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On the Periodicity of Solutions of the System of Rational Difference Equations $x_{n+1}=\frac{x_{n-1}+y_n}{y_n x_{n-1}-1},\ y_{n+1}=\frac{y_{n-1}+x_n}{x_n y_{n-1}-1}$

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Abstract

In this paper, we have investigated the periodicity of the solutions of the system of difference equations $x_{n+1}=\frac{x_{n-1}+y_n}{y_n x_{n-1}-1},\ y_{n+1}=\frac{y_{n-1}+x_n}{x_n y_{n-1}-1}$, where $x_0, x_{-1}, y_0, y_{-1}\in\mathbb{R}$.

Keywords: Difference Equation, Difference Equation Systems, Solutions

1. Introduction

Recently, there has been great interest in studying difference equation systems. One of the reasons for this is a necessity for some techniques which can be used in investigating equations arising in mathematical models describing real life situations in population biology, economic, probability theory, genetics, psychology etc. There are many papers with related to the difference equations system for example,

In [1] Cinar studied the solutions of the systems of the difference equations

$$x_{n+1}=\frac{1}{y_n},\ y_{n+1}=\frac{y_n}{x_{n-1}y_{n-1}}$$

In [2] Papaschinnopoulos and Schinas studied the oscillatory behavior, the boundedness of the solutions, and the global asymptotic stability of the positive equilibrium of the system of nonlinear difference equations

$$x_{n+1}=A+\frac{y_n}{x_{n-p}},\ y_{n+1}=A+\frac{x_n}{y_{n-q}},\ n=0,1,2,\cdots,p,q.$$

In [3] Papaschinnopoulos and Schinas proved the boundedness, persistence, the oscillatory behavior and the asymptotic behavior of the positive solutions of the system of difference equations

$$x_{n+1}=\sum_{i=0}^{k}\frac{A_i}{y_{n-i}^{p_i}},\ y_{n+1}=\sum_{i=0}^{k}\frac{B_i}{x_{n-i}^{q_i}}.$$

In [4,5] Özban studied the positive solutions of the system of rational difference equations and

$$x_{n+1}=\frac{a}{y_{n-k}},\ y_{n+1}=\frac{y_n}{x_{n-m}y_{n-m-k}}.$$

In [6,7] Clark and Kulenović investigate the global asymptotic stability

$$x_{n+1}=\frac{x_n}{a+cy_n},\ y_{n+1}=\frac{y_n}{b+dx_n}.$$

In [8] Camouzis and Papaschinnopoulos studied the global asymptotic behavior of positive solutions of the system of rational difference equations

$$x_{n+1}=1+\frac{x_n}{y_{n-m}},\ y_{n+1}=1+\frac{y_n}{x_{n-m}}.$$

In [9] Kurbanli, Çinar and Yalcinkaya studied On the behavaior of positive solutions of the system of rational difference equations

$$x_{n+1}=\frac{x_{n-1}}{y_n x_{n-1}+1},\ y_{n+1}=\frac{y_{n-1}}{x_n y_{n-1}+1}.$$

In [10] Kurbanli studied the behavaior of solutions of the system of rational difference

$$x_{n+1}=\frac{x_{n-1}}{y_n x_{n-1}-1},\quad y_{n+1}=\frac{y_{n-1}}{x_n y_{n-1}-1}.$$

In [11] Yang, Liu and Bai consided the behavior of the positive solutions of the system of the difference equations

$$x_n=\frac{a}{y_{n-p}},\quad y_n=\frac{by_{n-p}}{x_{n-q}y_{n-q}}.$$

In [12] Kulenović, Nurkanović studied the global asymptotic behavior of solutions of the system of difference equations

$$x_{n+1}=\frac{a+x_n}{b+y_n},\quad y_{n+1}=\frac{c+y_n}{d+z_n},\quad z_{n+1}=\frac{e+z_n}{f+x_n}.$$

