00065996$</td>
<td>$0.99992618$</td>
</tr>
<tr>
<td>6 term</td>
<td>$-1.88872513$</td>
<td>$1.00065996$</td>
<td>$0.99992618$</td>
</tr>
<tr>
<td>7 term</td>
<td>$-1.88872513$</td>
<td>$1.00065996$</td>
<td>$0.99992618$</td>
</tr>
</tbody>
</table>

**Table 3.** Comparison between new scheme and $(R-K4)$ scheme for the analytical solutions $f(\eta), \beta(\eta), \phi(\eta)$ when $Re = 30, Ha = 500, \alpha = 3^\circ, P_r = 0.1, S_r = S_f = 0.1, \gamma = 0.04, D_f = 0.04$.

<table>
<thead>
<tr>
<th>$\eta$</th>
<th>$f(\eta)$</th>
<th>$(R-K4)$</th>
<th>$\beta(\eta)$</th>
<th>$(R-K4)$</th>
<th>$\phi(\eta)$</th>
<th>$\eta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>1.00000000</td>
<td>1.00000000</td>
<td>1.00034465</td>
<td>1.00034465</td>
<td>0.99999077</td>
<td>0.99999077</td>
</tr>
<tr>
<td>0.1</td>
<td>0.9890195</td>
<td>0.9890195</td>
<td>1.0003445</td>
<td>1.0003445</td>
<td>0.9999908</td>
<td>0.9999908</td>
</tr>
<tr>
<td>0.2</td>
<td>0.9562698</td>
<td>0.9562698</td>
<td>1.0003437</td>
<td>1.0003437</td>
<td>0.9999910</td>
<td>0.9999910</td>
</tr>
</tbody>
</table>
Table 4. Comparison between new scheme and $(R-K4)$ scheme for the analytical solutions $f(\eta)$, $\beta(\eta)$, $\phi(\eta)$ when $Re = 10$, $Ha = 110$, $P_r = 0.2$, $D_r = 0.2$, $S_r = S_s = 0.2$, $\gamma = 0.4$, $\alpha = -2$.

<table>
<thead>
<tr>
<th>$\eta$</th>
<th>$f(\eta)$</th>
<th>$(R-K4)$</th>
<th>$\beta(\eta)$</th>
<th>$(R-K4)$</th>
<th>$\phi(\eta)$</th>
<th>$(R-K4)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>1.0000000</td>
<td>1.0000000</td>
<td>1.0003447</td>
<td>0.9999908</td>
<td>0.9999908</td>
<td></td>
</tr>
<tr>
<td>0.1</td>
<td>0.9905498</td>
<td>0.9905498</td>
<td>1.0006599</td>
<td>0.9999262</td>
<td>0.9999262</td>
<td></td>
</tr>
<tr>
<td>0.2</td>
<td>0.9621219</td>
<td>0.9621219</td>
<td>1.0006587</td>
<td>0.9999282</td>
<td>0.9999282</td>
<td></td>
</tr>
<tr>
<td>0.3</td>
<td>0.9144866</td>
<td>0.9144866</td>
<td>1.0006544</td>
<td>0.9999308</td>
<td>0.9999308</td>
<td></td>
</tr>
<tr>
<td>0.4</td>
<td>0.8472720</td>
<td>0.8472720</td>
<td>1.0006434</td>
<td>0.9999346</td>
<td>0.9999346</td>
<td></td>
</tr>
<tr>
<td>0.5</td>
<td>0.7599801</td>
<td>0.7599801</td>
<td>1.0006202</td>
<td>0.9999399</td>
<td>0.9999399</td>
<td></td>
</tr>
<tr>
<td>0.6</td>
<td>0.6520119</td>
<td>0.6520119</td>
<td>1.0005776</td>
<td>0.9999469</td>
<td>0.9999469</td>
<td></td>
</tr>
<tr>
<td>0.7</td>
<td>0.5226997</td>
<td>0.5226997</td>
<td>1.0005067</td>
<td>0.9999560</td>
<td>0.9999560</td>
<td></td>
</tr>
<tr>
<td>0.8</td>
<td>0.3713515</td>
<td>0.3713515</td>
<td>1.0003962</td>
<td>0.9999675</td>
<td>0.9999675</td>
<td></td>
</tr>
<tr>
<td>0.9</td>
<td>0.1973059</td>
<td>0.1973059</td>
<td>1.0002326</td>
<td>0.9999820</td>
<td>0.9999820</td>
<td></td>
</tr>
<tr>
<td>1.0</td>
<td>0.0000000</td>
<td>0.0000000</td>
<td>1.0000000</td>
<td>1.0000000</td>
<td>1.0000000</td>
<td></td>
</tr>
</tbody>
</table>

