The (2,1)-Total Labeling of $S_{n+1} \lor P_m$ and $S_{n+1} \times P_m$

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Abstract

The (2,1)-total labeling number $\lambda^T_2(G)$ of a graph $G$ is the width of the smallest range of integers that suffices to label the vertices and the edges of $G$ such that no two adjacent vertices have the same label, no two adjacent edges have the same label and the difference between the labels of a vertex and its incident edges is at least 2. In this paper, we studied the upper bound of $\lambda^T_2(G)$ of $S_{n+1} \lor P_m$ and $S_{n+1} \times P_m$.

Keywords: Total Labeling, Join of Graph, Path Graph

1. Introduction

Our terminology and notation will be standard. The reader is referred to [1] for the undefined terms. For a graph $G$, let $V(G)$, $E(G)$, $\Delta(G)$ and $\delta(G)$ denote, respectively, its vertex set, edge set, maximum degree and minimum degree. We use $N(v)$ to denote the neighborhood of $v$ and let $d(v) = |N(v)|$ be the degree of $v$ in $G$. Let $d(x,y)$ denotes the distance of vertices $x,y$ of $G$, $\lceil x \rceil$ is the smallest integer greater than $x$.

Motivated by the Frequency Channel assignment problem, Griggs and Yeh [2] introduced the $L(2,1)$-labeling of graphs. This notion was subsequently extended to a general form, named as $L(p,q)$-labeling of graphs. Let $p$ and $q$ be two nonnegative integers. An $L(p,q)$-labeling of graph $G$ is a function $f$ from its vertex set $V(G)$ to the set $\{0,1,2,\cdots,k\}$ for some positive integer $k$ such that $|f(x) - f(y)| \geq p$ if $x$ and $y$ are adjacent, and $|f(x) - f(y)| \geq q$ if $x$ and $y$ are at distance 2. The $L(p,q)$-labeling number $\lambda_{p,q}(G)$ of $G$ is the smallest $k$ such that $G$ has an $L(p,q)$-labeling $f$ with max $\{|f(v)| \mid v \in V(G)\} = k$.

Whittlesey et al. [3] investigated the $L(2,1)$-labeling of incidence graphs. The incidence graph of a graph $G$ is the graph obtained from $G$ by replacing each edge by a path of length 2. The $L(2,1)$-labeling of the incidence graph of $G$ is equivalent to an assignment of integers to each element of $V(G) \cup E(G)$ such that adjacent vertices have different labels, adjacent edges have different labels, and incident vertex and edge have the difference of labels by at least 2. This labeling is called $(2,1)$-total labeling of graphs, which was introduced by Havet and Yu [4], and generalized to $(d,1)$-total labeling form. Let $d \geq 1$ be an integer. A $k-(d,1)$-total labeling of graph $G$ is an integer-valued function $f$ defined on the set $V(G) \cup E(G)$ such that

$$|f(x) - f(y)| \geq \begin{cases} 1, & \text{if vertices } x \text{ and } y \text{ are adjacent;} \\ 1, & \text{if edges } x \text{ and } y \text{ are adjacent;} \\ d, & \text{if vertex } x \text{ incident to edge } y. \end{cases}$$

The $(d,1)$-total labeling number, denoted $\lambda^T_d(G)$, is the least integer $k$ such that $G$ has a $k-(d,1)$-total labeling.

When $d = 1$, the $(1,1)$-total labeling is the well-known total coloring of graphs, which has been intensively studied [5-7].

It was conjectured in [4] that $\lambda^T_1(G) \leq \Delta + 2d - 1$ for each graph $G$, which extends the well-known Total Coloring Conjecture in which $d = 1$. It was also shown in [4] $\lambda^T_d(G) \leq 2\Delta + d - 1$ for any graph $G$. The $(d,1)$-total labeling for some kinds of special graphs have been studied, e.g., complete graphs [4], outerplanar graphs for $d = 2$ [8], graphs with a given maximum average degree [9], etc.

In this paper, we studied the $(2,1)$-total labeling of joining graph with star and path $S_{n+1} \lor P_m$, and the cartesian product of star and path $S_{n+1} \times P_m$. The following two lemmas appeared in [4], which are very useful.

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Lemma 1.1. Let $G$ be a graph with maximum degree $\Delta$, then $\lambda_2^G (G) \geq \Delta + d - 1$.

Lemma 1.2. If $\lambda_2^G (G) = \Delta + d - 1$, then the vertices with maximum degree of $G$ must be labeled 0 or $\Delta + d - 1$.

2. The (2, 1)-Total Labeling of $S_{n+1} \cup P_m$

Let $G_1$ and $G_2$ be two graphs, by starting with a disjoint union of $G_1$ and $G_2$, adding edges by joining each vertex of $G_1$ to each vertex of $G_2$, we can obtain the join of graph $G_1$ and $G_2$, denoted $G_1 \cup G_2$.

