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A Note on Crank-Nicolson Scheme for Burgers' Equation

Kanti Pandey, Lajja Verma

Department of Mathematics and Astronomy, University of Lucknow, Lucknow, India
E-mail: *pandey_kanti@yahoo.co.in, lajjamaths@yahoo.co.in*
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Abstract

In this work we generate the numerical solutions of the Burgers' equation by applying the Crank-Nicolson method directly to the Burgers' equation, *i.e.*, we do not use Hopf-Cole transformation to reduce Burgers' equation into the linear heat equation. Absolute error of the present method is compared to the absolute error of the two existing methods for two test problems. The method is also analyzed for a third test problem, numerical solutions as well as exact solutions for different values of viscosity are calculated and we find that the numerical solutions are very close to exact solution.

Keywords: Hopf-Cole Transformation, Burgers' Equation, Crank-Nicolson Scheme, Nonlinear Partial Differential Equations

1. Introduction

Burgers' equation is one of the most important nonlinear partial differential equations governed by the following equation

$$w_t = \left(\frac{v_d}{2} w_x - \frac{w^2}{2} \right)_x,$$

$$w \equiv w(x,t), (x,t) \in [0,1] \times [0,T]. \tag{1}$$

This equation is nonlinear and can be considered as nonlinear analog of the Navier-Stokes equations. It has a convection term, a diffusion term and a time-dependent term. It also has a large variety of applications in modeling of water in unsaturated oil, dynamics of soil in water, statics of flow problems, mixing and turbulent diffusion, cosmology and seismology [1-3].

With viscous term the Burgers' Equation (1) is parabolic while without viscous term it is hyperbolic. In the later case it possesses discontinuous solutions due to the nonlinear term and even if smooth initial condition is considered the solution may be discontinuous after finite time. It also governs the phenomenon of shock waves [4].

In the present work we consider the Burgers' Equation (1) with the initial condition

$$w(x,0) = f(x), \quad 0 \le x \le 1, \tag{2}$$

and the boundary conditions

$$w(0,t) = g_1(t), \quad 0 \le t \le T,$$
$$w(0,t) = g_2(t), \quad 0 \le t \le T, \tag{4}$$

where $v_d > 0$ is a coefficient of viscous diffusion and f, g_1 and g_2 are the sufficiently smooth given functions.

Several researchers have successfully used Burgers' equation to develop new algorithms and to test various existing algorithms. In most of the cases researchers used Hopf-Cole [5,6] transformation to linearize the Burgers' equations into parabolic partial differential equation. Some of the researchers also tried to tackle the nonlinear Burgers' equation directly (without Hopf-Cole). Kadalbajoo *et al.* [7] applied Crank-Nicolson finite difference method to the linearized Burgers' equation by Hopf-Cole transformation which is unconditionally stable and is second order convergent in both space and time with no restriction on mesh size. Gorguis [8] applied the Adomian decomposition method on the Burgers' equation directly and compared the numerical result with the analytical result. In another result due to Kutluay *et al.* [9] a direct approach via least square quadratic B-spline finite element method is discussed. Recently Pandey *et al.* [10] discussed Douglas finite difference scheme on linearized Burgers' equation which is fourth order convergent in space and second order convergent in time.

In this paper we first apply Crank-Nicolson finite difference scheme directly on the nonlinear Equation (1) and derive a nonlinear finite difference scheme, and then use it to derive a system of linear equations which we

solve by using Mathematica (version 7.0). For two test problems we compare the absolute error of the numerical solutions to the absolute error of the numerical results established by Kadalbajoo *et al.* [7] and Pandey *et al.* [10]. For the third test problem we compare our numerical results with the analytical solution. The exact solution for the third problem is calculated on Mathematica (version 7.0).

In this paper we show that it is more appropriate to consider the Burgers' equation directly than reducing it into linear parabolic problem and then discretize it. Therefore this paper is an improvement over the results in [7] and [10].

This paper is divided in 4 sections. In Section 2 we give expression for exact solution by Hopf-Cole transformation which is used to calculate the exact solutions. In the same section we collocate and discretize to get a nonlinear finite difference equation and then through a simple approximation we deduce linear finite difference equation. In Section 3 we give three examples and in Section 4 we demonstrate properties of the computed numerical solutions in the form of **Tables 1-11** and **Figures 1-4**.

