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Existence of Competitive Equilibria without Standard Boundary Behavior

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Abstract

We study the existence of competitive equilibria when the excess demand function fails to satisfy the standard boundary behavior. We introduce alternative boundary conditions and we examine their role in proving the existence of strictly positive solutions to a system of non-linear equations (competitive equilibium prices). In addition, we slightly generalize a well-known theorem on the existence of maximal elements, and we unveil the link between the hypothesis of our theorem and one of the boundary conditions introduced in this work.

Keywords: Excess Demand Function, Competitive Equilibrium, Set-Valued Functions, Selections, Fixed Points, Maximal Elements

1. Introduction

The purpose of this paper is twofold. In general, we provide a set of sufficient conditions in order for a system of *N* non-linear equations in *N* unknown to possess a strictly positive solution. In particular, since we deal with excess demand functions (vector fields) defined on suitable price-domains, from the standpoint of Economic Theory the natural interpretation of our results is the existence of a price-vector that clears every markets that are assumed perfectly competitive. In other words, the existence of a vector of strictly positive prices such that demand equals supply on every market. Such state of the economy is called competitive equilibrium.

Note that the literature about the existence of competitive equilibria is vast, and a survey of the numerous and remarkable contributions would hardly do justice to them. So, why yet another paper on the existence of competitive equilibria? To answer this question, first it is worth recalling briefly the established literature.

Let $Z:\mathbb{R}^N_{++}\to\mathbb{R}^N$ be a function satisfying the following properties:

1) Z is continuous.

2) $Z(p)=Z(\lambda p)$ for all $p\in\mathbb{R}^N_{++}$ and all $\lambda>0$ (homogeneity of degree zero).

3) $p\cdot Z(p)=0$ for all $p\in\mathbb{R}^N_{++}$ (Walras law)

4) There exists a $s>0$ such that $Z_i(p)>-s$ for all $p\in\mathbb{R}^N_{++}$ and all $i=1,2,\cdots,N$.

5) If $p_n\to p$, where $p\neq 0$ and $p_i=0$ for some *i*, then $\operatorname{Max}\{Z_1(p_n),\cdots,Z_N(p_n)\}\to\infty$.

Recall that any finite-dimensional economy with continuous, strictly convex and strictly monotonic preferences, and with production sets that are closed, strictly convex, bounded above, and such that a strictly-positive aggregate consumption bundle is producible from the aggregate endowment, gives rise to an aggregate excess demand function enjoying the above properties (see, e.g., Aliprantis *et al.* [1], Arrow and Hahn [2], Mas-Colell *et al.* [3]). Notice that property 5) is the standard boundary behavior. Clearly, a competitive equilibrium price vector is a $p^*\in\mathbb{R}^N_{++}$ such that $Z(p^*)=0$.

Obviously, with constant returns-to-scale production, the production set is neither strictly convex nor bounded above. We borrow the formulation of the economy and the definition of competitive equilibrium from Geanakoplos [4].

Specifically, assume that preferences are continuous, strictly convex and strictly monotonic. Then, a constant returns-to scale economy can be formalized by an aggregate net demand function-technology pair (Z,Y), where $Z:\mathbb{R}^N_{++}\to\mathbb{R}^N$ and $Y\subset\mathbb{R}^N$ is a closed and convex cone that allows for free disposal. Clearly, we must restrict attention to the set of $p\in\mathbb{R}^N_{++}$ such that $pY\leq 0$, *i.e.*, $py\leq 0$ for all $y\in Y$. One can assume, without much loss in generality, that the set of $p\in\mathbb{R}^N_{++}$

such that $pY \leq 0$ is nonempty.[1] Under these assumptions, Z still satisfies properties 1) through 5) above. A competitive equilibrium for a constant returns-to-scale production economy can now be defined as a price $\overline{p} \in \mathbb{R}^N_{++}$ such that $Z(\overline{p}) \in Y$ and $\overline{p}Y \leq 0$.

To summarize: if one is dealing with general production economies, then Z defined above is the (production-inclusive) aggregate excess demand function. If, rather, one is dealing with constant returns-to-scale economies, then Z is the aggregate net demand function that stems solely from consumers' preference maximization. In either cases, when preferences are continuous, strictly convex and strictly monotonic, Z satisfies properties 1) through 5) above.

Now, let us turn to our contribution in this paper: we weaken the continuity of the excess demand function, we drop the standard boundary behavior (replacing it with alternative boundary conditions), and we prove a new mathematical theorem which is then utilized to study the existence of competitive equilibria. More precisely, following in Tian's footsteps [5], we do not assume that the aggregate excess demand function is lower semicontinuous, whereas in the literature the excess demand function is typically continuous (see above).

