On Complete Bicubic Fractal Splines

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Abstract

Fractal geometry provides a new insight to the approximation and modelling of experimental data. We give the construction of complete cubic fractal splines from a suitable basis and their error bounds with the original function. These univariate properties are then used to investigate complete bicubic fractal splines over a rectangle $\Omega$. Bicubic fractal splines are invariant in all scales and they generalize classical bicusplines. Finally, for an original function $f \in C^4(\Omega)$, upper bounds of the error for the complete bicubic fractal splines and derivatives are deduced. The effect of equal and non-equal scaling vectors on complete bicubic fractal splines were illustrated with suitably chosen examples.

Keywords: Fractals, Iterated Function Systems, Fractal Interpolation Functions, Fractal Splines, Surface Approximation.

1. Introduction

Schoenberg [1] introduced “spline functions” to the mathematical literature. In the last 60 years, splines have proved to be enormously important in different branches of mathematics such as numerical analysis, numerical treatment of differential, integral and partial differential equations, approximation theory and statistics. Also, splines play major roles in field of applications, such as CAGD, tomography, surgery, animation and manufacturing. In this paper, we discuss on complete fractal splines that generalize the classical complete splines.

Fractal interpolation functions (FIFs) were introduced by Barnsley [2,3] based on the theory of iterated function system (IFS). The attractor of the IFS is the graph of FIF that interpolates a given set of data points. Fractal interpolation constitutes an advance in the sense that the functions used are not necessarily differentiable and show the rough aspect of real-world signals [3,4]. A specific feature is the fact that the graph of these interpolants possesses a fractal dimension and this parameter provides a geometric characterization of the measured variable which may be used as an index of the complexity of a phenomenon. Barnsley and Harrington [5] first constructed a differentiable FIF or $C^r$-FIF $f$ that interpolates the prescribed data if values of $f^{(k)}$, $k = 1, 2, \ldots, r$, at the initial end point of the interval are given. In this construction, specifying boundary conditions similar to those of classical splines was found to be quite difficult to handle. The fractal splines with general boundary conditions are studied recently [6,7]. The power of fractal methodology allows us to generalize almost any other interpolation techniques, see for instance [8-10].

Fractal surfaces have proved to be useful functions in scientific applications such as metallurgy, physics, geology, image processing and computer graphics. Massopust [11] was first to put forward the construction of fractal interpolation surfaces (FISs) on triangular domains, where the interpolation points on the boundary of the domain are coplanar. Geronimo and Hardin [12], and Zhao [13] generalized this construction in different ways. The general bivariate FIS on rectangular grids are treated for instance in references [14,15]. Recently, Bouboulis and Dalla constructed fractal interpolation surfaces from FIFs through recurrent iterated function systems [16].

In this paper we approach the problem of complete cubic spline surface from a fractal perspective. In Section 2, we construct cardinal cubic fractal splines through moments and estimate the error bound of the complete cubic spline with the original function. The construction of bicubic fractal splines is carried out in Section 3 through tensor products. Finally, for an original function $f \in C^4(\Omega)$, upper bounds of the error for the complete
bicubic fractal splines and derivatives are deduced. The effect of scaling factors on bicubic fractal splines are demonstrated in the last section through various examples.

2. Complete Cubic Fractal Splines

We discuss on fractal interpolation based on IFS theory in Subsection 2.1 and construct cardinal cubic fractal spline through moments in Subsection 2.2. Upper bounds of \(L_n\)-norm of the error of a complete cubic spline FIF with respect to the original function are deduced in Subsection 2.3.

2.1. Fractal Interpolation Functions

Let \(\Delta_i: t_0 < t_1 < \ldots < t_N\) be a partition of the real compact interval \(I = [t_0, t_N]\). Let a set of data points \(D = \{(t_n, x_n) \in I \times \mathbb{R} : n = 0, 1, 2, \ldots, N\}\) be given. Set \(I_n = [t_n, t_{n+1}]\) and let \(L_n: I \rightarrow I_n, n = 1, 2, \ldots, N\) be contractive homeomorphisms such that

\[
L_n(t_0) = t_{n-1}, \quad L_n(t_N) = t_n,
\]

\[
\|L_n(c_1) - L_n(c_2)\| \leq \alpha \|c_1 - c_2\| \quad \forall c_1, c_2 \in I,
\]

for some \(0 \leq \alpha < 1\). Let \(C = I \times \mathbb{R}\) and \(N\) continuous mappings, \(F_n: C \rightarrow \mathbb{R}\), satisfying

\[
F_n(t_0, x_0) = x_{n-1}, \quad F_n(t_N, x_N) = x_n,
\]

\[
\|F_n(t, x) - F_n(t', y)\| \leq \|\alpha\| \|x - y\|, \quad t, t' \in I, \quad x, y \in \mathbb{R}, \quad -1 < \alpha < 1.
\]

Now, define functions

\[
\forall n = 1, 2, \ldots, N, \quad w_n(t, x) = \left(L_n(t), F_n(t, x), w_n: C \rightarrow I_n \times \mathbb{R} \right)
\]

**Proposition 2.1** (Barnsley [2]) The Iterated Function System (IFS) \(\{C; w_n : n = 1, 2, \ldots, N\}\) defined above admits a unique attractor \(G\). \(G\) is the graph of a continuous function \(f: I \rightarrow \mathbb{R}\) which obeys \(f(t_0) = x_0\) for \(n = 0, 1, 2, \ldots, N\).

