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On Hilbert's Integral Inequality and Its Applications

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Abstract

The main result of this paper is presented as follows *Let* $f, g, h, k \geq 0$, *h is homogeneous and symmetric of degree* λ *and* $F(x, y) = 1 - k(x) + k(y) \geq 0$. *Then*

$$\left(\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{h(x,y)} dxdy \right)^4 \leq \left(\left(C \int_0^\infty x^{1-\lambda} f^2(x)dx \right)^2 - \left(\int_0^\infty (Ck(x) - C(x)) x^{1-\lambda} f^2(x)dx \right)^2 \right)$$

$$\times \left(\left(C \int_0^\infty x^{1-\lambda} g^2(x)dx \right)^2 - \left(\int_0^\infty (Ck(x) - C(x)) x^{1-\lambda} g^2(x)dx \right)^2 \right),$$

where

$$C = \int_0^\infty \frac{t^a}{h(1,t)} dt, \quad C(x) = \int_0^\infty \frac{k(xt)t^a}{h(1,t)} dt,$$

provided the integrals on the RHS do exists. Some other special cases are also deduced.

1. Introduction

If

$$f, g \geq 0, \frac{1}{p} + \frac{1}{q} = 1,$$

$$0 < \int_0^\infty f^p(x)dx < \infty, \quad 0 < \int_0^\infty g^q(x)dx < \infty,$$

then

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dxdy$$

$$\leq \frac{\pi}{\sin(\pi/p)} \left(\int_0^\infty f^p(x)dx \right)^{1/p} \left(\int_0^\infty g^q(x)dx \right)^{1/q} \quad (1)$$

where the constant factor $\pi / \sin(\pi/p)$ is the best possible. Many mathematicians presented generalizations or new kinds of (1). Hardy inequality is very important in analysis theory and applications, it has been absorbing much interest of analysis see ([1,2]) .Very recently P. X. Ying and G. Mingzhe (see [3]) proved the following new kind

Theorem 1.1. *Let* $f(x)$ *be a real function. If*

$$0 < \int_0^\infty f^2(x)dx < \infty, \quad then$$

$$\left(\int_0^\infty \int_0^\infty \frac{f(x)f(y)}{x+y} dxdy \right)^2$$

$$< \pi^2 \left(\left(\int_0^\infty f^p(x)dx \right)^2 - \left(\int_0^\infty \varpi(x) f^p(x)dx \right)^2 \right) \quad (2)$$

where $\varpi(x) = \frac{1}{1+\sqrt{x}} - \frac{1}{1+x}$.

2. Lemma

The following lemma is needed for our aim.
  Lemma 2.1. *Let* $h(x, y)$ *be symmetric. Then*

$$\int_0^\infty \int_0^\infty \frac{f(x)f(y)}{h(x,y)} F(x,y) dxdy = \int_0^\infty \int_0^\infty \frac{f(x)f(y)}{h(x,y)} dxdy$$

$$\qquad (3)$$

where $F(x, y) = 1 - k(x) + k(y)$
  Proof.

$$\int_0^\infty \int_0^\infty \frac{f(x)f(y)}{h(x,y)} F(x,y) dxdy = \int_0^\infty \int_0^\infty \frac{f(x)f(y)}{h(x,y)} (1-k(x)+k(y)) dxdy$$

$$= \int_0^\infty \int_0^\infty \frac{f(x)f(y)}{h(x,y)} dxdy - \int_0^\infty \int_0^\infty \frac{f(x)f(y)k(x)}{h(x,y)} dxdy + \int_0^\infty \int_0^\infty \frac{f(x)f(y)k(y)}{h(x,y)} dxdy$$

$$= \int_0^\infty \int_0^\infty \frac{f(x)f(y)}{h(x,y)} dxdy - \int_0^\infty \int_0^\infty \frac{f(x)f(y)k(x)}{h(x,y)} dxdy + \int_0^\infty \int_0^\infty \frac{f(y)f(x)k(x)}{h(y,x)} dydx$$

$$= \int_0^\infty \int_0^\infty \frac{f(x)f(y)}{h(x,y)} dxdy - \int_0^\infty \int_0^\infty \frac{f(x)f(y)k(x)}{h(x,y)} dxdy + \int_0^\infty \int_0^\infty \frac{f(x)f(y)k(x)}{h(x,y)} dxdy = \int_0^\infty \int_0^\infty \frac{f(x)f(y)}{h(x,y)} dxdy$$

