In this paper we consider a problem of investigating the dependence of \( \|P(\beta z)\|_p - \beta \|P(z)\|_p \) on \( \|P(z)\|_p \) for every real or complex number \( \beta \) with \( |\beta| \leq 1 \), \( R > r \geq 1 \), \( p > 0 \) and present certain compact generalizations which, besides yielding some interesting results as corollaries, include some well-known results, in particular, those of Zygmund, Bernstein, De-Bruijn, Erdös-Lax and Boas and Rahman as special cases.

Keywords: \( L^p \)-Inequalities, Polynomials, Complex Domain

1. Introduction

Let \( P_n(z) \) denote the space of all complex polynomials \( \sum_{j=0}^{n} a_j z^j \) of degree at most \( n \). For \( P \in P_n \), define
\[
\|P(z)\|_p \coloneqq \left\{ \frac{1}{2\pi} \int_{0}^{2\pi} |P(e^{i\theta})|^p \, d\theta \right\}^{1/p}, 1 \leq p < \infty
\]
and
\[
\|P(z)\|_\infty \coloneqq \max_{\mathbb{C}} |P(z)|.
\]

A famous result known as Bernstein’s inequality (for reference, see [1] or [2]) states that if \( P \in P_n \), then
\[
\|P'(z)\|_p \leq n \|P(z)\|_p
\]
(1)
whereas concerning the maximum modulus of \( P(z) \) on the circle \( |z| = R > 1 \), we have
\[
\|P(Rz)\|_p \leq R^n \|P(z)\|_p,
\]
(2)
(for reference, see [3]). Inequalities (1) and (2) can be obtained by letting \( p \to \infty \) in the inequalities
\[
\|P'(z)\|_p \leq n \|P(z)\|_p, \quad p \geq 1
\]
(3)
and
\[
\|P(Rz)\|_p \leq R^n \|P(z)\|_p, \quad R > 1, \quad p > 0,
\]
(4)
respectively. Inequality (3) was found by Zygmund [4] whereas inequality (4) is a simple consequence of a result of Hardy [5] (see also [6]). Since Inequality (3) was deduced from M. Riesz’s interpolation formula [7] by means of Minkowski’s inequality, it was not clear, whether the restriction on \( p \) was indeed essential. This question was open for a long time. Finally Arestov [8] proved that (3) remains true for \( 0 < p < 1 \) as well. Both the Inequalities (3) and (4) can be sharpened if we restrict ourselves to the class of polynomials having no zero in \( |z| < 1 \). In fact, if \( P \in P_n \) and \( P(z) \neq 0 \) in \( |z| < 1 \), then Inequalities (3) and (4) can be respectively replaced by
\[
\|P'(z)\|_p \leq n \|P(z)\|_p, \quad p > 0
\]
(5)
and
\[
\|P(Rz)\|_p \leq R^n \|P(z)\|_p, \quad R > 1, \quad p > 0.
\]
(6)

Inequality (5) is due to De-Bruijn [9] for \( p \geq 1 \) and Rahman and Schmeisser [10] extended it for \( 0 < p < 1 \) whereas the Inequality (6) was proved by Boas and Rahman [11] for \( p \geq 1 \) and later it was extended for \( 0 < p < 1 \) by Rahman and Schmeisser [12]. For \( p = \infty \), the Inequality (5) was conjectured by Erdős and later verified by Lax [13] whereas Inequality (6) was proved by Ankeny and Rivlin [14].

Recently the Authors in [12] (see also [15]) investigated the dependence of
\[
\|P(Rz) - P(z)\|_p \text{ on } \|P(z)\|_p
\]
for \( R > 1, \quad p \geq 1 \). As a compact generalization of Inequalities (3) and (4), they have shown that if \( P \in P_n \), then for every \( R > 1 \) and \( p \geq 1 \),
It is natural to seek the corresponding analog of (7) for polynomials \( P \in P_n \) having no zero in \( |z| < 1 \) and which is a compact generalization of Inequalities (5) and (6). In the present paper we consider a more general problem of investigating the dependence of
\[
\|P(Rz) - P(z)\|_p \leq \left( R^n - 1 \right) \|P(z)\|_p. \tag{7}
\]
for every real or complex number \( \beta \) with \( |\beta| \leq 1 \), \( R > r \geq 1 \), \( p > 0 \) and develop a unified method for arriving at these results. We first present the following interesting result and a compact generalization of Inequalities (3) and (4), which also extends Inequality (7) for \( 0 < p < 1 \) as well.

**Theorem 1.** If \( P \in P_n \), then for every real or complex number \( \beta \) with \( |\beta| \leq 1 \), \( R > r \geq 1 \) and \( p > 0 \),
\[
\|P(Rz) - \beta P(z)\|_p \leq \left( R^n - r^n \right) \|P(z)\|_p. \tag{8}
\]
The result is best possible and equality in (8) holds for \( P(z) = az^n, a \neq 0 \).