In [13] Zhang, Yang, Megson and Evans investigated the behavior of the positive solutions of the system of difference equations

$$x_n=A+\frac{1}{y_{n-p}},\quad y_n=A+\frac{y_{n-1}}{x_{n-r}y_{n-s}}$$

In [14] Zhang, Yang, Evans and Zhu studied the boundedness, the persistence and global asymptotic stability of the positive solutions of the system of difference equations

$$x_{n+1}=A+\frac{y_{n-m}}{x_n},\quad y_{n+1}=A+\frac{x_{n-m}}{y_n}$$

In [15] Yalcinkaya and Cinar studied the global asmptotic stability of the system of difference equations

$$z_{n+1}=\frac{t_n z_{n-1}+a}{t_n+z_{n-1}},\quad t_{n+1}=\frac{z_n t_{n-1}+a}{z_n+t_{n-1}}.$$

In [16] Yalcinkaya, Cinar and Atalay investigated the solutions of the system of difference equations

$$x_{n+1}^{(1)}=\frac{x_n^{(2)}}{x_n^{(2)}-1},\quad x_{n+1}^{(2)}=\frac{x_n^{(3)}}{x_n^{(3)}-1},\cdots,\ x_{n+1}^{(k)}=\frac{x_n^{(1)}}{x_n^{(1)}-1}.$$

In [17] Yalcinkaya studied the global asmptotic stability of the system of difference equations

$$z_{n+1}=\frac{t_n z_{n-1}+a}{t_n+z_{n-1}},\quad t_{n+1}=\frac{z_n t_{n-1}+a}{z_n+t_{n-1}}.$$

In [18] Irićanin and Stević studied the positive solutions of the system of difference equations

$$x_{n+1}^{(1)}=\frac{1+x_n^{(2)}}{x_{n-1}^{(3)}},\quad x_{n+1}^{(2)}=\frac{1+x_n^{(3)}}{x_{n-1}^{(4)}},\ \cdots,\ x_{n+1}^{(k)}=\frac{1+x_n^{(1)}}{x_{n-1}^{(2)}}$$

$$x_{n+1}^{(1)}=\frac{1+x_n^{(2)}+x_{n-1}^{(3)}}{x_{n-2}^{(4)}},\ x_{n+1}^{(2)}=\frac{1+x_n^{(3)}+x_{n-1}^{(4)}}{x_{n-2}^{(5)}},$$

$$\cdots,\ x_{n+1}^{(k)}=\frac{1+x_n^{(1)}+x_{n-1}^{(2)}}{x_{n-2}^{(3)}}$$

In this paper, we investigated the periodicity of the solutions of the difference equation system

$$x_{n+1}=\frac{x_{n-1}+y_n}{y_n x_{n-1}-1},\quad y_{n+1}=\frac{y_{n-1}+x_n}{x_n y_{n-1}-1}\tag{1.1}$$

where the initial conditions are arbitrary real numbers.

2. Main Results

Theorem 1. Let $y_0=a,\ y_{-1}=b,\ x_0=c,\ x_{-1}=d$ be arbitrary real numbers and let $\{x_n,y_n,z_n\}$ be a solutions of the system (1.1). Also, assume that $ad\neq 1$ and $cb\neq 1$. All solutions of (1.1) are as following:

$$x_n=\begin{cases}\dfrac{d+a}{ad-1}, & n=6k+1\\ b, & n=6k+2\\ a, & n=6k+3\\ \dfrac{b+c}{cb-1}, & n=6k+4\\ d, & n=6k+5\\ c, & n=6k+6\end{cases},\quad k=0,\ 1,\ 2,\cdots\tag{1.2}$$

$$y_n=\begin{cases}\dfrac{b+c}{cb-1}, & n=6k+1\\ d, & n=6k+2\\ c, & n=6k+3\\ \dfrac{d+a}{ad-1}, & n=6k+4\\ b, & n=6k+5\\ a, & n=6k+6\end{cases},\quad k=0,\ 1,\ 2,\cdots\tag{1.3}$$