The object of this paper is to present the following general result

3. Main Results

Theorem 3.1. *Let* $f,g,h,k \geq 0$, *h is homogeneous and symmetric of degree* λ *and*

$$F(x,y) = 1 - k(x) + k(y) \geq 0.$$

Then

$$\left(\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{h(x,y)} dxdy \right)^4 \leq \left(\left(C \int_0^\infty x^{1-\lambda} f^2(x) dx \right)^2 - \left(\int_0^\infty (Ck(x) - C(x)) x^{1-\lambda} f^2(x) dx \right)^2 \right)$$

$$\times \left(\left(C \int_0^\infty x^{1-\lambda} g^2(x) dx \right)^2 - \left(\int_0^\infty (Ck(x) - C(x)) x^{1-\lambda} g^2(x) dx \right)^2 \right),$$

(4)

where

$$C = \int_0^\infty \frac{t^a}{h(1,t)} dt, \quad C(x) = \int_0^\infty \frac{k(xt)t^a}{h(1,t)} dt,$$

provided the integrals on the RHS do exists.
Proof.

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)F(x,y)}{h(x,y)} dxdy = \int_0^\infty \int_0^\infty \frac{f(x)\sqrt{F(x,y)}}{\sqrt{h(x,y)}} \times \frac{g(y)\sqrt{F(x,y)}}{\sqrt{h(x,y)}} dxdy$$

$$\leq \left(\int_0^\infty \int_0^\infty \frac{f^2(x)F(x,y)}{h(x,y)} dxdy \right)^{1/2} \left(\int_0^\infty \int_0^\infty \frac{g^2(y)F(x,y)}{h(x,y)} dxdy \right)^{1/2} = \sqrt{M} \sqrt{N}.$$

$$M = \int_0^\infty \int_0^\infty \frac{f(x)\sqrt{F(x,y)}}{\sqrt{h(x,y)}} \left(\frac{y}{x} \right)^{a/2} \times \frac{f(x)\sqrt{F(x,y)}}{\sqrt{h(x,y)}} \left(\frac{x}{y} \right)^{a/2} dxdy$$

$$\leq \left(\int_0^\infty \int_0^\infty \frac{f^2(x)F(x,y)}{h(x,y)} \left(\frac{y}{x} \right)^a dxdy \right)^{1/2} \left(\int_0^\infty \int_0^\infty \frac{f^2(x)F(x,y)}{h(x,y)} \left(\frac{y}{x} \right)^a dxdy \right)^{1/2} = \sqrt{M_1} \sqrt{M_2}.$$

$$M_1 = \int_0^\infty \int_0^\infty \frac{f^2(x)(1-k(x)+k(y))}{h(x,y)} \left(\frac{y}{x} \right)^a dxdy = \int_0^\infty f^2(x)(1-k(x)) \int_0^\infty \frac{(y/x)^a}{h(x,y)} dydx + \int_0^\infty f^2(x) \int_0^\infty \frac{k(y)(y/x)^a}{h(x,y)} dydx$$

$$= \int_0^\infty x^{1-\lambda} (1-k(x)) f^2(x) \int_0^\infty \frac{u^a}{h(1,u)} dudx + \int_0^\infty x^{1-\lambda} f^2(x) \int_0^\infty \frac{u^a k(xu)}{h(1,u)} dudx$$

$$= C \int_0^\infty x^{1-\lambda} f^2(x) dx - \int_0^\infty (Ck(x) - C(x)) x^{1-\lambda} f^2(x) dx.$$

Similarly,

$$M_2 = C\int_0^\infty x^{1-\lambda} f^2(x)dx + \int_0^\infty \left(Ck(x) - C(x)\right) x^{1-\lambda} f^2(x)dx.$$