Moreover, we address hypothetical economies in which the standard boundary behavior of the aggregate excess demand function (property 5) above) is not necessarily satisfied, and we prove two theorems on the existence of competitive equilibria. Indeed, to motivate our work, in Section 3 we exhibit two model-economies: in the former, the standard boundary behavior fails. In the latter, the sufficient conditions for the standard boundary behavior are violated, and therefore the standard boundary behavior may or may not be satisfied. On the other hand, it is well-known that, whenever the excess demand function is defined on a relatively-open price-domain[2] (as it is the case in this paper), some sort of boundary conditions are needed to demonstrate the existence of a zero for such a function. In fact, loosely put, proper boundary conditions remedy the lack of compactness of the price-domain, and thus enable the application of specific fixed-point theorems. For these reasons, we introduce two alternative boundary conditions and we study how they are related to one another. Our alternative assumptions on the boundary behavior formalize inward-pointing conditions of the aggregate excess demand function. The former condition is formalized by means of the projection mapping (see Section 2), and the latter by means of convex combinations. This comes in handy because this method of modelling the boundary conditions enables us to retain the central idea of the first existence theorem in the proof of the second one, which thus becomes a variant of the first theorem. Hence, this approach offers a somewhat unified framework for two seemingly different problems

Finally, in the context of Hilbert spaces, we prove a slight generalization of a theorem on the existence of maximal elements due to Yannelis and Prabhakar [6] (Theorem 5.1). Interestingly, one of the assumptions in our theorem lends itself to be interpreted in terms of boundary behavior 1 defined in Section 2. This strong analogy enables us to prove again the existence of a strictly positive equilibrium price vector as a short corollary of our new theorem.

Clearly, in this work we treat the excess demand function as the primitive of the economy at hand. This may be regarded by economists as an unconventional route to proving the existence of competitive equilibria. Nevertheless, the approach we follow, based directly upon the excess demand function, is well-suited to highlight the mathematical aspects of our contribution. Indeed, in Section 4 we shall develop a unifying treatment that can handle both general production economies and constant returns-to-scale economies. We shall detail the proof of the existence of competitive equilibria only for the former, since virtually the same method may be used to analyze constant returns-to-scale production economies as well.

The lay-out of the paper is as follows: In Section 2, we set our notation and we develop some background. Also, we introduce two alternative boundary conditions on the excess demand function. We also explain how our conditions relate to the literature we know of, and finally we state the main mathematical theorem that will be used in this paper. It is a celebrated selection theorem due to Michael [7]. In Section 3, we construct two model economies for which the standard boundary behavior of the excess demand function is not necessarily satisfied. This justifies our interest in proving existence of competitive equilibria under alternative boundary conditions. In Section 4, we prove two theorems on the existence of competitive equilibria or, rather, on the existence of a strictly positive solution to a finite system of non-linear equations. In the process, we also compare our approach to the relevant literature. In subsection 4.1, we prove a theorem on the existence of maximal elements for correspondences[3] whose domain is different from the range, and domain and range are both subset of a Hilbert space. It is a natural generalization of Yannelis and Prabhakar [6], and thus it may be interesting in its own right. In subsection 4.2 we employ our new theorem to study the existence of competitive equilibria. In Section 5 we conclude and we outline a few directions for future research.

[1]See [4].

[2]For instance, when the excess demand function stems from consumers with strictly-monotonic preferences.

[3]Throughout this paper, by correspondence we mean a set-valued function.

2. Background, Notation, and Boundary Conditions

Let $Z:\mathbb{R}^N_{++}\to\mathbb{R}^N$ be the aggregate excess demand function of a general production economy. The question we are after in this paper is:

Problem: *Does there exist some* $p^*\in\mathbb{R}^N_{++}$ *such that* $Z(p^*)=0$?

To set the stage for the subsequent analysis, let $\Delta=\left\{p\in\mathbb{R}^N_{+}:p\cdot\mathbf{1}=1\right\}$, where 1 is the $N-$dimensional vector $(1,1,\cdots,1)$, and let $Int\Delta=\left\{p\in\mathbb{R}^N_{++}:p\cdot\mathbf{1}=1\right\}$. Also, given any $\varepsilon>0$, let $\Delta_\varepsilon=\left\{p\in\Delta:p\geq\varepsilon\mathbf{1}\right\}$. Finally, given any $\varepsilon>0$, we let $\partial\Delta_\varepsilon=\left\{p\in\Delta_\varepsilon:p_i=\varepsilon \text{ for some } i=1,2\cdots,N\right\}$.

Normalization: Clearly, since we are searching for a $p^*\in\mathbb{R}^N_{++}$ such that $Z(p^*)=0$, by virtue of property 2) above we can restrict the domain of Z to $Int\Delta$. We choose this normalization over other admissible ones (for example, one might have Z defined on the intersection of the unit sphere with $\mathbb{R}^N_{++}$) because convexity is very handy in our proofs.

Recall that the standard boundary behavior (property 5)) plays a crucial role in proving the existence of a strictly positive vector, p^*, such that $Z(p^*)=0$ (see, e.g., Aliprantis *et al.* [1], and Mas-Colell *et al.* [3]). When production exhibits constant returns-to-scale, the standard boundary behavior of the aggregate net demand still comes into play to prove the existence of competitive equilibria. For details see, for example, Geanakoplos [4].