The above function \(f\) is called a Fractal Interpolation Function (FIF) corresponding to the IFS \(\{L_n(t), F_n(t, x)\}_n\). Let \(G = \{g: I \rightarrow \mathbb{R} \mid g\text{ is continuous}, g(t_0) = x_0 \text{ and } g(t_N) = x_N\}\) be a complete metric space with respect to the uniform norm. Define, the Read-Bajraktarević \(^c\) operator \(T\) on \((G, \|\cdot\|)\) by

\[
Tg(t) = F_n \left(L_n^{-1}(t), g\left(L_n^{-1}(t)\right)\right), \quad t \in I_n, \quad n = 1, 2, \ldots, N.
\]

According to (1)-(2), \(Tg\) is continuous on the interval \(I_n : n = 1, 2, \ldots, N\) and at each of the points \(t_1, t_2, \ldots, t_{n-1}\). \(T\) is a contraction mapping on the metric space \((G, \|\cdot\|)\) i.e.

\[
\|Tg - Tg\| \leq \|\alpha\| \|g - g\|,
\]

where \(\|\alpha\| = \max\{\|\alpha_n\| : n = 1, 2, \ldots, N\}\). Since \(\|\alpha\| < 1\), \(T\) possesses a unique fixed point \(f\) (say) on \(G\), that is to say, there is \(f \in G\) such that \((ff)(t) = f(t) \forall t \in I\).

This function is the FIF corresponding to \(w_n\) and according to (3), the FIF satisfies the functional equation:

\[
f(t) = F_n \left(L_n^{-1}(t), f \circ L_n^{-1}(t)\right), \quad t \in I_n, \quad n = 1, 2, \ldots, N.
\]

The most widely studied fractal interpolation functions so far are defined by the IFS

\[
\left\{L_n(t) = a_n t + b_n, \quad F_n(t, x) = \alpha_n x + q_n(t)\right\}
\]

where \(-1 < \alpha_n < 1\) and \(q_n : I \rightarrow \mathbb{R}\) are suitable continuous functions such that (2) are satisfied. \(\alpha_n\) is called a vertical scaling factor of the transformation \(w_n\) and \(\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)\) is the scale vector of IFS. The scale factors give a degree of freedom to the FIF and allow us to modify its properties. If \(q_n\) are affine in (6) for \(t \in I\), then the FIF is called affine [3]. Based on the principle of construction [6] of a \(C^1\)-FIF, \(r \in \mathbb{N}\), complete cardinal cubic fractal splines are constructed through their moments in the following.

2.2. Complete Cardinal Cubic Fractal Splines

A cubic spline is called complete if the values of its first derivative are prescribed at the end points. A function \(h(t)\) defined on the grid \(\Delta_i: t_0 < t_1 < \ldots < t_N\) is called an interpolating cubic spline function if the function. (1) is a cubic polynomial on each partial segment \([t_{n-1}, t_n]\), \(n = 1, 2, \ldots, N\). The function is of class \(C^0[t_0, t_N]\). 3) satisfies the conditions \(h(t) = x_n, n = 0, 1, 2, \ldots, N\). Two conditions are given in the form of restriction on the spline values and/or the values of its derivatives at the end points of the segment \([t_0, t_N]\).

**Definition 2.1** A function \(f_{m,n}(t)\) is called a cardinal cubic fractal spline if 1) \(f_{m,n}\) is a FIF associated with the set of data points \((t_m, \delta_{m,n})\) with mesh \(\Delta_j\), that is to say

\[
f_{m,n}(t) = \delta_{m,n} = 1, \quad m = n,
\]

\[
= 0, \quad m \neq n,
\]

\[
\forall m, n = 0, 1, 2, \ldots, N.
\]

Besides, 2) \(f_{m,n} \in C^1[t_0, t_N]\), 3) the corresponding IFS \(\omega_{m,n}(t, x) = \left(L_n(t), F_{m,n}(t, x)\right)\) is such that \(L_n(t)\) is defined by (6) and \(F_{m,n}(t, x) = \alpha_n x + q_{m,n}(t)\), \(\|\alpha_{m,n}\| < 1\), where \(q_{m,n}(t)\) is a suitable cubic polynomial so that the polynomial associated with the fractal function \(f_{m,n}\) on the mesh \(\Delta_j\) is affine.

In the construction of cardinal cubic fractal splines, we have taken \(\alpha_{m,n} = \alpha_n, \quad n = 1, 2, \ldots, N; \quad m = 0, 1, 2, \ldots, N.\)
A derivation of the defining equations for a cubic fractal spline through moments $M_{m,n} = f''(t_n)$ $n=0,1,\ldots,N$ can be found in [6], but for completeness and to set the terminology, it is outlined in the appendix.