Proof: For n = 0, 1, 2, 3, 4, 5, we have

$$x_1=\frac{x_{-1}+y_0}{y_0x_{-1}-1}=\frac{d+a}{ad-1},$$

$$y_1=\frac{y_{-1}+x_0}{x_0y_{-1}-1}=\frac{b+c}{cb-1},$$

$$x_2=\frac{x_0+y_1}{y_1x_0-1}=\frac{c+\dfrac{b+c}{cb-1}}{\dfrac{b+c}{cb-1}c-1}=\frac{\dfrac{c(cb-1)+b+c}{cb-1}}{\dfrac{c^2+1}{cb-1}}=\frac{c^2b+b}{c^2+1}=b,$$

$$y_2=\frac{y_0+x_1}{x_1y_0-1}=\frac{a+\dfrac{d+a}{ad-1}}{\dfrac{d+a}{ad-1}a-1}=\frac{a^2d+d}{a^2+1}=d,$$

$$x_3 = \frac{x_1 + y_2}{y_2 x_1 - 1} = \frac{\frac{d+a}{ad-1} + d}{d\frac{d+a}{ad-1} - 1} = \frac{\frac{a\left(1+d^2\right)}{ad-1}}{\frac{d^2+1}{ad-1}} = a,$$

$$y_3 = \frac{y_1 + x_2}{x_2 y_1 - 1} = \frac{\frac{b+c}{bc-1} + b}{b\frac{b+c}{bc-1} - 1} = \frac{c\left(1+b^2\right)}{b^2+1} = c,$$

$$x_4 = \frac{x_2 + y_3}{y_3 x_2 - 1} = \frac{b+c}{cb-1},$$

$$y_4 = \frac{y_2 + x_3}{x_3 y_2 - 1} = \frac{d+a}{ad-1},$$

$$x_5 = \frac{x_3 + y_4}{y_4 x_3 - 1} = \frac{a + \frac{d+a}{ad-1}}{\frac{d+a}{ad-1}a - 1} = \frac{a^2 d - a + d + a}{ad + a^2 - ad + 1} = d,$$

$$y_5 = \frac{y_3 + x_4}{x_4 y_3 - 1} = \frac{c + \frac{b+c}{cb-1}}{\frac{b+c}{cb-1}c - 1} = \frac{b + bc^2}{c^2+1} = b$$

and

$$x_6 = \frac{x_4 + y_5}{y_5 x_4 - 1} = \frac{\frac{b+c}{cb-1} + b}{b\frac{b+c}{cb-1} - 1} = \frac{c\left(b^2+1\right)}{b^2+1} = c,$$

$$y_6 = \frac{y_4 + x_5}{x_5 y_4 - 1} = \frac{\frac{d+a}{ad-1} + d}{d\frac{d+a}{ad-1} - 1} = \frac{ad^2 + a}{d^2+1} = a;$$

for n = 6, 7, 8, 9, 10, 11,

$$x_7 = \frac{x_5 + y_6}{y_6 x_5 - 1} = \frac{d+a}{ad-1} = x_1,$$

$$y_7 = \frac{y_5 + x_6}{x_6 y_5 - 1} = \frac{b+c}{cb-1} = y_1,$$

$$x_8 = \frac{x_6 + y_7}{y_7 x_6 - 1} = \frac{c + \frac{b+c}{cb-1}}{\frac{b+c}{cb-1}c - 1} = \frac{b\left(c^2+1\right)}{c^2+1} = b,$$

$$y_8 = \frac{y_6 + x_7}{x_7 y_6 - 1} = \frac{a + \frac{d+a}{ad-1}}{\frac{d+a}{ad-1}a - 1} = \frac{a^2 d + d}{a^2+1} = d = y_2,$$

$$x_9 = \frac{x_7 + y_8}{y_8 x_7 - 1} = \frac{\frac{d+a}{ad-1}}{d\frac{d+a}{ad-1} - 1} = \frac{ad^2 + 1}{d^2+1} = a = x_3,$$