For our purposes it will be convenient to use a formulation of the standard boundary behavior which does not involve asymptotic conditions. To this end, the following result is a straightforward consequence of the standard boundary behavior of the excess demand function (and of property 4). It is not difficult to prove:

Proposition 2.1: *Let* $Z:Int\Delta\to\mathbb{R}^N$ *be a map satisfying properties* 4) *and* 5) *listed above. Assume that* $(p_n)_{n=1}^\infty\subseteq Int\Delta$ *is such that* $p_n\to p\in\Delta$, *with* $p_i=0$ *for some i. Then, for any* $\pi\in Int\Delta$ *there exists a* n *such that* $\pi\cdot Z(p_n)>0$.

As the examples in Section 3 demonstrate, one can conceive of an economy for which the standard boundary behavior may *fail.*

Therefore, we still wish to provide an affirmative answer to Problem above, but we have to drop the standard boundary behavior hypothesis. To this end, we shall now introduce two alternative boundary conditions for the aggregate excess demand function, but first we need to provide some mathematical background.

For any $\varepsilon>0$, define the restriction of the (metric) projection mapping to Δ, that is $P:\Delta\to\Delta_\varepsilon$. It is known that P is well-defined and continuous, and that $P(\pi)=\pi$ for all $\pi\in\Delta_\varepsilon$ (see, e.g., Aliprantis and Border [8], pp. 247-249). It's easy to see that $P(\pi)\in\partial\Delta_\varepsilon$ for each $\pi\in\Delta\setminus\Delta_\varepsilon$. We are now ready to introduce:

Boundary behavior 1: *There exists a* $\varepsilon>0$ *such that* $p\cdot Z\left(P(p)\right)\leq 0$ *for all* $p\in\Delta$.

Remark 2.1: To the best of our knowledge, the projection mapping was introduced in Economics by Todd [9] in a general equilibrium model of production with activity analysis. It was used also by Kehoe [10]. In this paper we utilize the projection function in a different manner. Basically, boundary behavior 1 formalizes the assumption that the excess demand function is "inward-pointing" on Δ_ε. A different "inward-pointing" condition on the excess demand function was introduced by Neuefeind [11], and Husseinov [12]. We stress, however, that Neuefeind works with continuous excess demand functions, whereas in the next section we are able to address Problem above without assuming continuity of the excess demand function.

Boundary behavior 2: *There exists a* $\varepsilon>0$ *such that if* $\pi\in\partial\Delta_\varepsilon$ *and* $p\cdot Z(\pi)>0$, *with* $p\in\Delta$, *then there is a* $0<\lambda<1$ *such that* $\lambda p+(1-\lambda)\pi\in\Delta_\varepsilon$.

It is logical to investigate the relationship between boundary behavior 1 and boundary behavior 2. The next theorem, due to Iryna Topolyan[4], serves this purpose:

Theorem 2.1 (Topolyan): *If* $Z:Int\Delta\to\mathbb{R}^N$ *satisfies properties* 3) *above*, *and boundary behavior* 2, *then it also satisfies boundary behavior* 1.

Proof: Assume that Z satisfies boundary behavior 2. Let $\varepsilon>0$ be such that if $\pi\in\partial\Delta_\varepsilon$ and $p\cdot Z(\pi)>0$, with $p\in\Delta$, then there is a $0<\lambda<1$ such that $\lambda p+(1-\lambda)\pi\in\Delta_\varepsilon$. Begin by noticing that, by 3), for each $p\in\Delta_\varepsilon$ we have that $p\cdot Z(P(p))=0$. We claim that $p\cdot Z(P(p))\leq 0$ for all $p\in\Delta$. Indeed, suppose, by contradiction, that there exists a $\tilde{p}\in\Delta$ such that $\tilde{p}\cdot Z(P(\tilde{p}))>0$. Then, $\tilde{p}$ must lie in $\Delta\setminus\Delta_\varepsilon$. Put $\pi=P(\tilde{p})\in\partial\Delta_\varepsilon$. By assumption, there exists $0<\lambda<1$ such that $\lambda\tilde{p}+(1-\lambda)\pi\in\Delta_\varepsilon$. Hence, by definition of projection mapping, $\|\tilde{p}-\pi\|\leq\|\tilde{p}-(\lambda\tilde{p}+(1-\lambda)\pi)\|$, but the latter inequality implies that $\lambda\leq 0$, which is impossible. The proof is complete.

In the proofs presented in Section 4 we shall invoke the following selection theorem due to Michael [7], (Theorem 3.1'''):[5]

Theorem 2.2 (Michael)*: Let X be a perfectly normal* T_1 *topological space*, *and let* Y *be a separable Banach space. Let* $D(Y)$ *be the collection of all nonempty and convex subsets of Y which are either finite-dimensional,*

[4]Personal communication.