For a basis of complete cubic fractal spline space on $I$, we need $f''(t_n) = f''(t_{n+1}) = 0$ for $m=0,1,\ldots,N$ in the construction of cubic spline $FIF$, and two more complete cubic fractal splines $f_{i-1}$ and $f_{i+1}$ such that $f_i(t_0) = 0, n = 0, 1, \ldots, N; f_i(t_N) = 1, f_i'(t_N) = 0, f_{N+i}(t_0) = 0, f'_{N+i}(t_N) = 1.$

Denote, $x_n = f(t_n), x_{N+1} = f(t_{N+1}) = f'(t_N).$ Let $f$ be the original function providing the data $\{f(t_n), x_n\}_{n=0}^{N+1}$ and $f_c$ be the complete cubic fractal spline corresponding to this data. Let $U(I, \Delta_T) = \{h \mid h$ is a complete cubic fractal spline on $\Delta_T\}$ . If $h \in U(I, \Delta_T)$ interpolating the same data $\{f(t_n), x_n\}_{n=0}^{N+1}$, then due to the uniqueness of fixed point of Read-Bajraktarević operator, $h(t) = \sum_{n=1}^{N+1} x_n f_n(t).$ Also, none of the $f_n$ is a linear combination of other cardinal splines and hence $\{f_n\}_{n=1}^{N+1}$ is a basis for $U(I, \Delta_T)$. Define a complete cubic fractal spline operator $F : C^2(I) \to U(I, \Delta_T)$ as $F(f) = f_c$ such that $f_c(t_n) = f_c(L_n(t)) = \sum_{n=1}^{N+1} x_n f_n(L_n(t))$ $= \sum_{n=1}^{N+1} x_n f_n(t_n), t \in I, n = 1, 2, \ldots, N.$

It is easy to check that $F$ is linear and bounded operator on $C^2(I)$. According to (7) and (8), we have $f_c(t_0) = f_c(L_0(t_0)) = \sum_{n=1}^{N+1} x_n f_n(L_0(t_0)) = \sum_{n=1}^{N+1} x_n \delta_{n,0} = x_0$

and for $i = 1, 2, \ldots, N$,

$f_c(t_i) = f_c(L_i(t_i)) = \sum_{n=1}^{N+1} x_n f_n(L_i(t_i))$ $= \sum_{n=1}^{N+1} x_n f_n(t_0) = \sum_{n=1}^{N+1} x_n \delta_{n,i} = x_n.$

Also, $f_c'(t_0) = \sum_{n=1}^{N+1} x_n f_n'(t_0) = x_{-1} = f'(t_0)$ and $f_c'(t_N) = \sum_{n=1}^{N+1} x_n f_n'(t_N) = x_{N+1} = f'(t_N).$ If we choose $\alpha_n = 0; n = 1, 2, \ldots, N$, then from (26), it is clear that right side of cardinal spline $f_m$ reduces to a cubic polynomial in $t$ and hence, in this case $f_m$ reduces to a classical complete cardinal spline $S_m$ such that $S_m(t_n) = \delta_{m,n}$. The classical complete cubic spline $S(t)$ for the data $\{f(t_n), x_n\}_{n=0}^{N}$ is given by $S(t_n) = \sum_{n=1}^{N+1} x_n S_n(L_n(t)), t \in I, n = 1, 2, \ldots, N.$

### 2.3. Error Estimation with Univariate Fractal Splines

To estimate error bounds for the complete bicubic fractal spline, we need error bounds between a cubic fractal spline and the original function $f \in C^2(I), p = 2, 3, 4$. For given moments $\{M_{m,n}\}_{m=0}^{N}$, we can observe that $q_{m,n}$ is a function of the scaling factors $\alpha_n, n = 1, 2, \ldots, N$ for the cubic fractal spline equation (cf. (26)). We need the following proposition with the assumption $|\alpha_n| \leq \kappa < 1$, for fixed $\kappa$. 

**Proposition 2.2** Let $f_m$ and $S_m (m = -1, 0, \ldots, N+1)$ be the cardinal cubic fractal spline and the classical cardinal cubic spline respectively to the same set of data $\{f(t_n), \delta_{m,n}\}_{n=0}^{N}$. Let $h = \max\{t_n - t_{n-1}; n = 1, 2, \ldots, N\}$, $|\alpha|^2 = \max\{|\alpha_n|; n = 1, 2, \ldots, N\}$, and $|I|$ is the length of the interval $I$. Suppose the cubic polynomial $q_m(\alpha_n, t)$ associated with the IFS corresponding to the cardinal fractal spline $f_m$ satisfies

$$\left|\frac{\partial^{2} q_m(\alpha_n, t)}{\partial \alpha_n \partial \alpha_n^*}\right| \leq \Theta_{u,m}$$

for $|\alpha_n| \in (0, \kappa a_n^*), t \in I, u = 0, 1, 2$ and $n = 1, 2, \ldots, N$.