2. Description of the Method

2.1. Exact Solution

Hopf and Cole [5,6] suggested that (1) can be reduced to linear heat equation by the non-linear transformation. Let

$$\psi = -v_d \left(\log \phi \right), \tag{5}$$

and

$$w = \psi_x. \tag{6}$$

Putting in (1) we get

$$\psi_t + \frac{1}{2} \psi_x^2 = \frac{v_d}{2} \psi_{xx}. \tag{7}$$

Next applying the transformation (5) we get

$$\phi_t = \frac{v_d}{2} \phi_{xx}. \tag{8}$$

The fourier series solution to the linearized heat equation is

$$\phi(x,t) = A_0 + \sum_{n=1}^{\infty} A_n \exp\left(-\frac{v_d n^2 \pi^2 t}{2} \right) \cos n\pi x \tag{9}$$

with fourier coefficients at $t = 0$ as

$$A_0 = \int_0^1 E(x) \, dx,$$
$$A_n = 2\int_0^1 E(x) \cos(n\pi x) \, dx, \tag{11}$$

where $w_0(\xi) = w(\xi,0)$ and

$$E(x) = \exp\left(-\frac{1}{v_d} \int_0^x w_0(\xi) \, d\xi \right).$$ Using the Hopf-Cole

transformation we have the exact solution

$$w(x,t) = \pi v_d \frac{\sum_{n=1}^{\infty} A_n H_n(t) n \sin n\pi x}{A_0 + \sum_{n=1}^{\infty} A_n H_n(t) \cos n\pi x} \tag{12}$$

where $H_n(t) = \exp\left(-\frac{v_d n^2 \pi^2 t}{2} \right)$.

2.2. Discretization

First of all we divide the solution space into a uniform mesh. For this we divide the interval $[0, 1]$ into N equal subintervals and divide the interval $[0, T]$ into M equal subintervals. Let $h = 1/N$ be the mesh width in space and $x_i = ih$ for $i = 1(1)N$. Let $k = T/M$ be the mesh width in time and $t_j = jk$ for $j = 0(1)M$. Now Collocating the Burgers' Equation (1) we get

$$\frac{\partial w_{i,j}}{\partial t} + \frac{1}{2} \frac{\partial \left(w_{i,j} \right)^2}{\partial x} = \frac{v_d}{2} \frac{\partial^2 w_{i,j}}{\partial x^2}, \tag{13}$$
$$i = 1(1)N, \ j = 0(1)M.$$

where $w_{i,j} = w(x_i, t_j)$. The Crank-Nicolson method [11,12] gives the following system of nonlinear equations,

$$w_{i,j+1} - w_{i,j} + \frac{1}{2} \frac{k}{h} \mu_x \delta_x \left(\frac{w_{i,j+1} + w_{i,j}}{2} \right)^2$$
$$= \frac{v_d}{2} \frac{k}{h^2} \delta_x^2 \left(\frac{w_{i,j+1} + w_{i,j}}{2} \right). \tag{14}$$

where δ_x and μ_x are central difference operator and averaging operator respectively. To linearize we put $w_{i,j+1} = w_{i,j} + v_{i,j}$ into (14) where $v_{i,j} = w_{i,j+1} - w_{i,j}$ and neglecting $O\left(v_{i,j}^2 \right)$ we get

$$-\left\{ \frac{1}{4} s w_{i-1,j} + \frac{r}{2} \frac{v_d}{2} \right\} v_{i-1,j} + \left\{ 1 + r \frac{v_d}{2} \right\} v_{i,j}$$
$$+ \left\{ \frac{1}{4} s w_{i+1,j} - \frac{r}{2} \frac{v_d}{2} \right\} v_{i+1,j}$$
$$= \left\{ r \frac{v_d}{2} + \frac{1}{4} r w_{i-1,j} \right\} w_{i-1,j} - r v_d w_{i,j}$$
$$+ \left\{ r \frac{v_d}{2} - \frac{1}{4} s w_{i+1,j} \right\} w_{i+1,j} \tag{15}$$

where $s = (k/h)$ and $r = (k/h^2)$.

3. Numerical Results and Discussions

In this section we demonstrate the accuracy of the

present method by solving three test problems and compare the results with the two existing results. The computations are performed using Mathematica 7.0 and Origin 7.5.