[5]For our purposes it will suffice to state the theorem as done in Tian [5].

or closed, or have an interior point. Suppose that $F: X \to D(Y)$ *is a lower hemicontinuous correspondence. Then, there exists a continuous function* $f: X \to Y$ *such that* $f(x) \in F(x)$ *for all* $x \in X$.

3. Illustrative Examples

We now present two examples of economies whose excess demand function does not necessarily satisfy property 5). The former example is very simple, admittedly artificial, but its virtue is to convey the main ideas. As for the latter, we refer the reader to Impicciatore *et al.* [13]. The key ingredient in both models is that not all of the goods and services traded affect consumers preferences, while agents are endowed with strictly-monotonic preferences over a subset of commodities and services.

Example 1: Consider a competitive economy with one representative consumer and one representative firm. There are two commodities. A consumption good, denoted by c, which is produced by the firm with linear technology and consumed by the consumer. We let p be the price of the consumption good. The second commodity, denoted by x, is a production input, owned by the consumer, which is available in fixed and limited quantity, say $\bar{x}$. The production input is not produced and is supplied by the consumer to the firm. We let w be the price of the input production services. The consumer is endowed with $\bar{x}$ units of the production input, but she is not endowed with the consumption good. Consumer's preferences are represented by the utility function $u: \mathbb{R}_+ \to \mathbb{R}$ which is a function only of the consumption good and is assumed to be strictly increasing. The production technology is such that x units of the production input yield ax units of the consumption good, with $a > 0$.

Thus, the firm's profit-maximization problem is $\max_{x \ge 0} pax - wx$, and the consumer's preference maximization problem can be described as

$$\begin{aligned}&\max_{(c,x)\ge 0} u(c)\\ &s.t.\\ &pc = wx\\ &0 \le x \le \bar{x}\end{aligned}$$

It is very easy to see that, if a competitive equilibrium price vector (p^*, w^*) exists, then we must have $(p^*, w^*) \gg 0$. So, in equilibrium prices must be strictly positive. This is why in Section 4 we will be concerned with the existence of strictly positive equilibrium price vectors. It is routine matter to check that the consumer's net demand function is: $Z(p,w) = \left(\frac{w}{p}\bar{x}, -\bar{x}\right)$.

Now we show that for this economy the standard boundary behavior of the net demand function does not hold. This is why in Section 4 we shall put forward a method for proving existence of competitive equilibria under alternative boundary conditions of the aggregate net demand functions. To see that the standard boundary behavior fails, note that in view of Proposition 2.1 above it will suffice to exhibit a price vector $(\hat{p}, \hat{w}) \gg 0$ such that, for each $\varepsilon > 0$ with $\hat{p} > \varepsilon$ and $\hat{w} > \varepsilon$, there exists a vector $(p,w) = (1-\varepsilon, \varepsilon)$ satisfying

$$(\hat{p}, \hat{w}) \cdot \left(\frac{\varepsilon}{1-\varepsilon}\bar{x}, -\bar{x}\right) \le 0.$$

Indeed, simple calculations reveal that, if we take $(\hat{p}, \hat{w}) = \left(\frac{1}{2}, \frac{1}{2}\right)$, then $\left(\frac{1}{2}, \frac{1}{2}\right) \cdot \left(\frac{\varepsilon}{1-\varepsilon}\bar{x}, -\bar{x}\right) < 0$ for any $0 < \varepsilon < \frac{1}{2}$.

One might ask what goes wrong in this model, with regard to the standard boundary behavior. Basically, in this example what causes the standard boundary behavior of the net demand function to fail is the presence of a non-reproducible input available in fixed quantity. Also, consumer's preferences over both goods are convex, but not strictly-convex, and monotonic, but not strictly-monotonic (see Section 1).

Example 2: Another example one might think of, deals with a mainstream reformulation of the original Walras' theory of savings and capital accumulation. We refer the reader to Impicciatore *et al.* [13] for the details. Here we just sketch informally a few elements of the model.

Time-horizon is finite with two periods, $t = 0,1$. In each period there are C consumption goods, and J labor/leisure services; there are M capital goods, as well as a consistent number of capital goods' production services.

There exists a complete array of Arrow-Debreu forward markets open at $t = 0$. Consumers purchase capital goods produced at $t = 0$ in order to sell their production services at $t = 1$. We assume that consumers have to store capital goods in order to supply their services to the production sector in the next period.

There is a finite number H of consumers, indexed by $h = 1, \cdots, H$. We assume that capital goods are not consumed, nor do they affect agents' preferences. Hence, consumers' preferences are defined on the consumption set $X_h = \mathbb{R}_+^{2(C+J)}$. Preferences on X_h are continuous, strictly increasing and strictly quasi-concave.

At $t = 0$ each consumer is endowed with labor/leisure services and capital goods. Similarly, at $t = 1$ each consumer is endowed with labor/leisure services. At $t = 0$ services from owned capital goods are inelastically supplied. Capital goods purchased at $t=0$ are stored for one period; at $t=1$ their services are supplied to the production sector.