Then,

$$\left\|f_m^\alpha(u) - S_m^\alpha(u)\right\| \leq \frac{h^{2-\alpha}}{\left|\Gamma^{2-\alpha} - h^{2-\alpha}\right|} \left\|\left\|\left\|\left[\| S_m^\alpha\| \right\| + \Theta_{\alpha,m}\right\| \right\|\right\|$$

$$\leq 0, 1, 2,$$

The proof of the above proposition can be seen in [6]. Now, we derive an upper bound for the error between the classical complete cubic spline and a complete cubic fractal spline for the same set of interpolation data. According to (9) and (10), we get the bottom equation

$$\left|f_c^\alpha(L_n(t)) - S_c^\alpha(L_n(t))\right| = \sum_{n=1}^{N+1} f(t_n)(f_m^\alpha - S_m^\alpha)(L_n(t)) \leq \sum_{n=1}^{N+1} \left\|f_c^\alpha - S_m^\alpha\right\|$$

$$\leq \sum_{n=1}^{N+1} \left\|f_c^\alpha - S_m^\alpha\right\| \leq \sum_{n=1}^{N+1} \left\|f_m^\alpha - S_m^\alpha\right\|$$

$$\leq \sum_{n=1}^{N+1} \left\|f_m^\alpha - S_m^\alpha\right\|$$

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where 1-norm of \( f \) is \( \| f \|_1 = \max \{ \| f \|_{\L_1}, \| f \|_{\L_\infty} \} \). Let \( \Delta_u = \max \| \Theta_u \|_{m,m} \) and \( \Theta_u = \max \| \Theta_u \|_{m,m} \). Then, \( \Gamma_{u,a,n} = (N+3)(\Delta_u + \Theta_u) \). Since the above inequality is true for \( n = 1, 2, \ldots, N \), we have the following estimate.

\[
\| f_c^{(u)} - S^{(u)} \|_1 \leq \| f \|_{\L_\infty} \Gamma_{u,a,n} h_t^{k-u}, \quad u = 0, 1, 2. \tag{12}
\]

We need the error bound between the complete cubic fractal spline \( f_c \) and the original function \( f \in C^p(I) \), \( p = 2, 3, 4 \).

**Proposition 2.3** Let \( S \) be the complete cubic spline interpolant of \( f \in C^p[t_a,t_N] \) for \( p = 2, 3, 4 \) with the assumption \( h_1 \leq 1 \). Then

\[
\| (S - f)^{(u)} \|_1 \leq \varepsilon_{p,a} \| f \|_{p}\Gamma_{u,a,n} h_i^{k-u}, \quad 0 \leq u \leq \min \{ p, 3 \},
\]

where \( \varepsilon_{p,a} \) are given in Table 1 with \( \varepsilon_{1} = h_1 / \min (t_a - t_{a-1}) \).

If the original function \( f \) is such that \( f \in C^p[t_a,t_N] \) with \( \| f \|_{\L_\infty} = \max \{ \| f \|_{\L_1}, \| f \|_{\L_\infty}, \ldots, \| f \|_{p}\| \} \). According to (12) and (13), we have the following upper bound estimation for the error.

\[
\| f_c^{(u)} - f^{(u)} \|_1 \leq \| f \|_{\L_\infty} \Gamma_{u,a,n} h_i^{k-u} + \varepsilon_{p,a} h_i^{k-u}, \quad 0 \leq u \leq 2,
\]

and

\[
\| f_c^{(u)} - f^{(u)} \|_1 \leq \| f \|_{\L_\infty} \Gamma_{u,a,n} h_i^{k-u} + \varepsilon_{p,a} h_i^{k-u}, \quad 0 \leq u \leq 2. \tag{14}
\]

### 3. Fractal Splines in Two Variables

Using univariate complete cubic fractal spline results, we construct complete bicubic fractal splines in Subsection 3.1 through tensor product and the upper bounds of the \( L_u \)-norm of its error with the original functions in Subsection 3.2.

#### 3.1. Construction of Complete Bicubic Fractal Splines

Suppose that \( \Delta_t : a = t_0 < t_1 < \ldots < t_k = b \) and \( \Delta_s : c = s_0 < s_1 < \ldots < s_j = d \) form a rectangular mesh \( \pi : \Delta_t \times \Delta_s \) for a rectangular region \( \Omega = [a,b] \times [c,d] \). Let \( f(t,s) \) be a sufficiently smooth function in the domain \( \Omega \). Let \( \rho_j(t,s) \) be the complete bicubic spline fractal interpolation surface associated with the function \( f(t,s) \) and the mesh \( \pi \). Then, \( \rho_j(t,s) \) is a tensor product of univariate cubic fractal splines such that

\[
\rho_j(t_n,s_j) = f(t_n,s_j); \quad n = 0, 1, \ldots, N; \quad j = 0, 1, \ldots, J,
\]

where \( \rho_j^{(p)}(t_n,s_j) = f^{(p)}(t_n,s_j); \quad n = 0, 1, \ldots, N; \quad j = 0, 1, \ldots, J \),

\[
\rho_j^{(p)}(t_n,s_j) = f^{(p)}(t_n,s_j); \quad n = 0, 1, \ldots, N; \quad j = 0, J,
\]

\[
\rho_j^{(1)}(t_n,s_j) = f^{(1)}(t_n,s_j); \quad n = 0, 1, \ldots, N; \quad j = 0, J.
\]

where \( \rho_j^{(p)} = \partial^{p} / \partial t^p \partial s^p \). This definition is analogous to that of the classical complete bicubic spline in [19].