3.1. Problem 1

Consider Equation (1) with boundary conditions and initial condition as

$$w(0,t) = w(1,t) = 0, t > 0, \tag{16}$$

$$w(x,0) = \sin \pi x, \tag{17}$$

where $(v_d)/2$ is the coefficient of kinematic viscosity. We substitute

$$w(x,t) = \frac{-v_d \phi_x}{\phi} \tag{18}$$

in Equation (1) and get $\phi_t = \frac{v_d}{2}\phi_{xx}, 0 < x < 1, t > 0$ with initial condition

$$\phi(x,0) = \exp\left(\frac{-1}{\pi v_d}(1 - \cos \pi x)\right), 0 < x < 1. \tag{19}$$

and boundary condition

$$\phi_x(0,t) = \phi_x(1,t) = 0, t > 0.$$

The exact solution of the Burgers' Equation (1) is (12) with given Fourier coefficients:

$$A_0 = \int_0^1 \exp\left(\frac{-1}{\pi v_d}(1 - \cos \pi x)\right) dx,$$

$$A_n = 2\int_0^1 \exp\left(\frac{-1}{\pi v_d}(1 - \cos \pi x)\cos n\pi x\right) dx.$$

3.2. Problem 2

As a second example consider (1) with the boundary conditions (16) and initial condition

$$w(x,0) = 4x(1-x), 0 < x < 1. \tag{20}$$

The exact solution (12) can be obtained in the similar fashion as in Problem 3.1 with the Fourier coefficients as follows :

$$A_0 = \int_0^1 \exp\left(\frac{-2x^2}{3v_d}(3 - 2x)\right) dx,$$

$$A_n = 2\int_0^1 \exp\left(\frac{-2x^2}{3v_d}(3 - 2x)\right)\cos(n\pi x) dx.$$

3.3. Problem 3

Consider Equation (1) with boundary conditions (16) and

initial condition

$$w(x,0) = \frac{2\pi \sin \pi x}{2 + \cos \pi x}, \quad 0 < x < 1. \tag{21}$$

The exact solution can be calculated by using the formula (12) where

$$A_0 = \int_0^1 \left(\frac{2 + \cos \pi x}{2}\right)^{2/v_d} dx,$$

$$A_n = 2\int_0^1 \left(\frac{2 + \cos \pi x}{2}\right)^{2/v_d} \cos n\pi x dx. \tag{22}$$

4. Conclusions

We present Crank-Nicolson finite difference scheme for Burgers' equation without Hopf-Cole transformation. We claim that it is better to solve the nonlinear Burgers' equation directly, *i.e.*, without reducing it to linear heat equation by Hopf-Cole transformation. Our claim is very well supported by the **Tables 1-10** and Graphs 1-3. From **Table 11** and Graph 4 it is also proved that numerical results are in good agreement with the analytical solution. The exact solution for the Problem 3.3 is calculated by using Mathematica 7.0. **Figure 4** also depicts the physical behavior of the solutions and thus behavior of any physical system governed by Burgers' equation can be studied by this method.

4.1. Tables

Computed results are displayed in **Tables 1** to **11** at different nodal points for different values of viscosity. In **Tables 1** to **6** we compare the absolute error with the absolute error of [7,10] for the Problem 3.1. From **Table 7** to **Table 10** absolute error is compared to the absolute

Table 1. Comparison of the absolute error with the absolute error of [7,10] for Problem 3.1 at T = 0.01, for v_d = 20 and K = 0.0001 for N = 40.

x /Error	Without Hopf-Cole	Kadalbajoo *et al.*	Pandey *et al.*
0.1	5.78801E–05	6E–05	0.00016
0.2	0.000109996	0.00011	0.00031
0.3	0.000151182	0.00016	0.00044
0.4	0.000177403	0.00019	0.00051
0.5	0.000186149	0.0002	0.00054
0.6	0.000176667	0.00019	0.00051
0.7	0.000149991	0.00016	0.00044
0.8	0.000108805	0.00011	0.00031
0.9	5.71442E–05	6E–05	0.00016

Table 2. Comparison of the absolute error with the absolute error of [7,10] for Problem 3.1 at *T* = 0.01, for v_d = 20 and *K* = 0.0001 for *N* = 80.