Since we are concerned with the behavior of the aggregate net demand function, we shall omit the formalization of and the assumptions on the production sector.

We assume that consumers are endowed with individual storage technologies, formalized as follows: for each $h \in \{1,2,\cdots,H\}$ and each capital good $m \in \{1,2,\cdots,M\}$, the storage function $\sigma_m^h : \mathbb{R}_+ \to \mathbb{R}_+$ maps any feasible quantity of the capital good purchased and stored at $t=0$ into the quantity of services available to the production sector at $t=1$.

A capacity constraint on the storage technology is in place. That is, for each $h \in \{1,2,\cdots,H\}$ and each $m \in \{1,2,\cdots,M\}$, there exists a $\hat{k}_m^h > 0$ such that $\sigma_m^h : \left[0, \hat{k}_m^h\right] \to \mathbb{R}_+$.

Each consumer takes prices as given, and chooses a bundle of consumption goods, labor/leisure and capital goods that maximizes his utility. The consumer is subject to the storage capacity constraint and the budget constraint.

The authors then define the notion of virtual aggregate net demand function, which is instrumental in proving existence of equilibria. It is worth underscoring that in this model the virtual aggregate net demand function may fail to satisfy the standard boundary behavior. To see this, note that we may well think of each consumer as being equipped with *monotonic*, and *convex* preferences defined over *every* goods and services traded in the economy. On the other hand, we know from Section 1 that sufficient conditions for the standard boundary behavior are *strictly convex* and *strictly monotonic* preferences. In other words, the sufficient conditions for the standard boundary behavior are violated. Furthermore, suppose we are given an arbitrary sequence (π_n) of strictly positive prices that converges to $\pi \neq 0$, where π belongs to the boundary of $\mathbb{R}_+^N$. By the capacity constraint on storage, the demand for capital goods is bounded above, and one can prove that at least one consumers' income is finite and positive. In a nutshell, these are the reasons why the virtual aggregate net demand function does not necessarily satisfy the standard boundary behavior. Therefore, as we pointed out above, we seek a method to prove existence of equilibria that does not hinge on the standard boundary behavior.

4. Main Existence Theorems

Suppose we are given a function $Z : Int\Delta \to \mathbb{R}^N$. In this section we are concerned with the existence of some $p^* \in Int\Delta$ such that $Z(p^*) = 0$.

We begin by making the following assumption:

Assumption 4.1: $Z : Int\Delta \to \mathbb{R}^N$ *satisfies the Walras law*, *that is* $p \cdot Z(p) = 0$ *for all* $p \in Int\Delta$. *Also*, *Z satisfies boundary behavior* 1. *Moreover*, *Z is such that the correspondence*

$$U : \Delta_\varepsilon \twoheadrightarrow \Delta \text{ defined by } U(\pi) := \left\{ p \in \Delta : p \cdot Z(\pi) > 0 \right\} \tag{4.1}$$

is lower hemicontinuous.[6]

Remark 4.1: If $Z : Int\Delta \to \mathbb{R}^N$ is lower semicontinuous, then it's easy to see that $U : \Delta_\varepsilon \twoheadrightarrow \Delta$ is lower hemicontinuous. Thus, our assumption is weaker than assuming lower-semicontinuity of *Z*.

Before we state and prove Theorem 4.1 below, let us comment on the strategy of the proof, and on the relation to the established literature. By and large, the first part of our proof is inspired by Tian [5] and Ewald's approach to proving the basic Ky-Fan theorem (see Ewald [14], Lemma 3.6.1, and Theorem 3.6.5). Our proof, though, departs from Ewald's in two significant ways. First of all, the correspondences defined in [14] are assumed to have open lower sections. In contrast, we posit the assumption of lower hemicontinuity (see Assumption 4.1 above) because, in general, it is weaker. Moreover, we assume lower hemicontinuity to facilitate a comparison with the approach followed by Tian [5], and because we believe it is a more natural assumptions when dealing with Economic models.

Secondly, since Ewald deals with correspondences with open lower sections, he finds it natural to employ the finite-dimensional version of Browder's selection theorem. In contrast, we work directly with a lower hemicontinuous correspondence, *U* defined in (4.1), and therefore we shall employ Michael's selection theorem (Theorem 2.2 above). Incidentally, Theorem 2.2 above is a generalization of the finite-dimensional version of Browder's selection theorem used by Ewald.

Theorem 4.1: *If Assumption* 4.1 *holds*, *then there exists a* $\hat{\pi} \in \Delta_\varepsilon$ *such that* $Z(\hat{\pi}) = 0$.