$$y_9 = \frac{y_7 + x_8}{x_8 y_7 - 1} = \frac{\frac{b+c}{cb-1}}{b\frac{b+c}{cb-1} - 1} = \frac{c + cb^2}{b^2+1} = c = y_3,$$

$$x_{10} = \frac{x_8 + y_9}{y_9 x_8 - 1} = \frac{b+c}{cb-1} = x_4,$$

$$y_{10} = \frac{y_8 + x_9}{x_9 y_8 - 1} = \frac{d+a}{ad-1} = y_4,$$

$$x_{11} = \frac{x_9 + y_{10}}{y_{10} x_9 - 1} = \frac{a + \frac{d+a}{ad-1}}{\frac{d+a}{ad-1}a - 1} = \frac{a^2 d + d}{a^2+1} = d = x_5,$$

$$y_{11} = \frac{y_9 + x_{10}}{x_{10} y_9 - 1} = \frac{c + \frac{b+c}{cb-1}}{\frac{b+c}{cb-1}c - 1} = \frac{c^2 b + b}{c^2+1} = b = y_5$$

and

$$x_{12} = \frac{x_{10} + y_{11}}{y_{11} x_{10} - 1} = \frac{\frac{b+c}{cb-1} + b}{b\frac{b+c}{cb-1} - 1} = \frac{c + cb^2}{1+b^2} = c = x_6,$$

$$y_{12} = \frac{y_{10} + x_{11}}{x_{11} y_{10} - 1} = \frac{\frac{d+a}{ad-1} + d}{d\frac{d+a}{ad-1} - 1} = \frac{ad^2 + a}{d^2+1} = y_6.$$

Also, we have

$$x_1 = \frac{d+a}{ad-1} = x_7 = x_{13} = \cdots = x_{6n+1}, \quad n = 0,1,2,3,\cdots$$

$$x_2 = b = x_8 = x_{14} = \cdots = x_{6n+2}, \quad n = 0,1,2,3,\cdots$$

$$x_3 = a = x_9 = x_{15} = \cdots = x_{6n+3}, \quad n = 0,1,2,3,\cdots$$

$$x_4 = \frac{b+c}{cb-1} = x_{10} = x_{16} = \cdots = x_{6n+4}, \quad n = 0,1,2,3,\cdots$$

$$x_5 = d = x_{11} = x_{17} = \cdots = x_{6n+5}, \quad n = 0,1,2,3,\cdots$$

$$x_6 = c = x_{12} = x_{18} = \cdots = x_{6n+6}, \quad n = 0,1,2,3,\cdots$$

and

$$y_1 = \frac{b+c}{cb-1} = y_7 = y_{13} = \cdots = y_{6n+1}, \quad n = 0,1,2,3,\cdots$$

$$y_2 = d = y_8 = y_{14} = \cdots = y_{6n+2}, \quad n = 0,1,2,3,\cdots$$

$$y_3 = c = y_9 = y_{15} = \cdots = y_{6n+3}, \quad n = 0,1,2,3,\cdots$$

$$y_4 = \frac{d+a}{ad-1} = y_{10} = y_{16} = \cdots = y_{6n+4}, \quad n = 0,1,2,3,\cdots$$

$$y_5 = b = y_{11} = y_{17} = \cdots = y_{6n+5}, \quad n = 0,1,2,3,\cdots$$

$y_6 = a = y_{12} = y_{18} = \cdots = y_{6n+6}, \quad n = 0,1,2,3,\cdots. \square$

Theorem 2. Let $y_0 = a,\ y_{-1} = b,\ x_0 = c,\ x_{-1} = d$ be arbitrary real numbers and let $\{x_n, y_n, z_n\}$ be a solutions of the system (1.1). Also, assume that $ad \neq 1$ and $cb \neq 1$. The solutions of x_n and y_n are six periodic.

Proof. The proof is clear from Theorem 1.□

Corollary 1. If $n \in \mathbb{N}$ for $x_{6n+k} = y_{6n+k+3}$.

Proof. The proof is clear from Theorem 1.

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