x /Error	Without Hopf-Cole	Kadalbajoo *et al.*	Pandey *et al.*
0.1	1.37807E–05	1E–05	7E–05
0.2	2.61869E–05	3E–05	0.00014
0.3	3.59876E–05	5E–05	0.00021
0.4	4.22225E–05	5E–05	0.00024
0.5	4.42961E–05	6E–05	0.00025
0.6	4.20319E–05	5E–05	0.00024
0.7	3.56792E–05	5E–05	0.00021
0.8	2.58784E–05	3E–05	0.00014
0.9	0.00001359	1E–05	8E-05

Table 3. Comparison of the absolute error with the absolute error of [7,10] for Problem 3.1 at *T* = 0.1, for v_d = 2 and *K* = 0.001 for *N* = 40.

x /Error	Without Hopf-Cole	Kadalbajoo *et al.*	Pandey *et al.*
0.1	6.10384E–05	6E–05	0.00016
0.2	0.0001156	0.00011	0.0003
0.3	0.000157849	0.00016	0.00042
0.4	0.00018329	0.00018	0.0005
0.5	0.000189452	0.0002	0.00054
0.6	0.000176365	0.00019	0.00052
0.7	0.000146518	0.00017	0.00045
0.8	0.000104112	0.00048	0.00033
0.9	5.38605E–05	0.00026	0.00018

Table 4. Comparison of the absolute error with the absolute error of [7,10] for Problem 3.1 at *T* = 0.1, for v_d = 2 and *K* = 0.001 for *N* = 80.

x /Error	Without Hopf-Cole	Kadalbajoo *et al.*	Pandey *et al.*
0.1	1.45914E–05	2E–05	7E–05
0.2	2.76247E–05	3E–05	0.00014
0.3	3.76962E–05	4E–05	0.0002
0.4	4.37257E–05	4E–05	0.00023
0.5	4.51272E–05	5E–05	0.00025
0.6	4.19278E–05	5E–05	0.00024
0.7	3.47543E–05	5E–05	0.00021
0.8	0.000024642	3E–05	0.00016
0.9	1.27275E–05	2E–05	8E–05

Table 5. Comparison of the absolute error with the absolute error of [7,10] for Problem 3.1 for v_d = 20, *K* = 0.01, *N* = 80 at different times.

x /Error	Without Hopf-Cole	Kadalbajoo *et al.*	Pandey *et al.*
0.4	5.9E–05	8.00E–05	0.00023
0.6	4.4E–05	5.00E–05	0.00016
1	4E–05	2.00E–05	8E–05
3	1.48E–05	0	1E–05
0.4	0.000118	8.00E–05	0.00029
0.6	9.5E–05	7.00E–05	0.00024
1	7.1E–05	4.00E–05	0.00015
3	2.39E–05	1.00E–05	2E–05
0.4	3.7E–05	4.00E–05	0.00021
0.6	7.5E–05	6.00E–05	0.00021
1	7.2E–05	3.00E–05	0.00015
3	0.0010165	0.00101	0.0002

Table 6. Comparison of the absolute error with the absolute error of [7,10] for Problem 3.1 for v_d = 0.02, *K* = 0.01, *N* = 80 at different times.

x /Error	Without Hopf-Cole	Kadalbajoo *et al.*	Pandey *et al.*
0.4	7.2E–05	0.00038	0.00076
0.6	5E–05	6E–05	0.00012
1	2.7E–05	2E–05	0.00013
3	8E–06	0	6E–05
0.4	0.000167	0.00726	0.01517
0.6	0.000106	0.00269	0.00736
1	5.2E–05	0.00058	0.00229
3	7E–06	0	4E-05
0.4	0.000384	0.02654	0.04398
0.6	0.000234	0.01	0.02528
1	9.6E–05	0.00228	0.0093
3	3.3E–05	0	0.00047

Table 7. Comparison of the absolute error with the absolute error of [7,10] for Problem 3.2 at *T* = 0.1, for v_d = 2 and *K* = 0.001 for *N* = 40.

x /Error	Without Hopf-Cole	Kadalbajoo *et al.*	Pandey *et al.*
0.1	6.34389E–05	6E–05	0.00012
0.2	0.000119993	0.00011	0.00023
0.3	0.000163549	0.00016	0.00032
0.4	0.00018952	0.00019	0.00037
0.5	0.000195514	0.0002	0.00039
0.6	0.000181726	0.0002	0.00038
0.7	0.000150809	0.00017	0.00033
0.8	0.000107093	0.00012	0.00024
0.9	5.53858E–05	7E–05	0.00013