**Remark 1.** For \( \beta = 0 \), Theorem 1 reduces to Inequality (4) and for \( \beta = 1 \), \( r = 1 \), it validates Inequality (7) for each \( p > 0 \).

If we set \( \beta = 1 \) in Inequality (8), we immediately get the following generalization of Inequality (7).

**Corollary 1.** If \( P \in P_n \), then for \( R > r \geq 1 \) and \( p > 0 \),
\[
\|P(Rz) - P(z)\|_p \leq \left( R^n - r^n \right) \|P(z)\|_p. \tag{9}
\]
The result is best possible and equality in (9) holds for \( P(z) = az^n, a \neq 0 \).

For \( \beta > 0 \), we do not have the corresponding analog of (7) for polynomials \( \in P_n \) having no zero in \( |z| < 1 \) and which is a compact generalization of Inequalities (5) and (6). In the present paper we consider a more general problem of investigating the dependence of
\[
\|P(Rz) - \beta P(z)\|_p \leq \left( R^n - \beta r^n \right) \|P(z)\|_p. \tag{10}
\]
for every real or complex number \( \beta \) with \( |\beta| \leq 1 \), \( R > r \geq 1 \) and \( p > 0 \) and develop a unified method for (10) for \( 0 < p < 1 \) as well.

**Corollary 2.** If \( P \in P_n \), then for \( R > r \geq 1 \) and \( p > 0 \),
\[
\|P(Rz) - P(z)\|_p \leq \left( R^n - r^n \right) \|P(z)\|_p. \tag{11}
\]

**Remark 2.** For \( r = 1 \), Corollary 2 reduces to Zygmond’s Inequality (3) for each \( p > 0 \).

The following result which is a compact generalization of Inequalities of (1) and (2) follows from Theorem 1 by letting \( p \to \infty \) in Inequality (8).

**Corollary 3.** If \( P \in P_n \), then for every real or complex number \( \beta \) with \( |\beta| \leq 1 \) and \( R > r \geq 1 \),
\[
\|P(Rz) - \beta P(z)\|_p \leq \left( R^n - \beta r^n \right) \max_{|z|=1} |P(z)| \tag{12}
\]
for \( |z| = 1 \).

The result is best possible and equality in (12) holds for \( P(z) = az^n, a \neq 0 \).

**Remark 3.** For \( \beta = 0 \), Corollary 3 reduces to Inequality (2) and for \( \beta = 1 \), if we divide the two sides of (11) by \( R - r \) and let \( R \to r \), it follows that if \( P \in P_n \), then for \( r \geq 1 \),
\[
\|P(Rz) - P(z)\|_p \leq \max_{|z|=1} |P(z)| \tag{13}
\]
for \( |z| = 1 \).

Theorem 2. If \( P \in P_n \) and \( P(z) \) does not vanish in \( |z| < 1 \), then for every real or complex number \( \beta \) with \( |\beta| \leq 1 \), \( R > r \geq 1 \) and \( p > 0 \)
\[
\|P(Rz) - \beta P(z)\|_p \leq \frac{\left( R^n - \beta r^n \right) \|1 - \beta\|_p}{\|z\|_p} \|P(z)\|_p. \tag{14}
\]
The result is best possible and equality in (14) holds for \( P(z) = az^n + b, |a| = |b| = 1 \).

**Remark 4.** For \( p = 1 \), if we do not permit \( 0 < p < 1 \) in (14), we immediately get De-Bruijn’s theorem (Inequality (5)) for each \( p > 0 \).

Next we mention the following compact generalization of a theorem of Erdős and Lax (Inequality (5) for \( p = \infty \)) and a result of Ankeny and Rivlin (Inequality (5) for \( p = \infty \)) which immediately follows from Theorem 2 by letting \( p \to \infty \) in (13).

**Corollary 4.** If \( P \in P_n \) and \( P(z) \) does not vanish in \( |z| < 1 \), then for every real or complex number \( \beta \) with \( |\beta| \leq 1 \) and \( R > r \geq 1 \),
\[
\|P(Rz) - \beta P(z)\|_p \leq \frac{\left( R^n - \beta r^n \right) \|1 - \beta\|_p}{\|z\|_p} \|P(z)\|_p. \tag{15}
\]
for \( |z| = 1 \).

The result is best possible and equality in (15) holds for \( P(z) = az^n + b, |a| = |b| = 1 \).
Remark 5. For $\beta = 1$, if we divide the two sides of (15) by $R - r$ and let $R \to r$, we get
\[ |P'(rz)| \leq \frac{R}{2} r^{n-1} \max |P(z)| \text{ for } |z| = 1. \] (16)

For $r = 1$, Inequality (16) was conjectured by Erdős and later verified by Lax[10]. If we take $\beta = 0$ in (15), we immediately get
\[ |P(Rz)| \leq \frac{R^2 + 1}{2} |P(z)|, R > 1. \] (17)

Inequality (17) is due to Ankeny and Rivlin [1].