Table 5. Variation in Nusselt number and Sherwood number with varying $Re$ when $Ha = 100$, $E_r = 1$, $S_r = S_s = 0.2$, $D_r = P_r = 0.5$, $\gamma = 0.1$.

<table>
<thead>
<tr>
<th>$Re$</th>
<th>$\alpha Nu = \beta'(1)$</th>
<th>$\alpha Sh = \phi'(1)$</th>
<th>$\alpha Nu = \beta'(1)$</th>
<th>$\alpha Sh = \phi'(1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.015725</td>
<td>-0.025891</td>
<td>0.015725</td>
<td>-0.025890</td>
</tr>
<tr>
<td>10</td>
<td>0.015501</td>
<td>-0.023525</td>
<td>0.016295</td>
<td>-0.028446</td>
</tr>
<tr>
<td>20</td>
<td>0.015403</td>
<td>-0.021341</td>
<td>0.017951</td>
<td>-0.031205</td>
</tr>
<tr>
<td>30</td>
<td>0.015357</td>
<td>-0.019331</td>
<td>0.023679</td>
<td>-0.034198</td>
</tr>
<tr>
<td>40</td>
<td>0.015334</td>
<td>-0.017490</td>
<td>0.047894</td>
<td>-0.037470</td>
</tr>
<tr>
<td>50</td>
<td>0.015300</td>
<td>-0.015814</td>
<td>0.166257</td>
<td>-0.041098</td>
</tr>
</tbody>
</table>
Table 6. Variation in Nusselt number and Sherwood number with varying $\alpha$ when $Re = 25, Ha = 100, E_c = 1, S_c = S_s = 0.2, D_r = P_r = 0.5, \gamma = 0.1$.  

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\alpha Nu = \beta'(1)$</th>
<th>$\alpha Sh = \phi'(1)$</th>
<th>$\alpha$</th>
<th>$\alpha Nu = \beta'(1)$</th>
<th>$\alpha Sh = \phi'(1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0^\circ$</td>
<td>0.670000</td>
<td>$-0.133889$</td>
<td>$0^\circ$</td>
<td>0.670000</td>
<td>$-0.133889$</td>
</tr>
<tr>
<td>$-2^\circ$</td>
<td>0.698657</td>
<td>$-0.120838$</td>
<td>$2^\circ$</td>
<td>0.645605</td>
<td>$-0.146431$</td>
</tr>
<tr>
<td>$-4^\circ$</td>
<td>0.728034</td>
<td>$-0.107761$</td>
<td>$4^\circ$</td>
<td>0.628332</td>
<td>$-0.158178$</td>
</tr>
<tr>
<td>$-6^\circ$</td>
<td>0.754221</td>
<td>$-0.095311$</td>
<td>$6^\circ$</td>
<td>0.620146</td>
<td>$-0.168901$</td>
</tr>
</tbody>
</table>

Table 7. Variation in Nusselt number and Sherwood number with varying $Ha$ when $Re = 25, E_c = 1, S_c = S_s = 0.2, D_r = P_r = 0.5, \gamma = 0.1$.  

<table>
<thead>
<tr>
<th>$\alpha = -5^\circ$</th>
<th>$\alpha Nu = \beta'(1)$</th>
<th>$\alpha Sh = \phi'(1)$</th>
<th>$\alpha = 5^\circ$</th>
<th>$\alpha Nu = \beta'(1)$</th>
<th>$\alpha Sh = \phi'(1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Ha$</td>
<td></td>
<td></td>
<td>$Ha$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>0.733702</td>
<td>$-0.106982$</td>
<td>0</td>
<td>0.601853</td>
<td>$-0.171361$</td>
</tr>
<tr>
<td>200</td>
<td>0.747908</td>
<td>$-0.096138$</td>
<td>200</td>
<td>0.642952</td>
<td>$-0.156091$</td>
</tr>
<tr>
<td>400</td>
<td>0.754835</td>
<td>$-0.086380$</td>
<td>400</td>
<td>0.678122</td>
<td>$-0.141432$</td>
</tr>
<tr>
<td>600</td>
<td>0.755444</td>
<td>$-0.077668$</td>
<td>600</td>
<td>0.706215</td>
<td>$-0.127721$</td>
</tr>
</tbody>
</table>

Table 8. Variation in Nusselt number and Sherwood number with varying $S_c$ when $Re = 25, E_c = 1, S_c = S_s = 0.2, D_r = P_r = 0.5, \gamma = 0.1$.  