Table 8. Comparison of the absolute error with the absolute error of [7,10] for Problem 3.2 at *T* = 0.1, for v_d = 2 and *K* = 0.001 for *N* = 80.

x /Error	Without Hopf-Cole	Kadalbajoo *et al.*	Pandey *et al.*
0.1	0.000015144	1E–05	3E–05
0.2	2.86478E–05	3E–05	6E–05
0.3	3.90448E–05	5E–05	9E–05
0.4	4.52233E–05	5E–05	9E–05
0.5	0.000046599	5E–05	1E–04
0.6	4.32271E–05	5E–05	1E–04
0.7	3.57791E–05	5E–05	9E–05
0.8	2.53375E–05	3E–05	6E–05
0.9	1.30758E–05	2E–05	3E–05

Table 9. Comparison of the absolute error with the absolute error of [7,10] for Problem 3.2 for v_d = 0.2, *K* = 0.01, *N* = 80 at different times.

x /Error	Without Hopf-Cole	Kadalbajoo *et al.*	Pandey *et al.*
0.4	6.3E–05	9E–05	0.00017
0.6	4.6E–05	5E–05	0.00011
1	3.5E–05	2E–05	6E–05
3	1.16E–05	0	1E–05
0.4	0.000118	8E–05	0.00013
0.6	0.000101	7E–05	0.00012
1	8.1E–05	3E–05	8E–05
3	2.46E–05	1E–05	1E–05
0.4	3.6E–05	4E–05	8E–05
0.6	7.1E–05	7E–05	3E–05
1	6.8E–05	4E–05	6E–05
3	1.45E–05	0	1E–05

Table 10. Comparison of the absolute error with the absolute error of [7,10] for Problem 3.2 for v_d = 0.02, *K* = 0.01, *N* = 80 at different times.

x /Error	Without Hopf-Cole	Kadalbajoo *et al.*	Pandey *et al.*
0.4	0.00012	0.00047	0.00113
0.6	8.6E–05	8E–05	0.00024
1	4.9E–05	2E–05	0.00011
3	1.21E–05	0	6E–05
0.4	0.000167	0.00818	0.0172
0.6	0.000128	0.00293	0.00839
1	7.7E–05	0.00059	0.00258
3	1.6E–05	0	5E–05
0.4	0.000286	0.0289	0.04617
0.6	0.000212	0.011	0.02718
1	0.000109	0.00238	0.0101
3	4.4E–05	4E–05	0.0005

Table 11. Comparison of the numerical solution with the exact solution for Problem 3.3 at different space points at *T* = 0.01, for v_d = 20 and *K* = 0.0001 for different values of *N*.

x	*N* =10	*N* = 20	*N* = 40	*N* = 80	Exact
0.1	0.36871	0.377309	0.370262	0.370173	0.370141
0.2	0.705991	0.720825	0.708758	0.708574	0.70851
0.3	0.981812	0.99888	0.985219	0.984931	0.984834
0.4	1.16936	1.18429	1.17275	1.17236	1.17224
0.5	1.24751	1.25697	1.25035	1.24988	1.24973
0.6	1.20385	1.20667	1.20583	1.20532	1.20517
0.7	1.03763	1.0351	1.03875	1.03827	1.03813
0.8	0.761828	0.757042	0.762304	0.761929	0.761818
0.9	0.403226	0.399692	0.403361	0.403154	0.403094

(a)