A polynomial $P \in P_n$ is said to be self-inversive if $P(z) = uQ(z)$ for all $z \in C$ where $|u| = 1$ and $Q(z) = z^n P(1/z)$.$^1$ It is known[16, 17] that if $P \in P_n$ is self-inversive polynomial, then for every $p \geq 1$,
\[ \left\| P'(z) \right\|_{p} \leq n \left\| P(z) \right\|_{p} \left\| z \right\|_{p}. \] (18)

Finally, we present the following result which include some well-known results for self-inversive polynomials as special cases.

**Theorem 3.** If $P \in P_n$ is self-inversive polynomial, then for every real or complex number $\beta$ with $|\beta| \leq 1$, $R > r \geq 1$ and $p > 0$,
\[ \left\| P(Rz) - \beta P(rz) \right\|_{p} \leq \frac{\left\| R^n - \beta r^n \right\|}{\left\| 1 + z \right\|_{p}} \left\| P(z) \right\|_{p}. \] (19)

The result is best possible and equality in (19) holds for $P(z) = z^n + 1$.

**Remark 6.** Taking $\beta = 0$ in Theorem 3, it follows that if $P \in P_n$ is self-inversive polynomial, then for $R > 1$ and $p > 0$,
\[ \|P(Rz)\|_p \leq \frac{\|R^n z + 1\|_p}{\|1 + z\|_p} \|P(z)\|_p. \] (20)

The result is sharp.

Many interesting results can be deduced from Theorem 3 in exactly the same way as we have deduced from Theorem 2.

**2. Lemmas**

For the proofs of these theorems, we need the following lemmas.

**Lemma 1.** If $P \in P_n$ and $P(z)$ has all its zeros in $|z| \leq k$ where $k \leq 1$, then for every $R \geq r \geq 1$ and $|z| = 1$,
\[ \|P(Rz)\|_p \geq \frac{(R + k)^n}{(r + k)^n} \|P(rz)\|_p. \] (21)

**Proof of Lemma 1.** Since all the zeros of $P(z)$ lie in $|z| \leq k$, we write
\[ P(z) = C \prod_{j=1}^{n} (z - r_j e^{i\theta_j}) \]
where $r_j \leq k$. Now for $0 \leq \theta < 2\pi$, $R \geq r \geq 1$, we have
\[ \left| \frac{Re^{i\theta} - r_j e^{i\theta_j}}{re^{i\theta} - r_j e^{i\theta_j}} \right| = \left| \frac{R^2 + r_j^2 - 2Rr \cos(\theta - \theta_j)}{r^2 + r_j^2 - 2rr_j \cos(\theta - \theta_j)} \right| \geq \frac{R + r_j}{r + r_j} \left( \frac{R + k}{r + k} \right)^n, j = 1, 2, \ldots, n. \]

Hence
\[ \frac{\|P(Re^{i\theta})\|}{\|P(re^{i\theta})\|} \geq \prod_{j=1}^{n} \frac{R + r_j}{r + r_j} \left( \frac{R + k}{r + k} \right)^n \]
for $0 \leq \theta < 2\pi$. This implies for $|z| = 1$ and $R \geq r \geq 1$,
\[ \|P(Rz)\|_p \geq \frac{(R + k)^n}{(r + k)^n} \|P(rz)\|_p, \]
which completes the proof of Lemma 1.

**Lemma 2.** If $P \in P_n$ and $P(z)$ does not vanish in $|z| < 1$, then for every real or complex number $\beta$ with $|\beta| \leq 1$, $R > r \geq 1$, and $|z| = 1$,
\[ \|P(Rz) - \beta P(rz)\|_p \leq \|Q(z) - \beta P(z)\|_p \] (22)
where $Q(z) = z^n P(1/z)$. The result is sharp and equality in (22) holds for $P(z) = z^n + 1$.

**Proof of Lemma 2.** For the case $R = r$, the result follows by observing that $\|P(z)\| \leq \|Q(z)\|$ for $|z| \geq 1$. Henceforth, we assume that $R > r$. Since the polynomial $P(z)$ has all its zeros in $|z| \geq 1$, therefore, for every real or complex number $\alpha$ with $|\alpha| > 1$, the polynomial $f(z) = P(z) - \alpha Q(z)$, where $Q(z) = z^n P(1/z)$, has all its zeros in $|z| \leq 1$. Applying Lemma 1 to the polynomial $f(z)$ with $k = 1$, we obtain for every $R > r \geq 1$ and $0 \leq \theta < 2\pi$,
\[ \left| f(Re^{i\theta}) \right| \geq \frac{(R + 1)^n}{(r + 1)^n} \left| f(re^{i\theta}) \right|. \] (23)

Since $f(Re^{i\theta}) \neq 0$ for every $R > r \geq 1, 0 \leq \theta < 2\pi$ and $R + 1 > r + 1$, it follows from (23) that
\[ |f(Re^{i\theta})| > \left( \frac{r+1}{r+1} \right)^n |f(Re^{i\theta})| \geq |f(0)| \]
for every \( R > r \geq 1 \) and \( 0 \leq \theta < 2\pi \). This gives
\[ |f(Rz)| < |f(Rz)|, \]
for \( |z| = 1 \) and \( R > r \geq 1 \).