Let $S_{n+1}$ be a star with $n+1$ vertices $v_0, v_1, \cdots, v_n$, in which $d(v_0) = n$, we call $v_0$ the center of $S_{n+1}$. Let $P_m$ be a path with $m$ vertices $u_1, u_2, \cdots, u_m$. Then $G = S_{n+1} \cup P_m$ has the following propositions:

1) $\Delta(G) = d(v_0) = m + n$;
2) $d(u_i) = d(u_0) = \cdots = d(u_{m-1}) = n + 3$;
3) $d(v_j) = d(v_1) = \cdots = d(v_m) = m + 1$.

For $n \geq 1$, $S_{n+1}$ is a path, S. M. Zhang [10] had studied the $(2, 1)$-total labeling of $P_m \cup P_n$. So in the sequel, we only consider the case $n \geq 3$.

Theorem 2.1. Let $G = S_{n+1} \cup P_m$, if $m \geq 2 + n$, then $\lambda_2^G (G) = \Delta + 1$.

Proof. By lemma 1.1, it’s need to prove $\lambda_2^G (G) \leq \Delta + 1 = m + n + 1$.

Now, we give a $(m + n + 1)$ - $(2, 1)$-total labeling of $G$ as follows:

For $i = 1, 2, \cdots, n$ and $j = 1, 2, \cdots, m$, let

$$f(v_i) = (i + j - 1) \mod (m + 1),$$

$$f(v_i) = m + n + 1;$$

$$f(v_i) = n + j (j = 1, 2, \cdots, m),$$

$$f(v_i) = n + 1;$$

$$f(u_{i,j}) = \begin{cases} m + n, & 1 \leq j \leq m - 2 \text{ and } j \text{ is odd} \ 1 \leq j \leq m - 2 \text{ and } j \text{ is even} \\
1 + m, & \end{cases}$$

It’s easy to see that $f$ is a $(m + n + 1)$ - $(2, 1)$-total labeling of $S_{n+1} \cup P_m$, so we have $\lambda_2^G (G) \leq \Delta + 1 = m + n + 1$.

Theorem 2.2. Let $G = S_{n+1} \cup P_m$, if $m = n + 1 \geq 7$ then $\lambda_2^G (G) = \Delta + 1$

Proof. By lemma 1.1, it’s need to prove $\lambda_2^G (G) \leq \Delta + 1 = 2n + 2$.

Now, we give a $(2n + 2)$ - $(2, 1)$-total labeling of $G$ as follows:

For $i = 1, 2, \cdots, n$ and $j = 1, 2, \cdots, m + 2$, let

$$f(v_i) = (i + j - 1) \mod (n + 2)$$

$$f(v_i) = m + n + 1;$$

$$f(v_i) = n + j (j = 1, 2, \cdots, n);$$

$$f(u_{i,j}) = \begin{cases} m + n, & 1 \leq j \leq n - 2 \text{ and } j \text{ is odd} \\
1 \leq j \leq n - 2 \text{ and } j \text{ is even} \\
1 + m, & \end{cases}$$

For $n \geq 6$, it’s easy to see that $f$ is a $(m + n + 1)$ - $(2, 1)$-total labeling of $G$, so we have $\lambda_2^G (G) \leq \Delta + 1 = n + 4$.

Then $f$ is a $(m + n + 1)$ - $(2, 1)$-total labeling of $G$, so we have $\lambda_2^G (G) \leq \Delta + 1 = n + 4$.

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The cartesian product of graph $G$ and $H$, denoted $G \times H$, which vertex set and edge set are the follows: $V(G \times H) = \{(u,v) | u \in G, v \in H\}$, $E(G \times H) = \{(u,v)(u',v') | v = v', uu' \in E(G); or u = u', vv' \in E(H)\}$.

Let $w_{ij}(j = 0, 1, \ldots, n; j = 1, 2, \ldots, m)$ denote the vertex $(v_{ij}, u_{ij})$ of the graph $S_{n+1} \times P_m$.

Obviously, for $m = 2$, $\Delta(G \times H) = n + 1 = d(w_{0ij})(j = 1, 2)$, the degree of other vertexes is 2. For $m \geq 3$,

$$\Delta(G \times H) = n + 2 = d(w_{0ij})(j = 1, 2, \ldots, m),$$

where $d(w_{0ij}) = d(w_{0ij})(j = 1, 2, \ldots, n)$, the degree of other vertexes is 3.

For $n = 1, 2$, $S_{n+1}$ is a path, S. M. Zhang [11] had studied the $(2, 1)$-total labeling of $P_m \times P_n$. So in the sequel, we only consider the case $n \geq 3$.