<table>
<thead>
<tr>
<th>$\alpha = -5^\circ$</th>
<th>$\alpha Nu = \beta'(1)$</th>
<th>$\alpha Sh = \phi'(1)$</th>
<th>$\alpha = 5^\circ$</th>
<th>$\alpha Nu = \beta'(1)$</th>
<th>$\alpha Sh = \phi'(1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_c$</td>
<td></td>
<td></td>
<td>$S_c$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.5</td>
<td>0.744752</td>
<td>$-0.101423$</td>
<td>0.5</td>
<td>0.630359</td>
<td>$-0.163681$</td>
</tr>
<tr>
<td>1.0</td>
<td>0.749722</td>
<td>$-0.203428$</td>
<td>1.0</td>
<td>0.642586</td>
<td>$-0.329127$</td>
</tr>
<tr>
<td>1.5</td>
<td>0.754690</td>
<td>$-0.306011$</td>
<td>1.5</td>
<td>0.654809</td>
<td>$-0.496338$</td>
</tr>
<tr>
<td>2.0</td>
<td>0.759656</td>
<td>$-0.409171$</td>
<td>2.0</td>
<td>0.667028</td>
<td>$-0.665313$</td>
</tr>
</tbody>
</table>

Table 9. Variation in Nusselt number and Sherwood number with varying $S_s$ when $Re = 25, E_c = 1, S_c = S_s = 0.2, D_r = P_r = 0.5, \gamma = 0.1$.  

<table>
<thead>
<tr>
<th>$\alpha = -5^\circ$</th>
<th>$\alpha Nu = \beta'(1)$</th>
<th>$\alpha Sh = \phi'(1)$</th>
<th>$\alpha = 5^\circ$</th>
<th>$\alpha Nu = \beta'(1)$</th>
<th>$\alpha Sh = \phi'(1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_s$</td>
<td></td>
<td></td>
<td>$S_s$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.5</td>
<td>0.744695</td>
<td>$-0.050600$</td>
<td>0.5</td>
<td>0.630303</td>
<td>$-0.081579$</td>
</tr>
<tr>
<td>1.0</td>
<td>0.749573</td>
<td>$-0.101190$</td>
<td>1.0</td>
<td>0.642440</td>
<td>$-0.163447$</td>
</tr>
<tr>
<td>1.5</td>
<td>0.754450</td>
<td>$-0.151922$</td>
<td>1.5</td>
<td>0.654576</td>
<td>$-0.245755$</td>
</tr>
<tr>
<td>2.0</td>
<td>0.759327</td>
<td>$-0.202797$</td>
<td>2.0</td>
<td>0.666713</td>
<td>$-0.328503$</td>
</tr>
</tbody>
</table>

Table 10. Variation in Nusselt number and Sherwood number with varying $P_r$ when $Re = 25, E_c = 1, S_c = S_s = 0.2, D_r = P_r = 0.5, \gamma = 0.1$.  