Using Rouche’s theorem and noting that all the zeros of \( f(Rz) \) lie in \( |z| \leq \frac{1}{R} < 1 \), we conclude that the polynomial
\[ T(z) = f(Rz) - \beta f(z) \]
\[ = \{P(Rz) - \beta P(z)\} - \alpha \{Q(Rz) - \beta Q(z)\} \quad (24) \]
has all its zeros in \( |z| < 1 \) for every real or complex number \( \beta, \alpha \) with \( |\beta| \leq 1, |\alpha| > 1 \) and \( R > r \geq 1 \). This implies
\[ |P(Rz) - \beta P(z)| \leq |Q(Rz) - \beta Q(z)| \quad (25) \]
for \( |z| \geq 1 \) and \( R > r \geq 1 \). If Inequality (25) is not true, then exist a point \( z = w \) with \( |w| \geq 1 \) such that
\[ |P(Rw) - \beta P(w)| > |Q(Rw) - \beta Q(w)|. \]
But all the zeros of \( Q(z) \) lie in \( |z| \leq 1 \), therefore, it follows (as in case of \( f(z) \) ) that all the zeros of \( Q(Rz) - \beta Q(z) \) lie in \( |z| < 1 \). Hence \( Q(Rw) - \beta Q(w) \neq 0 \) with \( |w| \geq 1 \). We take
\[ \alpha = \frac{P(Rw) - \beta P(w)}{Q(Rw) - \beta Q(w)}, \]
then \( \alpha \) is a well defined real or complex number with \( |\alpha| > 1 \) and with this choice of \( \alpha \), from (24) we obtain
\[ T(w) = 0 \]
where \( |w| \geq 1 \). This contradicts the fact that all the zeros of \( T(z) \) lie in \( |z| < 1 \). Thus
\[ |P(Rz) - \beta P(z)| \leq |Q(Rz) - \beta Q(z)| \]
for \( |z| \geq 1 \) and \( R > r \geq 1 \). This proves Lemma 2.

Next we describe a result of Arestov.

For \( \gamma = (\gamma_0, \gamma_1, \ldots, \gamma_n) \) and \( P(z) = \sum_{j=0}^{n} a_j z^j \in P_n \), we define
\[ \Lambda_{\gamma} P(z) = \sum_{j=0}^{n} \gamma_j a_j z^j. \]

The operator \( \Lambda_{\gamma} \) is said to be admissible if it preserves one of the following properties:
1) \( P(z) \) has all its zeros in \( \{ z \in \mathbb{C} : |z| \leq 1 \} \),
2) \( P(z) \) has all its zeros in \( \{ z \in \mathbb{C} : |z| \geq 1 \} \).

The result of Arestov may now be stated as follows.

**Lemma 3.** [8] Let \( \phi(x) = \psi(\log x) \) where \( \psi \) is a convex nondecreasing function on \( R \). Then for all \( P \in P_n \) and each admissible operator \( \Lambda_{\gamma} \),
\[ \int_0^{2\pi} \phi (|\Lambda_{\gamma} P(e^{i\theta})|) d\theta \leq \int_0^{2\pi} \phi (C(\gamma, n)|P(e^{i\theta})|) d\theta \]
where \( C(\gamma, n) = \max (\{|\gamma_0|, |\gamma_n|\}) \).

In particular, Lemma 3 applies with \( \phi : x \rightarrow x^p \) for every \( p \in (0, \infty) \). Therefore, we have
\[ \int_0^{2\pi} \phi (|\Lambda_{\gamma} P(e^{i\theta})|) d\theta \leq (C(\gamma, n) \int_0^{2\pi} \phi (|P(e^{i\theta})|) d\theta)^{1/p}. \]

We use (26) to prove the following interesting result.