**Theorem 3.1.** Let $G = S_{n+1} \times P_m$ then $\lambda^T (G) = \Delta + 1$.

**Proof.** By lemma 1.1, it’s need to prove $\lambda^T (G) \leq \Delta + 1$.

**Case 1.** if $m = 2, n \geq 4$, then $\Delta = n + 1$.

We give a $(n + 2) - (2, 1)$-total labeling of $G$ as follows:

By lemma 1.2, we let $f(w_{0ij}) = 0$, $f(w_{0ij}) = n + 2$,

$$f(w_{0ij}) = \begin{cases} i + 1, & i = 1, 2, \ldots, \frac{n}{2} \text{ and } j = 1, 2 \\ i + 2, & i = \frac{n}{2} + 1, \frac{n}{2} + 2, \ldots, n - 2; j = 1, 2 \\ i, & i = n, j = 2. \end{cases}$$

$$f(w_{0ij}) = \begin{cases} i + 1, & i = 1, 2, \ldots, \frac{n}{2} \text{ and } j = 1, 2 \\ i + 3, & i = \frac{n}{2} + 1, \frac{n}{2} + 2, \ldots, n - 2; j = 1, 2, \ldots, m. \end{cases}$$

$$f(w_{ij}) = \begin{cases} 4, & j \text{ is odd, } \end{cases}$$

$$f(w_{ij}) = \begin{cases} 5, & j \text{ is odd, } \end{cases}$$

$$f(w_{ij}) = \begin{cases} 6, & j \text{ is even}, \end{cases}$$

For $1 \leq j \leq m - 1$,

let $f(w_{ij}, w_{ij + 1}) = \begin{cases} 0, & i = 1, 2 \text{ and } j \text{ is odd, } \\
1, & i = 1, 2 \text{ and } j \text{ is even}. \end{cases}$

For $1 \leq j \leq m$,

let $f(w_{ij}) = \begin{cases} 1, & i = 3, 4, \ldots, n - 1 \text{ and } j \text{ is odd, } \\
2, & i = 3, 4, \ldots, n - 1 \text{ and } j \text{ is even}. \end{cases}$

$$f(w_{ij}) = \begin{cases} 3, & j \text{ is odd, } \end{cases}$$

For $1 \leq j \leq m - 1$,
$f(w_j, w_{i+j}) = \begin{cases} 
\left\lfloor \frac{n}{2} \right\rfloor + 2, & i = 0, 3, 4 \cdots n; \text{and } j \text{ is odd,} \\
\left\lfloor \frac{n}{2} \right\rfloor + 3, & i = 0, 3, 4 \cdots n; \text{and } j \text{ is even.} 
\end{cases}$

If $n \geq 4, 1 \leq j \leq m$ and $j$ is even, we can see that

$$f(w_{0,j}) - f(w_{0,j}w_{i+j}) = n + 3 - \left\lfloor \frac{n}{2} \right\rfloor + 3 \geq 2,$$

$$f(w_{0,j}) - f(w_{2,j}) = n + 3 - 6 \geq 1,$$

then $f$ is a $(n + 3) - (2, 1)$-total labeling of $S_{n+1} \times P_m$.

**Case 3.** If $m \geq n = 3$, then $\Delta = 5$.

We give a $6 - (2, 1)$-total labeling of $S_2 \times P_m$ as follows:

$$f(w_{0,j}) = 4(j = 1, 2 \cdots m),$$

For $1 \leq j \leq m$, let

$$f(w_{0,j}) = \begin{cases} 
0, & \text{j is odd,} \\
6, & \text{j is even.} 
\end{cases}$$

$$f(w_{0,j}w_{j}) = \begin{cases} 
5, & \text{j is odd,} \\
0, & \text{j is even.} 
\end{cases}$$

$$f(w_{0,j}w_{3,j}) = \begin{cases} 
6, & \text{j is odd,} \\
1, & \text{j is even.} 
\end{cases}$$

$$f(w_{i,j}) = \begin{cases} 
1, & i = 1, 2; \text{and j is odd,} \\
2, & i = 1, 2; \text{and j is even.} 
\end{cases}$$

$$f(w_{i,j}) = \begin{cases} 
2, & j \text{ is odd,} \\
3, & j \text{ is even.} 
\end{cases}$$

For $1 \leq j \leq m - 1$,

let $f(w_{0,j}w_{i+j}) = \begin{cases} 
2, & j \text{ is odd,} \\
3, & j \text{ is even.} 
\end{cases}$

$$f(w_{1,j}w_{i+j}) = \begin{cases} 
5, & j \text{ is odd,} \\
6, & j \text{ is even.} 
\end{cases}$$

$$f(w_{2,j}w_{i+j}) = \begin{cases} 
4, & j \text{ is odd,} \\
6, & j \text{ is even.} 
\end{cases}$$

$$f(w_{3,j}w_{i+j}) = \begin{cases} 
0, & j \text{ is odd,} \\
5, & j \text{ is even.} 
\end{cases}$$

It is easy to see that $\lambda_2^T(S_{n+1} \times P_m) = \Delta + 1$. This proves the theorem.

### 4. References