<table>
<thead>
<tr>
<th>$\alpha = -5^\circ$</th>
<th>$\alpha Nu = \beta'(1)$</th>
<th>$\alpha Sh = \phi'(1)$</th>
<th>$\alpha = 5^\circ$</th>
<th>$\alpha Nu = \beta'(1)$</th>
<th>$\alpha Sh = \phi'(1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_r$</td>
<td></td>
<td></td>
<td>$P_r$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.5</td>
<td>0.741769</td>
<td>$-0.020314$</td>
<td>0.5</td>
<td>0.623021</td>
<td>$-0.032670$</td>
</tr>
<tr>
<td>1.0</td>
<td>1.487440</td>
<td>$-0.040499$</td>
<td>1.0</td>
<td>1.255751</td>
<td>$-0.065259$</td>
</tr>
<tr>
<td>1.5</td>
<td>2.237014</td>
<td>$-0.060706$</td>
<td>1.5</td>
<td>1.898192</td>
<td>$-0.097917$</td>
</tr>
<tr>
<td>2.0</td>
<td>2.990488</td>
<td>$-0.080937$</td>
<td>2.0</td>
<td>2.550341</td>
<td>$-0.130647$</td>
</tr>
</tbody>
</table>
In addition to this section highlights the major outcomes of the analytical study presented by new algorithm. The analysis of the variations in temperature and concentration profiles for different parameters is prepared. For that purpose, Figures are plotted for varying several parameters. Moreover we have been divided this section into two subsections follow as.

- **Channel divergent** \( (\alpha > 0) \).

  In **Figures 2-9** are plotted to show the behavior curves of velocity, temperature and concentration profiles under the impact of different physical parameters. An increasing the opening angle \( \alpha \) gives variations in velocity, temperature and concentration profiles as displayed in **Figure 2**. The influence of parameter \( \alpha \) on the velocity field \( f(\eta) \) for divergent channel causes more effect at the middle channel as well as it represents as a maximum position at the central line (when \( \eta = 0 \)). It also has least effect in part near the walls (when...
$\eta = -1, 1$). High temperature in the central region of the channel is conspicuously clear and the maximum temperature lies there too. While the results of angle opening $\alpha$ on concentration profile show that the increase in $\alpha$ gives a decreased concentration profile. The central portion of the channel is more effect, while the portion of the near of walls the concentration is less affected. Effects of the increase Reynolds number $Re$ in $f(\eta), \beta(\eta)$ and $\phi(\eta)$ are observed in Figure 3. The velocity field $f(\eta)$ is decreasing with increasing Reynolds number and clearly shows that the highest level reaches the central part (when $\eta = 0$). A rise in temperature profile $\beta(\eta)$ leads to an increase in $Re$ as for the concentration profile $\phi(\eta)$, it is in a state of decrease when there is an increase in $Re$. The prominent appearance in the central part can be observed in the lowest effect. The behavior of temperature distribution under the influence of Hartmann number $Ha$ can be seen from Figure 4. The velocity distribution remains unchanged with the increase Hartmann. This figure gives a clear picture of how a stronger magnetic field can lead to a change in the temperature of the fluid. But Hartmann number gives a simple rise to the concentration $\phi(\eta)$ although this increase is simple, however can be a way to control the concentration of the fluid. In Figure 5 the change in temperature and concentration with an increase in Soret number $Sr$ are plotted. A rise in temperature and a drop in concentration at the central portion of the channel are observed. In fact that stronger viscous forces are responsible for these phenomena. Impact of the Schmidt number $Sc$ on temperature and concentration are demonstrated in Figure 6, with note that the effect $Sr$ is similar to the effect of the Soret number $Sr$. Figure 7 is plotted the effect of growing values of Prandtl number on temperature profile. Arise in temperature of the fluid with increasing $Pr$ can be seen. Figure 8 explains that change in temperature with rising Eckert number $Ec$. Again the increase in temperature was observed to increase $Ec$. The effect of Eckert number on $\phi(\eta)$ for diverging channel, an increase in $Ec$ decreases the concentration profile. This shows, the strong viscous forces are responsible for a rise temperature of the fluid and this rise is a lot of effect at the central portion. In Figure 9 the changes of chemical reaction $\gamma$ and Dufour number $D_{\gamma}$ on $\beta$ and $\phi(\eta)$ are presented. Dufour number arises due to the concentration gradient present in energy equation. The main variations in temperature are in central portion of the channel, where the variation in temperature near the walls is almost negligible, which can be said, the stronger concentration results in higher temperature values. Also this figure demonstrates that the increasing of $\gamma$ lead to decrease the concentration profile.

• **Channel convergent** ($\alpha < 0$).