### Table 1. Coefficients associated with the error of classical complete cubic spline.

<table>
<thead>
<tr>
<th>( \varepsilon_{p,a} )</th>
<th>( u = 0 )</th>
<th>( u = 1 )</th>
<th>( u = 2 )</th>
<th>( u = 3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p = 2 )</td>
<td>9/8</td>
<td>4</td>
<td>10</td>
<td>-</td>
</tr>
<tr>
<td>( p = 3 )</td>
<td>71/216</td>
<td>31/27</td>
<td>5</td>
<td>((63 + 80\eta')/9 )</td>
</tr>
<tr>
<td>( p = 4 )</td>
<td>5/384</td>
<td>1/24*</td>
<td>3/8*</td>
<td>((9\eta + 10\eta')/2* )</td>
</tr>
</tbody>
</table>

*See [18]
\[ \rho_f \left( (t_n,s_j), (\bar{s}_j) \right) = \sum_{m=1}^{N+1} \sum_{i=1}^{J+1} f(t_m,s_j) \]

\[ f_m \left( L_n(t), \bar{L}_j(s) \right), (t,s) \in \Omega, \]

where \( f(t_i,s_j) = \frac{\partial^2 f(t_i,s_j)}{\partial t \partial s} = \frac{\partial f(t_i,s_j)}{\partial t} \) with the analogues for \( f(t_i,s_j), \) \( f(t_i,s_j), \) \( f(t_i,s_j) \) etc.

We now show that the function \( \rho_f \) satisfies the interpolation conditions. According to (18), for \( n \in \{1,2,\ldots,N\} \) and \( j \in \{1,2,\ldots,J\}, \)

\[ \rho_f \left( (t_n,s_j), (\bar{s}_j) \right) = \left( (P \circ Q)f \right) \left( L_n(t), \bar{L}_j(s) \right) \]

\[ = \sum_{m=1}^{N+1} \sum_{i=1}^{J+1} f(t_m,s_j) f_m(t_n) \bar{f}_i(s_j) \]

\[ = \sum_{m=1}^{N+1} \sum_{i=1}^{J+1} f(t_m,s_j) \delta_{m,n} \delta_{i,j} = f(t_n,s_j) \]

Similarly, \( \rho_f(t_n,s_j) = f(t_n,s_j), \) \( j = 1,2,\ldots,J; \)

\[ \rho_f(t_n,s_j) = f(t_n,s_j), \quad n = 1,2,\ldots,N \quad \text{and} \quad \rho_f(t_n,s_j) = f(t_n,s_j). \]

Since \( \bar{f}_j(t_n) = f_m(t_n) = 0 \) for \( m = 0,1,\ldots,N, \)

(18) can be used. For \( n = 0,N;j = 1,\ldots,J. \) Analogously, the rest of conditions of definition (15) are true. Since, \( f_m = S_n \) if \( \alpha_{m,j} = 0 \) \( \forall \) \( n = 1,2,\ldots,N \) and \( \bar{S}_i = \bar{S} \) if \( \beta_i = 0 \) \( \forall \) \( j = 1,2,\ldots,J, \)

we can retrieve classical complete bicubic spline \( S_f \) to the original function \( f \) from (18).

### 3.2. Upper Bounds of \( L_{\infty} \)-Norm of Error

We will prove the \( L_{\infty} \)-norm error of complete bicubic splines with the original function by using the following notations analogous to those of Proposition 2.2 for the \( t \)-variable.

Let \( \hat{\alpha}_{i,j} ; \quad j = 1,2,\ldots,J \) be cubic polynomials associated with the IFSs for \( \bar{f}_i \) such that

\[ \frac{\partial^{1+1} \hat{\alpha}_{i,j}}{\partial \beta \partial \gamma} \left( \bar{f}_i(s) \right) \]

\[ \leq \hat{\Theta}_{i,j} \quad \text{for} \quad i = -1,0,1,\ldots,J + 1, \]

\[ |\bar{f}_i| \leq (0, \kappa^c) \] with \( |\beta| \leq \kappa < 1 \) for fixed real \( \kappa^c \).