Therefore

$$M^2 = \left(C\int_0^\infty x^{1-\lambda} f^2(x)dx\right)^2$$

$$- \left(\int_0^\infty \left(Ck(x) - C(x)\right) x^{1-\lambda} f^2(x)dx\right)^2,$$

and

$$N^2 = \left(C\int_0^\infty x^{1-\lambda} g^2(x)dx\right)^2$$

$$- \left(\int_0^\infty \left(Ck(x) - C(x)\right) x^{1-\lambda} g^2(x)dx\right)^2.$$

Applying lemma 2.1 to have

$$\left(\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{h(x,y)}dxdy\right)^4 = \left(\int_0^\infty\int_0^\infty \frac{f(x)g(y)F(x,y)}{h(x,y)}dxdy\right)^4$$

$$\leq \left(\left(C\int_0^\infty x^{1-\lambda} f^2(x)dx\right)^2 - \left(\int_0^\infty \left(Ck(x)-C(x)\right) x^{1-\lambda} f^2(x)dx\right)^2\right)$$

$$\times \left(\left(C\int_0^\infty x^{1-\lambda} g^2(x)dx\right)^2 - \left(\int_0^\infty \left(Ck(x)-C(x)\right) x^{1-\lambda} g^2(x)dx\right)^2\right).$$

Remark. It may be mentioned that theorem 1.1 follows from corollary 4.1 by putting

$$h(x,y) = x+y, \quad a = -1/2, \quad \lambda = 1, \quad k(x) = 1/(1+x)$$

as follows

$$C = \int_0^\infty \frac{u^{-1/2}}{1+u}du = 2\int_0^\infty \frac{du}{1+u^2} = \pi.$$

$$C(x) = \int_0^\infty \frac{k(xu)u^{-1/2}}{1+u}du = \int_0^\infty \frac{k(xu^2)}{1+u^2}du$$

$$= \int_0^\infty \frac{du}{\left(1+xu^2\right)\left(1+u^2\right)} = \frac{\pi}{1+\sqrt{x}},$$

and

$$Ck(x) - C(x) = \pi\left(\frac{1}{1+x} - \frac{1}{1+\sqrt{x}}\right)$$

Theorem 3.2. Let $f \geq 0$, $0 \leq g \leq 1$, $\lambda > 0$. Then

$$\left(\int_0^\infty\int_0^\infty \frac{f(x)f(y)\sqrt{1-g(x)}\sqrt{1+g(y)}}{(x+y)^\lambda}dxdy\right)^2 \leq B^2\left(\tfrac{\lambda}{2}, \tfrac{\lambda}{2}\right)\left(\left(\int_0^\infty x^{1-\lambda} f^2(x)dx\right)^2 - \left(\int_0^\infty x^{1-\lambda} f^2(x)g(x)dx\right)^2\right). \qquad (5)$$

Proof.

$$\int_0^\infty\int_0^\infty \frac{f(x)f(y)\sqrt{1-g(x)}\sqrt{1+g(y)}}{(x+y)^\lambda}dxdy = \int_0^\infty\int_0^\infty \frac{f(x)\sqrt{1-g(x)}}{(x+y)^{\lambda/2}}\left(\frac{y}{x}\right)^{\frac{\lambda}{4}-\frac{1}{2}} \times \frac{f(y)\sqrt{1+g(y)}}{(x+y)^{\lambda/2}}\left(\frac{x}{y}\right)^{\frac{\lambda}{4}-\frac{1}{2}}dxdy$$

$$\leq \left(\int_0^\infty\int_0^\infty \frac{f^2(x)(1-g(x))}{(x+y)^\lambda}\left(\frac{y}{x}\right)^{\frac{\lambda}{2}-1}dxdy\right)^{1/2}\left(\int_0^\infty\int_0^\infty \frac{f^2(y)(1+g(y))}{(x+y)^\lambda}\left(\frac{x}{y}\right)^{\frac{\lambda}{2}-1}dxdy\right)^{1/2} = \sqrt{G}\sqrt{H}.$$

$$G = \int_0^\infty f^2(x)(1-g(x))\left(\int_0^\infty \frac{(y/x)^{\frac{\lambda}{2}-1}}{(x+y)^\lambda}dy\right)dx = \int_0^\infty f^2(x)(1-g(x))\left(\int_0^\infty \frac{x^{1-\lambda}t^{\frac{\lambda}{2}-1}}{(1+t)^\lambda}dt\right)dx = B\left(\tfrac{\lambda}{2}, \tfrac{\lambda}{2}\right)\int_0^\infty x^{1-\lambda} f^2(x)(1-g(x))dx$$