For the converging channel, the variations in velocity, temperature and concentration profile due to the varying parameters are depicted in Figures 10-17, the behavior of velocity and temperature for changing angle opening $\alpha$ and Reynolds number $Re$ is quite opposite to the behavior of $f(\eta), \beta(\eta)$ and $\phi(\eta)$ in diverging channel as seen in Figure 10 and Figure 11. On that other hand, Figures 11-17 tell the effect of Hartmann, Eckert.
Figure 2. $f(\eta)$, $\beta(\eta)$, $\phi(\eta)$ for the value $Re = 50$, $Ha = 100$, $Pr = D_{j} = 0.5$, $Ec = 1$, $Sc = Sr = 0.2$, $\gamma = 0.1$, when the angle $\alpha$ is varied.

Figure 3. $f(\eta)$, $\beta(\eta)$, $\phi(\eta)$ for the value $Ha = 100$, $Pr = D_{j} = 0.5$, $Ec = 1$, $Sc = Sr = 0.2$, $\gamma = 0.1$, $\alpha = 3^\circ$ when the Reynolds number $Re$ is varied.
Figure 4. $f(\eta), \beta(\eta), \phi(\eta)$ for the value $Re = 50$, $Pr = D_f = 0.5$, $Ec = 1$, $Sc = Sr = 0.2$, $\gamma = 0.1$, $\alpha = 3^\circ$ when the Harmann number $Ha$ is varied.

Figure 5. $f(\eta), \beta(\eta), \phi(\eta)$ for the value $Re = 50$, $Ha = 100$, $Pr = D_f = 0.5$, $Ec = 1$, $Sc = 0.2$, $\gamma = 0.1$, $\alpha = 3^\circ$ when the Soret number $Sr$ is varied.
Figure 6. $f(\eta), \beta(\eta), \phi(\eta)$ for the value $Re = 50, Ha = 100, Pr = D_f = 0.5, Ec = 1, Sr = 0.2, \gamma = 0.1, \alpha = 3^\circ$ when the Schmidt number $Sc$ is varied.

Figure 7. $f(\eta), \beta(\eta), \phi(\eta)$ for the value $Re = 50, Ha = 100, D_f = 0.5, Ec = 1, Sc = Sr = 0.2, \gamma = 0.1, \alpha = 3^\circ$ when the Prandtl number $Pr$ is varied.

Figure 8. $\beta(\eta), \phi(\eta)$ for the value $Re = 50, Ha = 100, Pr = D_f = 0.5, Sc = Sr = 0.2, \gamma = 0.1, \alpha = 3^\circ$ when the Eckert number $Ec$ is varied.
Figure 9. \( \beta(\eta), \phi(\eta) \) for the value \( Re = 50, \ Ha = 100, \ Pr = 0.5, \ Ec = 1, \ Sc = Sr = 0.2, \) \( \gamma = 0.1, \ \alpha = 3' \) when the Dufour number \( D_f \) and chemical reaction parameter \( \gamma \) are varied.

Figure 10. \( f(\eta), \beta(\eta), \phi(\eta) \) for the value \( Ha = 100, \ Pr = D_f = 0.5, \ Ec = 1, \ Sc = Sr = 0.2, \) \( \gamma = 0.1, \ \alpha = -3' \) when the Reynolds number \( Re \) is varied.
Figure 11. $f(\eta), \beta(\eta), \phi(\eta)$ for the value $Re = 50, Ha = 100, Pr = D_r = 0.5, Ec = 1, Sc = Sr = 0.2, \gamma = 0.1$, when the angle $\alpha$ is varied.

Figure 12. $f(\eta), \beta(\eta), \phi(\eta)$ for the value $Re = 30, Pr = D_r = 0.5, Ec = 1, Sc = Sr = 0.2, \gamma = 0.1, \alpha = -3$ when the Harmann number $Ha$ is varied.
Figure 13. $f(\eta), \beta(\eta), \phi(\eta)$ for the value $Re = 50, Ha = 100, Pr = D_f = 0.5, Ec = 1, Sc = 0.2, \gamma = 0.1, \alpha = -3^\circ$ when the Soret number $Sr$ is varied.

Figure 14. $f(\eta), \beta(\eta), \phi(\eta)$ for the value $Re = 50, Ha = 100, Pr = D_f = 0.5, Ec = 1, Sr = 0.2, \gamma = 0.1, \alpha = -3^\circ$ when the Schmidt number $Sc$ is varied.