Let, \( \bar{L}_j = \max \left\{ \hat{\Theta}_{i,j} : i = -1,0,1,\ldots,J + 1 \right\}, \)

\[ \hat{\Theta}_v = \max \left\{ \hat{\Theta}_{i,j} : i = -1,0,1,\ldots,J + 1 \right\} \]

\[ \hat{\Gamma}_{v\beta\gamma} = \frac{1}{\left| J \right|^{1/2} - h_{1/2} - h} \left| \beta \right|_{\infty}. \]

Suppose \( C^1[\Omega] = \{ g : \Omega \rightarrow \mathbb{R} : g^{(u,v)} \in C[\Omega], 0 \leq u + v \leq 4 \} \) and the norm corresponding to this space is

\[ \| g \|_{C^1[\Omega]} = \max \| g^{(u,v)} \|_{\infty} : 0 \leq u + v \leq 4 \].

**Theorem 3.1** Let \( \rho_f \) be the complete bicubic fractal spline to the original function \( f \in C^1[\Omega] \). Then for an arbitrary sequence of partitions,

\[ \left\| f - \rho_f \right\|^{(u,v)} \leq \left\| f \right\| \left\{ \Gamma_{u,v,\alpha,\beta} h^{1/2} + h \right\} \]

\[ + \hat{\Gamma}_{v\beta\gamma} h^{1/2} + h \] (20)

\[ + \left( \Gamma_{u,v,\alpha,\beta} h^{1/2} + h \right) \left( \frac{\Gamma_{u,v,\alpha,\beta} h^{1/2} + h}{\left| \beta \right|_{\infty}} \right) \]

\[ 0 \leq u,v \leq 2. \]

**Proof.** In order to calculate the error, we will use the generic transformations \( P \) and \( Q \). Suppose that

\[ f - \rho_f = \left( f - Q(f) \right) - \left( f - Q(f) - (P(f) - \rho_f) \right) \]

\[ + (f - P(f)) \]

(19)

Consider

\[ \left( f - P(f) \right) \left( L_n(t), \bar{L}_j(s) \right) = f \left( L_n(t), \bar{L}_j(s) \right) \]

\[ - \sum_{m=1}^{N+1} f(t_m, \bar{L}_j(s)) f_m(L_n(t)) \]

For \( s = s^* \) fixed, \( \left( f - P(f) \right) \left( L_n(t), \bar{L}_j(s^*) \right) \) is the spline of \( f^{(u,v)} \left( L_n(t), \bar{L}_j(s^*) \right) \) (with respect to the first variable) and we can apply (14) for \( p = 4 - u \) since \( f^{(u,v)} \left( L_n(t), \bar{L}_j(s^*) \right) \in C^{4-u} [a,b] \). For \( 0 \leq u \leq 2, \)

\[ \left\| f^{(u,v)} \left( \bar{L}_j(s^*) \right) \right\|^{(u,v)} \leq \left\| f^{(0,v)} \left( \bar{L}_j(s^*) \right) \right\|^{(u,v)} \]

\[ \leq \left\| f \right\| \left\{ \Gamma_{u,v,\alpha,\beta} h^{1/2} + h \right\} \]

Since the last term of (20) does not depend on \( s^* \),

\[ \left\| f^{(u,v)} - P(f) \right\|^{(u,v)} \leq \left\| f \right\| \left\{ \Gamma_{u,v,\alpha,\beta} h^{1/2} + h \right\} \]

(21)

In the same way,

\[ \left\| f^{(u,v)} - Q(f) \right\|^{(u,v)} \leq \left\| f \right\| \left\{ \Gamma_{u,v,\alpha,\beta} h^{1/2} + h \right\} \]

(22)

Consider \( P(f) - \rho_f \) and using their definitions,

\[ \left( P(f) - \rho_f \right) \left( L_n(t), \bar{L}_j(s) \right) \]

\[ = \sum_{m=1}^{N+1} \left( f(t_m, \bar{L}_j(s)) - \sum_{j=1}^{J+1} f(t_m, s_j) \bar{f}_i(L_j(s)) \right) f_m(L_n(t)) \]

\[ = \sum_{m=1}^{N+1} R_t(t_m, \bar{L}_j(s)) f_m(L_n(t)) = \left( P(R_t) \right) \left( L_n(t), \bar{L}_j(s) \right) \]

where \( R_t = f - Q(f) \). Hence, we have

\[ f - Q(f) - (P(f) - \rho_f) = R_t - P(R_t). \]

For \( s = s^* \) fixed, \( P(R_t) \) is the cubic spline FIF of \( R_t^{(u,v)} \) with respect to the first variable and we can apply (14) taking \( p = 4 - u \).
\[
\left\| R_{t}^{(n)} \left( \cdot, \tilde{L}_{i} \left( s \right) \right) - P \left( R_{t}^{(n)} \right) \left( \cdot, \tilde{L}_{i} \left( s \right) \right) \right\|_{C}\leq \left\| R_{t}^{(n)} \left( \cdot, \tilde{L}_{i} \left( s \right) \right) \right\|_{C}\left( \Gamma_{u,a,N} h_{i}^{2-u} + e_{k,v} h_{i}^{4-v-u}\right)
\]