Proof: Clearly, $U : \Delta_\varepsilon \twoheadrightarrow \Delta$ is convex-valued. By Walras law we have that

$$\pi \notin U(\pi) \text{ for all } \pi \in \Delta_\varepsilon \tag{4.2}$$

Put $W = \left\{ \pi \in \Delta_\varepsilon : U(\pi) \neq \varnothing \right\}$. If $W = \varnothing$, then it's easy to see that we are done. So, assume, without loss in generality, that $W \neq \varnothing$. Pick any arbitrary $\tilde{\pi} \in W$. By definition of W, there exists a $\tilde{\tau} \in U(\tilde{\pi})$. Now, take any open neighborhood of $\tilde{\tau}$ in Δ, say $\mathcal{N}(\tilde{\tau})$. Clearly, $\mathcal{N}(\tilde{\tau}) \cap U(\tilde{\pi}) \neq \varnothing$. Since *U* is lower hemicontinuous, there exists an open neighborhood of $\tilde{\pi}$ in Δ_ε,

[6]ε appearing in the definition of the above correspondence is the ε involved in the definition of boundary behavior 1.

say $V(\tilde{\pi})$, such that

$$U(\pi)\cap\mathcal{N}(\tilde{\tau})\neq\varnothing \text{ for every } \pi\in V(\tilde{\pi}). \qquad (4.3)$$

It follows from (4.3) that $V(\tilde{\pi})\subset W$. Hence, W is open in Δ_{ε}.[7] Next, consider the restriction of U to W, that is

$$U|_W : W \twoheadrightarrow \Delta.$$

It should be clear that $U|_W$ meets the conditions of Theorem 2.2 above. Therefore, $U|_W$ admits a continuous selection, that is there exists a continuous function

$$f: W\to\Delta \text{ such that } f(\pi)\in U(\pi) \text{ for all } \pi\in W.$$

Now define a new correspondence $\gamma:\Delta_{\varepsilon}\twoheadrightarrow\Delta$ as follows:

$$\pi\mapsto\begin{cases}\{f(\pi)\} & \text{if } \pi\in W\\ \Delta & \text{if } \pi\notin W\end{cases}$$

Clearly, γ is convex and compact valued. We wish to prove that γ is upper hemicontinuous. To this end, by the closed graph theorem it will suffice to show that γ has closed graph. To see this, let $\{(\pi_n,\tau_n)\}$ be a sequence satisfying $\tau_n\in\gamma(\pi_n)$, for all n, $\pi_n\to\pi$, and $\tau_n\to\tau$. We must show that $\tau\in\gamma(\pi)$. So, we consider two cases: 1) If $\pi\notin W$, then $\gamma(\pi)=\Delta$. Since Δ is closed, it follows at once that $\tau\in \Delta=\gamma(\pi)$. 2) $\pi\in W$: because $\pi_n\to\pi$ and W is open in Δ_{ε} (see above), there is a N such that $\pi_n\in W$ for all $n\geq N$. Thus, for all $n\geq N$ we have that $\tau_n\in\gamma(\pi_n)=f(\pi_n)$. Hence, by continuity of f, we get

$$\tau=\lim_{n\to\infty}f(\pi_n)=f(\pi).$$

Thus, $\tau\in\{f(\pi)\}=\gamma(\pi)$. Now, take the composition of γ with P (P is the projection function defined in Section 2), and define the new correspondence

$$\gamma\circ P:\Delta\twoheadrightarrow\Delta$$

Since P is continuous and γ is upper hemicontinuous, $\gamma\circ P$ is upper hemicontinuous. By construction, $\gamma\circ P$ is non-empty valued, convex-valued and closed-valued. It then follows from Kakutani's fixed point theorem that there exists a $\pi^*\in\Delta$ such that $\pi^*\in\gamma(P(\pi^*))$. Now, set $P(\pi^*)=\hat{\pi}$. We claim that $\hat{\pi}\notin W$. To see this, assume, by contradiction, that $\hat{\pi}\in W$. Then, by construction, $\gamma(\hat{\pi})=\{f(\hat{\pi})\}$. Hence, $\pi^*=f(\hat{\pi})\in U(\hat{\pi})$, which implies that $\pi^*\cdot Z(\hat{\pi})>0$. But the preceding inequality contradicts boundary behavior 1. Therefore, $\hat{\pi}\notin W$ and so $U(\hat{\pi})=\varnothing$, which entails $p\cdot Z(\hat{\pi})\leq 0$ for all $p\in\Delta$. Thus, $Z(\hat{\pi})\leq 0$, and since $\hat{\pi}\gg 0$, Walras law immediately implies that $Z(\hat{\pi})=0$. The proof is finished.

Remark 4.2: Our Theorem 4.1 should be compared to Theorem 4.1 and Theorem 6.1 in Tian [5]. Tian relaxes the assumption of lower semicontinuity of Z too. However, he assumes that the excess demand function is defined on the whole Δ, and demonstrates the existence of a $q^*\in\Delta$ such that $Z(q^*)\leq 0$. Neuefeind [11] formalizes a condition that the excess demand function is "inward-pointing" close to the boundary of the price-simplex. His assumption does not require the excess demand function to satisfy property 5) above. However, Neuefeind assumes that the excess demand function is continuous. We dispense with the standard boundary condition on the excess demand function too, but unlike Neuefeind we relax the assumption of continuity of the excess demand function.