(b)
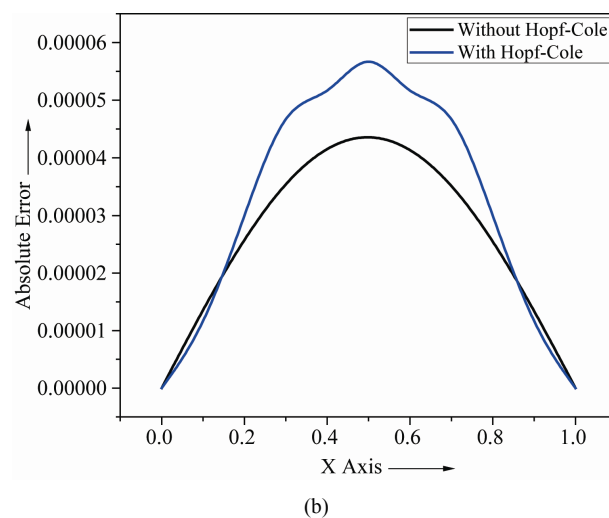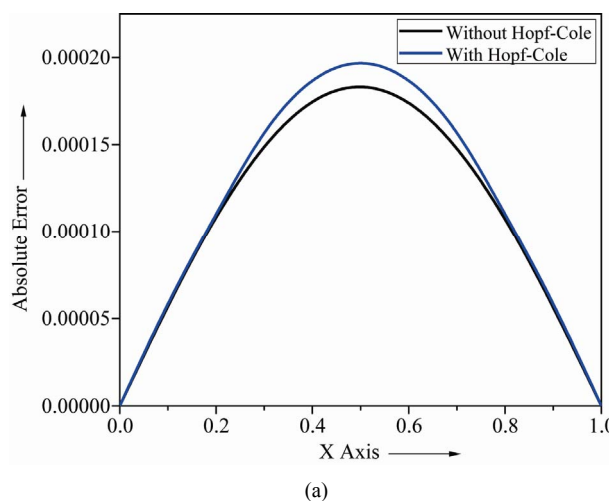
Figure 1. Absolute errors of Problem 3.1 at different times for v_d = 20 and *K* = 0.0001, (a) *N* = 40, (b), *N* = 80.
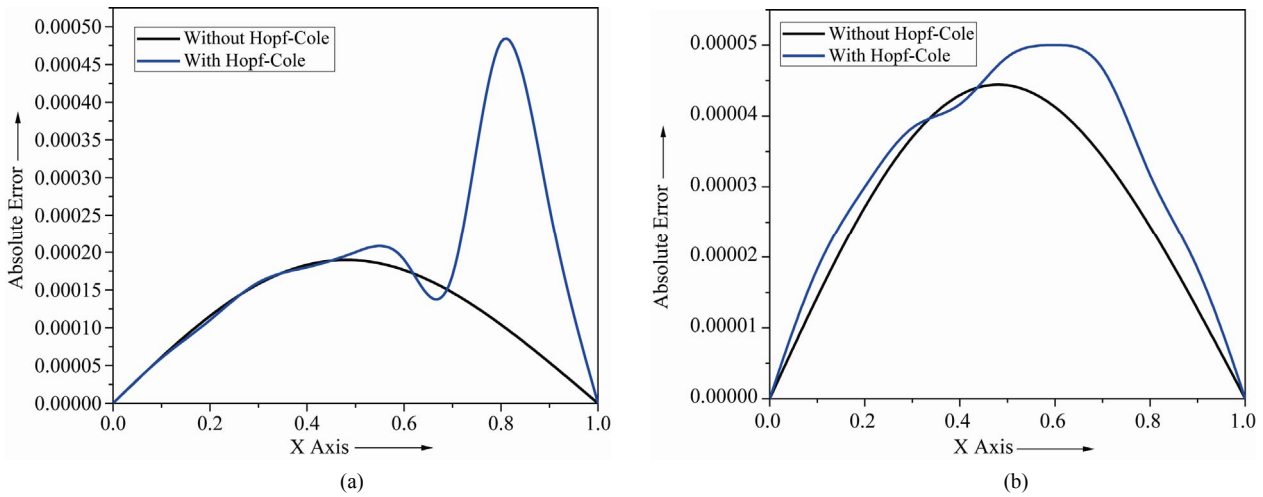
Figure 2. Absolute errors of Problem 3.1 at different times for $v_d = 2$ and $K = 0.001$, (a) $N = 40$, (b) $N = 80$.
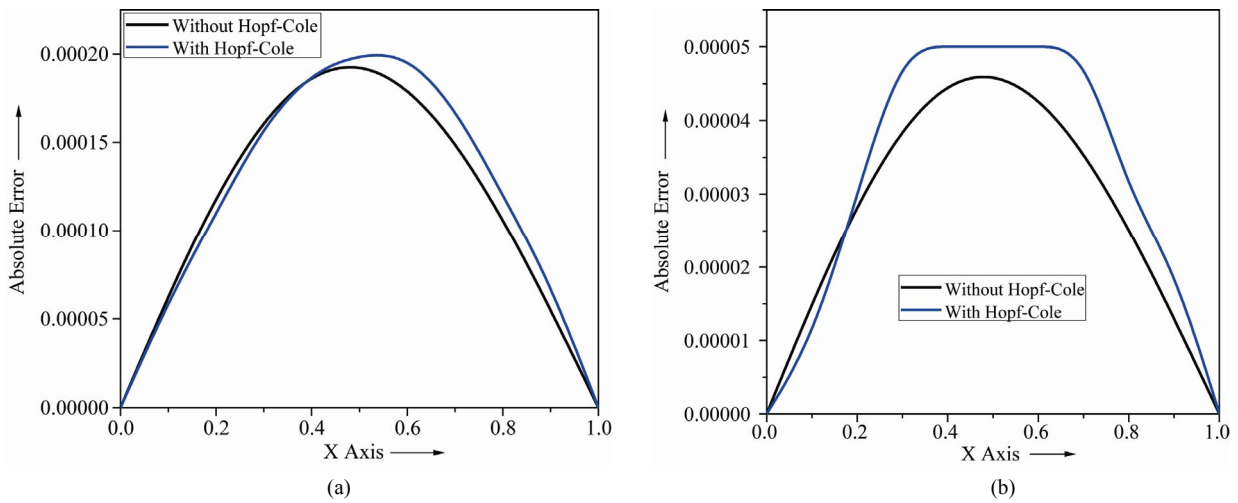