For \(0 \leq v \leq 2\),

\[
\left\| R_{t}^{(n)} \left( \cdot, \tilde{L}_{i} \left( s \right) \right) \right\|_{C}\leq \left\| R_{t}^{(n)} \left( \cdot, \tilde{L}_{i} \left( s \right) \right) \right\|_{C}\left( \Gamma_{u,a,N} h_{i}^{2-u} + e_{k,v} h_{i}^{4-v-u}\right)
\]

and similarly to (22) for \(0 \leq u \leq 2\),

\[
\left\| R_{t}^{(n)} \left( \cdot, \tilde{L}_{i} \left( s \right) \right) \right\|_{C}\leq \left\| R_{t}^{(n)} \left( \cdot, \tilde{L}_{i} \left( s \right) \right) \right\|_{C}\left( \Gamma_{u,a,N} h_{i}^{2-u} + e_{k,v} h_{i}^{4-v-u}\right)
\]

and by (23), we get the first equation of the bottom ones.

The inequality of Theorem 3.1 follows from (21)-(24).

**Remark.** From Theorem 3.1, it can be observed that the convergence of the bicubic fractal spline \(\rho_{f}\) is slower than that of the case of the classical bicubic spline \(S_{f}\) (see [20]). Since the classical bicubic spline is a particular case of bicubic fractal spline, Theorem 3.1 generalizes the classical result. If there exist positive reals \(k\) and \(l\) such that \(\Gamma_{u,a,N} < \frac{1}{(N+3)^{k}}\) and

\[
\Gamma_{v,b,j} < \frac{1}{(J+3)^{l}}
\]

then the complete bicubic fractal spline \(\rho_{f}\) converges to the original function \(f\) in \(C^{2}\)-norm for uniform partitions.

### 4. Examples

First, we construct three different bases for complete cubic fractal spline space with \(I = [0,3]\), \(N = 3\) and three different sets of scaling vectors. These scaling vectors play important role over classical splines in overall shape of fractal approximants. The scaling vectors are chosen for a basis of complete fractal splines as 1) Set I: \(\alpha_{n} = 0.9, i = 1,2,3\); 2) Set II: \(\alpha_{n} = -0.9, i = 1,2,3\); 3) Set III: \(\alpha_{n} = -0.5, \alpha_{2} = 0.9, \alpha_{3} = 0.7\). In our examples,

\[
L_{1}(t) = \frac{1}{3} t, L_{2}(t) = \frac{1}{3} t + \frac{1}{3}, \text{ and } L_{3}(t) = \frac{1}{3} t + \frac{2}{3}.
\]

We compute the moments \(M_{n,a,n}\), \(n = 1,2,3\); \(m = -1,0,...,4\) from Equations (26)-(28). These values of moments for three sets of bases are given in Table 2. These moments are then used in IFS

\[
\{ \mathbb{R}^{2}; w_{n}(t,x) = (L_{n}(t), F_{n}(x,t)), n = 1,2,3\}
\]

to compute \(F_{n}(t,x)\). From (5) and (25), we have the bottom equation. where \(x_{n}\) depends on the cardinal condition of the basis elements \(f_{n}\), i.e., \(x_{n} = a_{n,m,n}, n = 0,1,2,3; m = -1,0,...,4\).

Using the above IFS, we compute basis elements for complete cubic fractal spline with non-zero and zero scaling vectors. When scaling factors are same in Set I and Set II, the values of moments of \(f_{1}\) and \(f_{4}\); \(f_{0}\) and \(f_{1}\); \(f_{2}\) and \(f_{3}\) follow a particular pattern (see Table 2). This pattern is very close to the moments pattern of classical complete cardinal splines (with zero scale vector). That’s why pair-wise similarity between complete cardinal fractal splines \(f_{1}\) and \(f_{4}\); \(f_{0}\) and \(f_{1}\); \(f_{2}\) and \(f_{3}\) in such cases (see Figure 1(a) and Figure 1(b)) observed as in classical complete cardinal splines (see Figure 1(d)). But for unequal scaling factors in Set III, there is no pattern between moments of complete cardinal fractal splines and hence, their shapes are completely different (see Figure 1(c)). The unequal scaling factors provides an additional advantage of complete cardinal fractal splines over their classical counterparts in smooth object modelling in engineering applications like computer graphics, CAD/CAM.

The non-zero scale vectors gives irregular shape to fractal splines because \(f_{n}\) are typical fractal functions, i.e., fractal dimension of graph of \(f_{n}\) is non-integer. These cardinal fractal splines differ from their classical interpolants in the sense that they obey a functional relation related to self-similarity on smaller scales. Hence, cardinal fractal splines are defined globally on the entire domain. Moreover, classical complete splines are defined piecewisely between consecutive nodes and hence, their shapes can be defined locally. Importantly, if \(\alpha_{n} = 0, n = 1,2,3\), then we can retrieve the basis elements (see Figure 1(d)) for the classical complete cubic spline space. Using these three sets of vertical scaling vectors, we have constructed complete bicubic splines in the following.