Now, we assume that $Z: Int\Delta\to\mathbb{R}^N$ satisfies boundary behavior 2. Specifically:

Assumption 4.2: *$Z: Int\Delta\to\mathbb{R}^N$ satisfies the Walras law, that is $p\cdot Z(p)=0$ for all $p\in Int\Delta$. Also, Z satisfies boundary behavior 2. Moreover, Z is such that the correspondence*

$$U:\Delta_{\varepsilon}\twoheadrightarrow\Delta \text{ defined by } U(\pi):=\{p\in\Delta: p\cdot Z(\pi)>0\}$$

is lower hemicontinuous.

Remark 4.3: In light of Theorem 2.1, and Theorem 4.1, the following theorem is obvious, and requires no proof. However, we will give a direct proof in order to show that our way of proving existence is quite general, and can handle different types of boundary conditions. Incidentally, in the proof of Theorem 4.2 we will make use of a fixed point theorem due to Halpern and Bergman; it is of some interest to observe that, to the best of our knowledge, such a theorem has never been used before to prove existence of competitive equilibria.

Theorem 4.2: *If Assumption 4.2 holds, then there exists a $\hat{\pi}\in\Delta_{\varepsilon}$ such that $Z(\hat{\pi})=0$.*

Proof: The first part of the proof up to the construction of the correspondence $\gamma:\Delta_{\varepsilon}\twoheadrightarrow\Delta$ is identical to the proof of Theorem 4.1. Recall that $\gamma:\Delta_{\varepsilon}\twoheadrightarrow\Delta$ is defined as follows:

$$\pi\mapsto\begin{cases}\{f(\pi)\} & \text{if } \pi\in W\\ \Delta & \text{if } \pi\notin W\end{cases}$$

Clearly, γ is convex and compact valued. Furthermore, γ is upper hemicontinuous (see the proof of Theorem 4.1). Now, we will prove that γ is inward pointing.[8] To this end, pick any $\pi\in\Delta_{\varepsilon}$. If $\pi\notin W$, then $\gamma(\pi)=\Delta$, and therefore $\pi+\lambda(\pi-\pi)\in\Delta_{\varepsilon}$ for any $\lambda>0$. On the other hand, if $\pi\in W$, then $\gamma(\pi)=\{f(\pi)\}$, and therefore $f(\pi)\cdot Z(\pi)>0$. So, we

[7] So far we have demonstrated that, if U is lower hemicontinuous, then its domain is open. This is a well-known fact, but we have worked out the proof for the sake of completeness.

[8] For the definition of inward pointing correspondence see Aliprantis and Border [8], Definition 17.53.

consider two cases. 1): If $\pi \in \Delta_\varepsilon \setminus \partial\Delta_\varepsilon$, then clearly $\pi + \lambda(f(\pi) - \pi) \in \Delta_\varepsilon$ for some $\lambda > 0$ small enough. 2): If $\pi \in \partial\Delta_\varepsilon$ then, since $f(\pi)\cdot Z(\pi) > 0$, boundary behavior 2 immediately implies the existence of a $\lambda > 0$ such that $\pi + \lambda(f(\pi) - \pi) \in \Delta_\varepsilon$. Hence, γ is inward pointing. Therefore we have established that γ meets the sufficient conditions of Halpern-Bergman fixed point theorem (see Aliprantis and Border [8], Theorem 17.54). Thus, γ has a fixed point.[9] That is, there exists a $\hat{\pi} \in \Delta_\varepsilon$ such that $\hat{\pi} \in \gamma(\hat{\pi})$. Now, if $\hat{\pi} \in W$, then $\hat{\pi} \in \gamma(\hat{\pi}) = f(\hat{\pi}) \in U(\hat{\pi})$, which contradicts (4.2) above. Thus, we must have that $\hat{\pi} \notin W$. Hence, by definition of W and U, we obtain $p\cdot Z(\hat{\pi}) \le 0$ for all $p \in \Delta$, and so $Z(\hat{\pi}) \le 0$. On the other hand, since $\hat{\pi} \gg 0$, Walras law immediately implies that $Z(\hat{\pi}) = 0$, as was to be proven.

4.1. Some Instrumental Results

We begin this section by reminding the reader an open graph theorem for correspondences which will be used later on. It was first proven by Zhou [16] (Proposition 2).

Theorem 4.1.1: *Let* $U : X \twoheadrightarrow \mathbb{R}^N$ *be a correspondence, where* X *is a topological space. Assume that* U *has convex and open upper sections. Then,* U *is lower hemicontinuous if and only if* U *has open graph.*

In the context of Hilbert spaces, we can now prove a theorem which is a natural generalization of Theorem 5.1 in Yannelis and Prabhakar [6]. It is a generalization in that we do not require the domain and range of the correspondence at hand to be the same. Moreover, in place of the first condition in Theorem 5.1 of Yannelis and Prabhakar [6], we posit a more general assumption (see assumption 1) below). It is more general in the sense that it collapses to Yannelis and Prabhakar's condition whenever the domain and range of the correspondence coincide. Interestingly, our assumption 1) below bears a natural economic interpretation in terms of boundary behavior 1 defined in Section 2. Consequently, we shall show, in Section 4.2, that our Theorem 4.1.2 can be employed to provide another short proof of Theorem 4.1.