**Lemma 4.** Let \( Q(z) = z^n P(1/z) \). Since \( P(z) \) does not vanish in \( |z| < 1 \), by Lemma 2, for every real or complex number \( \beta \) with \( |\beta| \leq 1 \), \( R > r \geq 1 \) and \( |z| = 1 \), we have
\[ |P(Rz) - \beta P(z)| \leq |Q(Rz) - \beta Q(z)| = |R^n P(z/R) - \beta R^n P(z/R)|. \]

Now (as in the proof of Lemma 2), the polynomial
\[ H(z) = Q(Rz) - \beta Q(z) = R^n z^n P(1/Rz) - \beta R^n z^n P(1/z) \]
has all its zeros in \( |z| < 1 \) for every real or complex number \( \beta \) with \( |\beta| \leq 1 \) and \( R > r \), it follows that the polynomial
\[ z^n H(1/z) = R^n P(z/R) - \beta R^n P(z/R) \]
has all its zeros in \( |z| > 1 \). Hence the function
\[ f(z) = \frac{P(Rz) - \beta P(z)}{R^n P(z/R) - \beta R^n P(z/R)} \]
is analytic in \( |z| \leq 1 \) and \( |f(z)| \leq 1 \) for \( |z| = 1 \). Since \( f(z) \) is not a constant, it follows by the Maximum Modulus Principle that
\[ |f(z)| < 1 \]
or equivalently,
\[ |P(Rz) - \beta P(z)| < |R^n P(z/R) - \beta R^n P(z/R)| \]
for \( |z| < 1 \).

(28)

A direct application of Rouche’s theorem shows that
where \( \alpha \), \( \beta \), and \( \gamma \) are real numbers. If \( \beta \leq 1 \), \( R > r \geq 1 \), and \( \alpha \) real, then the desired result follows immediately for each \( p > 0 \). This completes the proof of Lemma 4.

From Lemma 4, we deduce the following more general lemma which is a result of independent interest with a variety of applications.

**Lemma 5.** If \( P \in P_2 \), then for every real or complex number \( \beta \) with \( |\beta| \leq 1 \), \( R > r \geq 1 \), and \( \alpha \) real,

\[
\int_0^{2\pi} \left| P(R e^{i\theta}) - \beta P(r e^{i\theta}) \right|^2 d\theta \\
+ \left| e^{i\alpha} \left( R^2 P(e^{i\theta}/R) - \overline{\rho} r^2 P(e^{i\theta}/r) \right) \right|^2 d\theta \\
\leq \left| \left( R^2 - \beta r^2 \right) + e^{i\alpha} \left( 1 - \beta \right) \right|^2 \int_0^{2\pi} \left| P(e^{i\theta}) \right|^2 d\theta.
\]

The result is sharp and equality in (29) holds for \( P(z) = \lambda z^n \), \( \lambda \neq 0 \).

**Proof of Lemma 5.** Since \( P(z) \) is a polynomial of degree at most \( n \), we can write

\[
P(z) = P_1(z)P_2(z) = \prod_{j=1}^k (z - z_j) \prod_{j=4+1}^n (z - z_j), \quad k \geq 1
\]

where all the zeros of \( P_1(z) \) lie in \( |z| > 1 \) and all the zeros of \( P_2(z) \) lie in \( |z| < 1 \). First we suppose that \( P_1(z) \) has no zero on \( |z| = 1 \) so that all the zeros of \( P_1(z) \) lie in \( |z| > 1 \). Let \( Q_2(z) = z^{n-k} \prod_{j=4+1}^n (1 - z z_j) \), then all the zeros of \( Q_2(z) \) lie in \( |z| > 1 \) and \( |Q_2(z)| = |P_2(z)| \) for \( |z| > 1 \). Now consider the polynomial

\[
g(z) = P_1(z)Q_2(z) = \prod_{j=1}^k (z - z_j) \prod_{j=4+1}^n (1 - z z_j),
\]

then all the zeros of \( g(z) \) lie in \( |z| > 1 \) and for \( |z| = 1 \), \( |g(z)| = |P_1(z)|Q_2(z)| = |P_1(z)|P_2(z) = |P(z)|. \quad (30)

By the Maximum Modulus Principle, it follows that

\[
|P(z)| \leq |g(z)| \quad \text{for} \quad |z| \leq 1.
\]

We claim that the polynomial \( h(z) = P(z) + \lambda g(z) \) does not vanish in \( |z| \leq 1 \) for every \( \lambda \) with \( |\lambda| > 1 \). If this is not true, then \( h(z_0) = 0 \) for some \( z_0 \) with \( |z_0| \leq 1 \). This gives

\[
|P(z_0)| = |\lambda| |g(z_0)|.
\]

Since \( g(z_0) \neq 0 \) and \( |\lambda| > 1 \), it follows that

\[
|P(z_0)| > |g(z_0)| \quad \text{with} \quad |z_0| \leq 1,
\]

which clearly contradicts (31). Thus \( h(z) \) does not vanish in \( |z| \leq 1 \) for every \( \lambda \) with \( |\lambda| > 1 \), so that all the zeros of \( h(z) \) lie in \( |z| > \rho \) for some \( \rho > 1 \) and hence all the zeros of \( h(\rho z) \) lie in \( |z| > 1 \). Applying (28) to the polynomial \( h(\rho z) \), we get

\[
|h(R \rho z) - \beta h(\rho z)| \leq |R^2 h(\rho z/r) - \overline{\rho} r^2 h(\rho z/r)|
\]

for \( |z| < 1 \), \( R > r \geq 1 \).