Similarly

$$H = B\left(\tfrac{\lambda}{2}, \tfrac{\lambda}{2}\right) \int_0^\infty y^{1-\lambda} f^2(y)\left(1+g(y)\right) dy,$$

and hence

$$\left(\int_0^\infty\int_0^\infty \frac{f(x)f(y)\sqrt{1-g(x)}\sqrt{1+g(y)}}{(x+y)^\lambda} dxdy\right)^2 \le GH = B^2\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right)\left(\left(\int_0^\infty x^{1-\lambda} f^2(x) dx\right)^2 - \left(\int_0^\infty x^{1-\lambda} f^2(x) g(x) dx\right)^2\right).$$

4. Applications

Corollary 4.1. *Let $f, h, k \ge 0$, h is homogeneous and symmetric of degree λ.*
Then

$$\left(\int_0^\infty\int_0^\infty \frac{f(x)f(y)}{h(x,y)} dxdy\right)^2 \le \left(\left(C\int_0^\infty x^{1-\lambda} f^2(x) dx\right)^2 - \left(\int_0^\infty \left(Ck(x)-C(x)\right) x^{1-\lambda} f^2(x) dx\right)^2\right) \tag{6}$$

where

$$C = \int_0^\infty \frac{t^a}{h(1,t)} dt, \quad C(x) = \int_0^\infty \frac{k(xt)t^a}{h(1,t)} dt,$$

provided the integrals on the RHS do exists.

Proof. The proof follows from theorem 3.1 by putting $g = f$

Corollary 4.2. *Let $f, h, k \ge 0$, h is homogeneous and symmetric of degree λ.*
Then

$$\left(\int_0^\infty\int_0^\infty \frac{f(x)f(y)}{(x+y)^\lambda} dxdy\right)^2 \le \left(\left(C\int_0^\infty x^{1-\lambda} f^2(x) dx\right)^2 - \left(\int_0^\infty \left(Ck(x)-C(x)\right) x^{1-\lambda} f^2(x) dx\right)^2\right) \tag{7}$$

where

$$C = \int_0^\infty \frac{t^a}{(1+t)^\lambda} dt, \quad C(x) = \int_0^\infty \frac{k(xt)t^a}{(1+t)^\lambda} dt,$$

provided the integrals on the RHS do exists.

Proof. The proof follows from corollary 4.1 by putting $h(x,y) = (x+y)^\lambda$.

Corollary 4.3. Let $f, h, k \ge 0$, h is homogeneous and symmetric of degree λ.
Then

$$\left(\int_0^\infty\int_0^\infty \frac{f(x)f(y)}{x^\lambda+y^\lambda} dxdy\right)^2 \le \left(\left(C\int_0^\infty x^{1-\lambda} f^2(x) dx\right)^2 - \left(\int_0^\infty \left(Ck(x)-C(x)\right) x^{1-\lambda} f^2(x) dx\right)^2\right) \tag{8}$$

where

$$C = \int_0^\infty \frac{t^a}{1+t^\lambda} dt, \quad C(x) = \int_0^\infty \frac{k(xt)t^a}{1+t^\lambda} dt,$$

provided the integrals on the RHS do exists.

Proof. The proof follows from corollary 4.1 by putting $h(x,y) = x^\lambda + y^\lambda$.

5. References

[1]   G. H. Hardy, J. E. Littlewood and G. Polya, "Inequalities," Cambridge University Press, Cambridge, 1952.

[2]   J. E. Mitronovic, J. E. Pecaric and A. M. Fink, "Inequalities Involving Functions and Their Integrals and Derivatives," Boston Kluwer Academic Publishers, 1991.

[3]   P. X. Ying and G. Mingzhe, "On Hilbert's Integral Inequality and Its Applications," *Journal of Applied Mathematics*, Preprint.