Let $\mathcal{H}$ be a Hilbert space, and let X and Y be non-empty, convex and compact subsets of $\mathcal{H}$, with $X \subset Y$. Let $P : Y \to X$ be the (metric) projection mapping defined in Section 2. For any subset A of $\mathcal{H}$, we denote by $conA$ the convex-hull of A.

Theorem 4.1.2: *Let* $U : X \twoheadrightarrow Y$ *be a correspondence such that*: 1) *For each* $y \in Y$, $y \notin conU(P(y))$; 2) U *has open lower sections in* X. *Then, there exists a* $\hat{x} \in X$ *such that* $U(\hat{x}) = \varnothing$.

Proof: Assume, by way of obtaining a contradiction, that for every $x \in X$, $U(x) \ne \varnothing$. Then, the correspondence $\Phi : X \twoheadrightarrow Y$, defined by $\Phi(x) = conU(x)$ for every x in X, is convex-valued and nonempty-valued. By assumption 2), it's easy to see that Φ has open lower sections in X.[10] Hence, by Browder selection theorem (see Browder [17]) there exists a continuous function $f : X \to Y$ such that $f(x) \in \Phi(x)$ for any $x \in X$. Now, consider the composition of mappings $f \circ P : Y \to Y$. By Brouwer-Schauder-Tychonoff theorem, $f \circ P : Y \to Y$ has a fixed point. That is, there exists a $y^* \in Y$ such that $y^* = f(P(y^*)) \in \Phi(P(y^*)) = conU(P(y^*))$, which contradicts assumption 1). The proof is finished.

4.2. Boundary Behavior and Existence of Maximal Elements

Consider, again, the correspondence U defined in (4.1) above. Recall that ε therein is the positive number whose existence is guaranteed by boundary behavior 1. Define the auxiliary correspondence

$$\tilde{U} : \Delta_\varepsilon \twoheadrightarrow \mathbb{R}^N \text{ by } \tilde{U}(\pi) := \{p \in \mathbb{R}^N : p\cdot Z(\pi) > 0\}$$

and let $S : \Delta_\varepsilon \twoheadrightarrow \mathbb{R}^N$ be defined by $S(\pi) = \Delta$ for every $\pi \in \Delta_\varepsilon$. Clearly,

$$U = \tilde{U} \cap S.$$

Assumption 4.2.1: $Z : Int\Delta \to \mathbb{R}^N$ *satisfies the Walras law and boundary behavior* 1. *Moreover,* Z *is such that* $\tilde{U} : \Delta_\varepsilon \twoheadrightarrow \mathbb{R}^N$ *is lower hemicontinuous.*

Next we show that Theorem 4.1 can be established as a simple and short corollary of Theorem 4.1.2. In this regard, it is interesting to notice that boundary behavior 1 corresponds to assumption 1) in Theorem 4.1.2.

Corollary 4.2.1: *If Assumption* 4.2.1 *holds, then there exists a* $\hat{\pi} \in \Delta_\varepsilon$ *such that* $Z(\hat{\pi}) = 0$.

Proof: By Theorem 4.1.1, $\tilde{U} : \Delta_\varepsilon \twoheadrightarrow \mathbb{R}^N$ has open graph. Obviously, $S : \Delta_\varepsilon \twoheadrightarrow \mathbb{R}^N$ has open lower sections. Thus, $U = \tilde{U} \cap S$ has open lower sections as well. Now, notice that boundary behavior 1 implies that U satisfies assumption 1) of Theorem 4.1.2. Hence, by Theorem 4.1.2, there exists a $\hat{\pi} \in \Delta_\varepsilon$ such that $U(\hat{\pi}) = \varnothing$. That is, $p\cdot Z(\hat{\pi}) \le 0$ for all $p \in \Delta$. As in the proof of Theorem 4.1, Walras law readily implies that $Z(\hat{\pi}) = 0$.

5. Concluding Remarks

From the standpoint of applied mathematics, we believe that this work is self-contained. From the perspective of economic theory, this paper can be improved and ex-

[9]We remark that the existence of a fixed point for γ may be proven also by invoking Theorem 4 in Tian [15].

[10]See the proof of Lemma 5.1 in Yannelis and Prabhakar [6].

tended. Let us outline what it would be worth undertaking for future research. First of all, one should investigate the relationship between the standard boundary behavior and our boundary behavior 1. Secondly, if it turns out that neither of them implies the other, or that the standard boundary behavior implies our boundary behavior 1, then it would be interesting to construct relevant economic models for which the standard boundary behavior does not hold, but our boundary conditions are satisfied by the excess demand function of the model itself.

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