Taking \( z = e^{i\rho}/\rho, 0 \leq \theta < 2\pi \), then \( |z| = (1/\rho) < 1 \) as \( \rho > 1 \) and we get

\[
|h(R e^{i\theta}) - \beta h(e^{i\theta})| \leq |R^2 h(e^{i\theta}/R) - \overline{\rho} r^2 h(e^{i\theta}/r)|,
\]

\[0 \leq \theta < 2\pi, \quad R > r \geq 1 \quad \text{and} \quad |\beta| < 1. \]

This implies

\[
h(R-z) - \beta h(z) < |R^2 h(z/R) - \overline{\rho} r^2 h(z/r)| \quad \text{for} \quad |z| = 1.
\]

An application of Rouche's theorem shows that the polynomial

\[
T(z) = h(Rz) - \beta h(z) + e^{i\alpha} \left( R^2 h(z/R) - \overline{\rho} r^2 h(z/r) \right)
\]

does not vanish in \( |z| \leq 1 \) for every real or complex number \( \beta \) with \( |\beta| < 1 \), \( R > r \geq 1 \) and \( \alpha \) real. Replacing \( h(z) \) by \( P(z) + \lambda h(z) \), it follows that the polynomial

\[
T(z) = \left\{ \left| P(Rz) - \beta P(z) \right| + e^{i\alpha} \left( R^2 P(z/R) - \overline{\rho} r^2 P(z/r) \right) \right\}
\]

\[+ \left| e^{i\alpha} \left( R^2 P(z/R) - \overline{\rho} r^2 P(z/r) \right) \right|^2 d\theta
\]

\[
\leq \left( |g(Rz) - \beta g(z)| + e^{i\alpha} \left( R^2 g(z/R) - \overline{\rho} r^2 g(z/r) \right) \right)
\]

\[
\left| \left( R^2 - \beta r^2 \right) + e^{i\alpha} \left( 1 - \beta \right) \right|^2 \int_0^{2\pi} \left| P(e^{i\theta}) \right|^2 d\theta.
\]

This completes the proof of Lemma 5.

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e in $|z| > 1$ for every real or complex number $\beta$ with $|\beta| \leq 1$, $R > r \geq 1$ and $\alpha$ real. Hence
\[
g(Rz_0) - \beta g(rz_0) + e^{\alpha} (R^*g((z_0)/R) - \bar{\beta} r^*g((z_0)/r)) \neq 0
\]
with $|z_0| \leq 1$.

We take
\[
\lambda = \left( P(Rz_0) - \beta P(rz_0) \right) + e^{\alpha} (R^* P(z_0/R) - \bar{\beta} r^* P(z_0/r))
\]
so that $\lambda$ is a well-defined real or complex number with $|\lambda| > 1$ and with this choice of $\lambda$, from (32) we get $T(z_0) = 0$ with $|z_0| \leq 1$. This clearly is a contradiction to the fact that $T(z)$ does not vanish in $|z| \leq 1$. Thus for every $\beta$ with $|\beta| \leq 1$, $R > r \geq 1$ and $\alpha$ real,
\[
\begin{align*}
&\left| \left( P(Rz) - \beta P(rz) \right) + e^{\alpha} \left( R^* P(z/R) - \bar{\beta} r^* P(z/r) \right) \right| \\
&\leq \left( |g(Rz) - \beta g(rz)| + e^{\alpha} \left( R^* g((z)/R) - \bar{\beta} r^* g((z)/r) \right) \right)
\end{align*}
\]
for $|z| \leq 1$, which in particular gives for each $p > 0$ and $0 \leq \theta < 2\pi$,
\[
\int_{0}^{2\pi} \left| \left( P(Re^{i\theta}) - \beta P(re^{i\theta}) \right) + e^{\alpha} \left( R^* P(e^{i\theta}/R) - \bar{\beta} r^* P(e^{i\theta}/r) \right) \right| d\theta
\]
\[
\leq \int_{0}^{2\pi} \left( |g(Re^{i\theta}) - \beta g(re^{i\theta})| + e^{\alpha} \left( R^* g(e^{i\theta}/R) - \bar{\beta} r^* g(e^{i\theta}/r) \right) \right) d\theta.
\]
Using lemma 4 and (30), it follows that for every $\beta$ with $|\beta| \leq 1$, $R > r > 0$ and $\alpha$ real,
\[
\int_{0}^{2\pi} \left( P(Re^{i\theta}) - \beta P(re^{i\theta}) \right) + e^{\alpha} \left( R^* P(e^{i\theta}/R) - \bar{\beta} r^* P(e^{i\theta}/r) \right) d\theta
\]
\[
\leq \left( |R^* - \beta r^*| + e^{\alpha} \left( 1 - \bar{\beta} \right) \right) \int_{0}^{2\pi} \left| g(e^{i\theta}) \right| d\theta
\]
\[
= \left( |R^* - \beta r^*| + e^{\alpha} \left( 1 - \bar{\beta} \right) \right) \int_{0}^{2\pi} \left| P(e^{i\theta}) \right| d\theta.
\]
Now if $P_1(z)$ has a zero on $|z| = 1$, then applying (34) to the polynomial $P'(z) = P_1(z)P_2(z)$ where $t < 1$, we get for every $\beta$ with $|\beta| \leq 1$, $R > r \geq 1$, $p > 0$ and $\alpha$ real,
\[
\int_{0}^{2\pi} \left( P'(Re^{i\theta}) - \beta P'(re^{i\theta}) \right) + e^{\alpha} \left( R^* P'(e^{i\theta}/R) - \bar{\beta} r^* P'(e^{i\theta}/r) \right) d\theta
\]
\[
\leq \left( |R^* - \beta r^*| + e^{\alpha} \left( 1 - \bar{\beta} \right) \right) \int_{0}^{2\pi} \left| P'(e^{i\theta}) \right| d\theta.
\]
Letting $t \to 1$ in (35) and using continuity, the desired result follows immediately and this proves Lemma 5.