Figure 15. $f(\eta), \beta(\eta), \phi(\eta)$ for the value $Re = 50, Ha = 100, D_f = 0.5, Ec = 1, Sc = Sr = 0.2, \gamma = 0.1, \alpha = -3^\circ$ when the Prandtl number $Pr$ is varied.
Figure 16. $f(\eta), \beta(\eta), \phi(\eta)$ for the value $Re = 50$, $Ha = 100$, $Pr = D_j = 0.5$, $Sc = Sr = 0.2$, $\gamma = 0.1$, $\alpha = -3$ when the Eckert number $Ec$ is varied.

Figure 17. $f(\eta), \beta(\eta), \phi(\eta)$ for the value $Re = 50$, $Ha = 100$, $Pr = 0.5$, $Ec = 1$, $Sc = Sr = 0.2$, $\gamma = 0.1$, $\alpha = -3$ when the Dufour number $D_j$ and chemical reaction parameter $\gamma$ are varied.

Prandtl and Dufour numbers on the temperature profile is similar for the effect in diverging channel. Also these Figures demonstrated the concentration profile possess same effect when there are changing in Hartmann number, Schmidt number, Soret number and chemical reaction parameter. Physical explanations can be provided the temperature profile show that the temperature at the central region increases with increasing angle opening. This can be attributed to that for fixed Reynold number, increasing angle opening leads to increase the cross-sectional flow area. This in turn leads to decrease the flow velocity and this mean the flow will be decelerated. Therefore, the heat dissipation will be reduced which leads to increase the temperature of the fluid. The inertia force of the fluid increases with increasing Reynold number which leads to enhance the parabolic behavior (increasing central temperature) for diverging channel with the opposite view in converging channel. Hartmann number increase in this case Lorentz force is also increasing for diverging and converging.
channels. This force imports extra drag to the flow. Therefore the temperature profile becomes more flat which means decreasing the temperature within the central region. The thin boundary layers that are near to the wall lead to that the temperature gradient at the highest level. Furthermore to the existence of the thick boundary layer in central region lead to that the temperature gradient at low level. Eckert number increases with increased temperature and thus produces an increase in kinetic energy. The change of the temperature profile with Prandtl number, and the increase of temperature with Prandtl number result from increasing of the momentum diffusivity. The Dufour number shows to increase shows less effect on temperature, increasing Dufour leads to increase the thermal energy of the fluid thus the temperature increase. The rate of most chemical reactions increases with a decrease the concentration of reactants. As for temperature, it increases if a reaction is heat-emitting and decreases when the reaction absorbs heat.

7. Conclusions

In this paper, the unsteady and two-dimensional magneto hydrodynamic (MHD) flow of viscous fluid in a channel with non-parallel plates is studied analytically using a new algorithm. The solution obtained by new algorithm is an infinite power series for appropriate initial approximation. The construction of this algorithm possessed good convergent series and the convergence of the results is explicitly shown. Graphical results and tables are presented to investigate the influence of physical parameters on velocity, temperature and concentration. Analysis of the converge confirms that the new algorithm is an efficient technique as compared to Range-Kutta algorithm with help of Shooting algorithm. The new algorithm that is widely applied to solve ordinary differential equations lead to the solutions resulting from this algorithm is compatible with numerical solution. Effects of different parameters on temperature and concentration profiles are analyzed and presented graphically. The conclusions can be drawn from the analysis presented:

- The behavior of temperature and concentration profiles are the same results \( \alpha, \ Re, E, P_c, D_f \) for diverging channel.
- Hartmann number \( Ha \) can be used to reduce the temperature of the flow fluid. Also, concentration of the fluid can also be controlled by employing a strong magnetic field.
- For converging channel, the variations in temperature are opposite for diverging channel with an increase in channel opening \( \alpha \) and \( Re \).
- For diverging channel, Nusselt number drops with a rise in angle opening and increases with a rise in Reynolds number and behaves oppositely for convergent channel.
- Increase in heat transfer rate is observed for increasing \( P_c, E, S, S_c, D_f, \) and \( \gamma \) in both channels.
- Increase in Reynolds number and Angle opening gives a drop to mass trans-
fer rate for diverging channel and a rise for the converging channel.

- The rate of mass transfer decreased for both channels with an increase in Schmidt, Soret, Prandtl, Eckert, Dufour numbers and chemical reaction parameter.
- Results obtained by new algorithm are in excellent agreement with numerical solution obtained.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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