Figure 3. Absolute errors of Problem 3.2 at different times for $v_d = 2$ and $K = 0.001$, (a) $N = 40$, (b) $N = 80$.
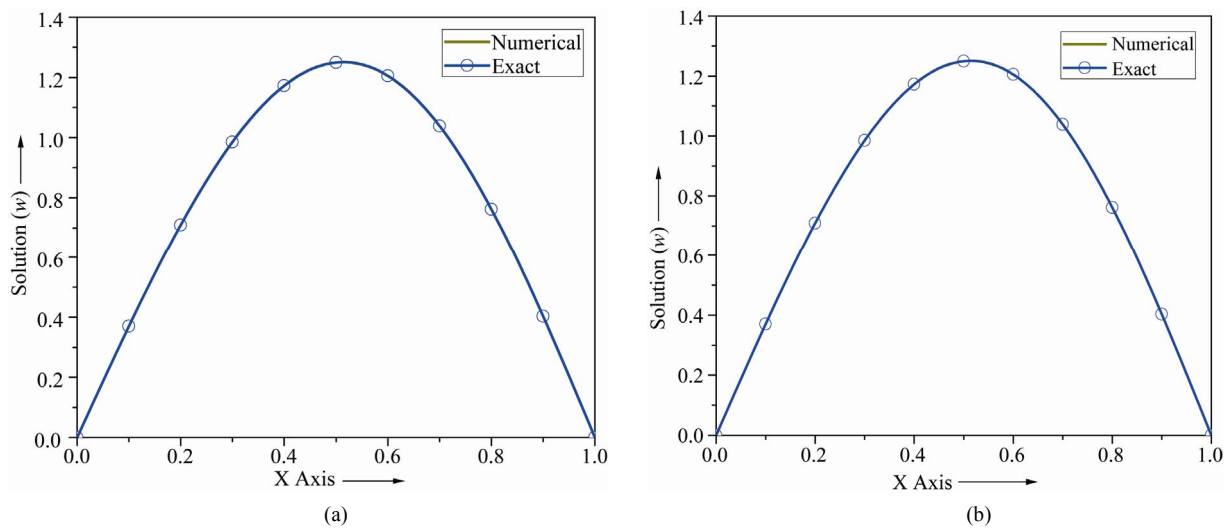

Figure 4. Numerical solutions of Problem 3.3 at different times for $v_d = 2$ and $K = 0.001$, (a) $N = 40$, (b) $N = 80$.

error of [7,10] for the Problem 3.2. In **Table 11** for Problem 3.3 we compare the numerical solution to the exact solution and it is observed that computed result shows greater agreement with the exact solution as the mesh size is refined.

4.2. Figures

In **Figures 1** and **2** we compare the absolute error with the absolute error of [7] for Problem 3.1 and in **Figure 3** we compare the absolute error with the absolute error of [7] for Problem 3.2. Finally in **Figure 4** we compare the exact solution with the numerical solution for Problem 3.3.

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