3. Proofs of the Theorems

Proof of Theorem 1. Since $P(z)$ is a polynomial of degree at most $n$, we can write
\[
P(z) = P_1(z)P_2(z) = \prod_{j=1}^{k} (z - z_j) \prod_{j=k+1}^{n} (z - z_j), k \geq 1
\]
where all the zeros of $P_1(z)$ lie in $|z| > 1$ and all the zeros of $P_2(z)$ lie in $|z| < 1$. First we suppose that all the zeros of $P_1(z)$ lie in $|z| < 1$. Let $Q_j(z) = z^{n_j} P_j(1/z)$, then all the zeros of $Q_j(z)$ lie in $|z| < 1$ and $Q_j(z) = P_2(z)$ for $|z| = 1$. Now consider the polynomial
\[
F(z) = P_1(z)Q_j(z) = \prod_{j=1}^{k} (z - z_j) \prod_{j=k+1}^{n} (1 - 1/z_j),
\]
then all the zeros of $F(z)$ lie in $|z| < 1$ and for $|z| = 1$,
\[
|F(z)| = |P_1(z)||Q_j(z)| = |P_1(z)||P_2(z)| = |P(z)|.
\]

By the Maximum Modulus Principle, it follows that
\[
|P(z)| \leq |F(z)|
\]
for $|z| = 1$ and $R > r \geq 1$.

Since all the zeros of $H(Rz)$ lie in $|z| < 1$, we conclude that for every $\beta$, $\lambda$ with $|\beta| \leq 1$ and $|\lambda| > 1$, the zeros of polynomial
\[
G(z) = H(Rz) - \beta H(Rz)
\]
\[
= (P(Rz) - \beta P(Rz)) + \lambda (F(Rz) - \beta F(Rz))
\]
lie in $|z| < 1$. This implies (as in the case of Lemma 2)
\[
|P(Rz) - \beta P(Rz)| \leq |F(Rz) - \beta F(Rz)|
\]
for $|z| \geq 1$ and $R > r \geq 1$,

which in particular gives for $R > r$ and $p > 0$,
\[
\int_{0}^{2\pi} \left| P(Re^{i\theta}) - \beta P(re^{i\theta}) \right|^p d\theta
\]
\[
\leq \int_{0}^{2\pi} \left| F(Re^{i\theta}) - \beta F(re^{i\theta}) \right|^p d\theta
\]
Again, since all the zeros of \( F(z) \) lie in \( |z| < 1 \), as before, \( F(Rz) - \beta F(z) \) has all its zeros in \( |z| < 1 \) for every real or complex number \( \beta \) with \( |\beta| \leq 1 \). Therefore, the operator \( \Lambda_\gamma \) defined by
\[
\Lambda_\gamma F(z) = F(Rz) - \beta F(z)
\]

is admissible. Hence by (26) of Lemma (3), for each \( p > 0 \), we have
\[
\int_0^{2\pi} \left[ F(Re^{i\theta}) - \beta F(re^{i\theta}) \right]^p d\theta \\
\leq \left| R^n - \beta r^n \right| \int_0^{2\pi} \left[ F(e^{i\theta}) \right]^p d\theta.
\]

Combining Inequalities (37) and (38) and noting that
\[
\left| F(e^{i\theta}) \right| = \left| P(e^{i\theta}) \right|
\]
we obtain for \( R > r \geq 1 \) and \( p > 0 \)
\[
\left[ \int_0^{2\pi} \left| P(Re^{i\theta}) \right|^p d\theta \right]^{1/p} \\
\leq \left| R^n - \beta r^n \right| \left[ \int_0^{2\pi} \left| P(e^{i\theta}) \right|^p d\theta \right]^{1/p}.
\]

In case \( P_i(z) \) has a zero on \( |z| = 1 \), the Inequality (39) follows by using similar argument as in the case of Lemma 5. This completes the proof of Theorem 1.

**Proof of Theorem 2.** By hypothesis \( P \in P_n \) and \( P(z) \) does not vanish in \( |z| < 1 \), therefore, by Lemma 2 for every real or complex number \( \beta \) with \( |\beta| \leq 1 \), \( 0 \leq \theta < 2\pi \) and \( R > r \geq 1 \),
\[
\left| P(Re^{i\theta}) - \beta P(re^{i\theta}) \right|^p \\
\leq \left| R^n P(e^{i\theta} / R) - \beta r^n P(e^{i\theta} / r) \right|^p
\]

Also, by Lemma 5,
\[
\int_0^{2\pi} \left| F(\theta) + e^{i\alpha} G(\theta) \right|^p d\theta \\
\leq \left| R^n - \beta r^n \right| + e^{i\alpha} \left( 1 - \beta \right) \left[ \int_0^{2\pi} \left| P(e^{i\theta}) \right|^p d\theta \right]
\]

where
\[
F(\theta) = P(Re^{i\theta}) - \beta P(re^{i\theta}) \quad \text{and} \quad G(\theta) = R^n P(e^{i\theta} / R) - \beta r^n P(e^{i\theta} / r).
\]

Integrating both sides of (41) with respect to \( \alpha \) from 0 to \( 2\pi \), we get for each \( p > 0 \), \( R > r \geq 1 \) and \( \alpha \) real,
\[
\int_0^{2\pi} \int_0^{2\pi} \left| F(\theta) + e^{i\alpha} G(\theta) \right|^p d\alpha d\theta \\
\leq \left[ \int_0^{2\pi} \left| R^n - \beta r^n \right| + e^{i\alpha} \left( 1 - \beta \right) \right] \left[ \int_0^{2\pi} \left| P(e^{i\theta}) \right|^p d\theta \right]
\]

Now for every real \( \alpha \), \( t \geq 1 \) and \( p > 0 \), we have
\[
\int_0^{2\pi} \left| t + e^{i\alpha} \right|^p d\alpha \geq \int_0^{2\pi} \left| t + e^{i\alpha} \right|^p d\alpha.
\]

If \( F(\theta) \neq 0 \), we take \( t = \left| G(\theta) / F(\theta) \right| \), then by (40) \( t \geq 1 \) and we get
\[
\int_0^{2\pi} \left| F(\theta) + e^{i\alpha} G(\theta) \right|^p d\alpha \\
= \left| F(\theta) \right|^p \int_0^{2\pi} \left| 1 + e^{i\alpha} G(\theta) \right|^p d\alpha \\
= \left| F(\theta) \right|^p \int_0^{2\pi} \left| G(\theta) / F(\theta) \right| + e^{i\alpha} \right|^p d\alpha \\
= \left| F(\theta) \right|^p \int_0^{2\pi} \left| G(\theta) / F(\theta) \right|^p + e^{i\alpha} \right|^p d\alpha.
\]

For \( F(\theta) = 0 \), this inequality is trivially true. Using this in (42), we conclude that for every real or complex number \( \beta \) with \( |\beta| \leq 1 \), \( R > r \geq 1 \) and \( \alpha \) real,
\[
\int_0^{2\pi} \left| R^n - \beta r^n \right| + e^{i\alpha} \left( 1 - \beta \right) \left[ \int_0^{2\pi} \left| P(e^{i\theta}) \right|^p d\theta \right]
\]

Since
\[
\left[ \int_0^{2\pi} \left| R^n - \beta r^n \right| + e^{i\alpha} \left( 1 - \beta \right) \right] \left[ \int_0^{2\pi} \left| P(e^{i\theta}) \right|^p d\theta \right]
\]

the desired result follows immediately by combining (43) and (44). This completes the proof of Theorem 2.

**Proof of Theorem 3.** Since \( P(z) \) is a self-inversive polynomial, we have \( P(z) = u Q(z) \) for all \( z \in C \) where \( u = 1 \) and \( Q(z) = z^n T(1/z) \). Therefore, for every real or complex number \( \beta \) and \( R > r \geq 1 \),
\[
\left| P(Rz) - \beta P(rz) \right| = \left| Q(Rz) - \beta Q(rz) \right|
\]
so that
\[
\left| G(\theta) / F(\theta) \right| = \frac{P(Re^{i\theta}) - \beta P(re^{i\theta})}{R^n P(e^{i\theta} / R) - \beta r^n P(e^{i\theta} / r)} = 1.
\]
Using this in (41) with $|\beta| \leq 1$ and proceeding similarly as in the proof of Theorem 2, we get the desired result. This completes the proof of Theorem 3.

4. References