Some complete bicubic fractal splines are constructed using the tensor product of univariate fractal splines for the interpolation data given in Table 3. In all our examples, we assume the same boundary conditions for complete

\[
\left\| R_{t}^{(n)} \right\|_{C}\leq \left\| R_{t}^{(n)} \right\|_{C}\left( \Gamma_{u,a,N} h_{i}^{2-u} + e_{k,v} h_{i}^{4-v-u}\right)
\]

\[
F_{n}(t,x) = \frac{1}{9} \left\{ \alpha_{n} x + \frac{(M_{n,m,n} - \alpha_{n} M_{n,m,n})}{18} + \frac{(M_{n,m,n} - \alpha_{n} M_{n,m,n})}{18} (3-t)^{3} - \frac{(M_{n,m,n} - \alpha_{n} M_{n,m,n})}{2} \right\}
\]

\[
\left\{ \alpha_{n} x + \frac{(M_{n,m,n} - \alpha_{n} M_{n,m,n})}{18} + \frac{(M_{n,m,n} - \alpha_{n} M_{n,m,n})}{18} (3-t)^{3} - \frac{(M_{n,m,n} - \alpha_{n} M_{n,m,n})}{2} \right\}
\]

\[
\left\{ \alpha_{n} x + \frac{(M_{n,m,n} - \alpha_{n} M_{n,m,n})}{18} + \frac{(M_{n,m,n} - \alpha_{n} M_{n,m,n})}{18} (3-t)^{3} - \frac{(M_{n,m,n} - \alpha_{n} M_{n,m,n})}{2} \right\}
\]
### Table 2. Moments for cardinal complete cubic fractal splines.

<table>
<thead>
<tr>
<th>Basis Elements</th>
<th>Moments</th>
<th>Set I</th>
<th>Set II</th>
<th>Set III</th>
<th>Classical Case</th>
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<tbody>
<tr>
<td>( f_i )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( M_{1,0} )</td>
<td></td>
<td>-16.4058</td>
<td>-2.5546</td>
<td>-2.4586</td>
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<td>1.4714</td>
<td>0.6318</td>
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<tr>
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<td>0.4584</td>
<td>-1.4137</td>
<td>-0.2667</td>
</tr>
<tr>
<td>( M_{1,3} )</td>
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<td>-10.9275</td>
<td>0.4844</td>
<td>-0.5122</td>
<td>0.1333</td>
</tr>
<tr>
<td>( M_{1,4} )</td>
<td></td>
<td>-24.4348</td>
<td>-2.8448</td>
<td>-2.7653</td>
<td>-4.4000</td>
</tr>
<tr>
<td>( f_0 )</td>
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<td></td>
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<td></td>
</tr>
<tr>
<td>( M_{0,0} )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>-24.4348</td>
</tr>
<tr>
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<td>-14.5217</td>
<td>3.5448</td>
<td>2.0938</td>
<td>2.8000</td>
</tr>
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<td>( f_\lambda )</td>
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<td></td>
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</tr>
<tr>
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<td>22.6087</td>
<td>1.8319</td>
<td>3.6812</td>
<td>3.2000</td>
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<td>( f_\sigma )</td>
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</tr>
<tr>
<td>( M_{1,2} )</td>
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<td></td>
<td></td>
<td>-15.5652</td>
</tr>
<tr>
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<td>( f_\theta )</td>
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<td></td>
<td>-24.4348</td>
</tr>
<tr>
<td>( f_\chi )</td>
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<td>-15.5652</td>
</tr>
<tr>
<td>( f_\xi )</td>
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<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( M_{1,6} )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>12.5797</td>
</tr>
</tbody>
</table>

### Table 3. Interpolation data for complete bicubic splines.

<table>
<thead>
<tr>
<th>( f(t_i,s_j) )</th>
<th>( s_j = 1 )</th>
<th>( s_j = 2 )</th>
<th>( s_j = 3 )</th>
<th>( s_j = 3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( t_i = 1 )</td>
<td>-1</td>
<td>11</td>
<td>-5</td>
<td>10</td>
</tr>
<tr>
<td>( t_i = 2 )</td>
<td>2</td>
<td>14</td>
<td>3</td>
<td>15</td>
</tr>
<tr>
<td>( t_i = 3 )</td>
<td>0</td>
<td>9</td>
<td>1</td>
<td>17</td>
</tr>
<tr>
<td>( t_i = 4 )</td>
<td>1</td>
<td>10</td>
<td>3</td>
<td>13</td>
</tr>
</tbody>
</table>
bicubic splines: \( f^{(1,0)}(t_n, s_j) = 5; n = 0, 3, j = 0, 1, 2, 3 \), \( f^{(0,3)}(t_n, s_j) = 3; n = 0, 1, 2, 3, j = 0, 3 \), and \( f^{(1,3)}(t_n, s_j) = 2; j = 0, 3 \). The scaling vectors are same in both directions in our first three examples, i.e., \( \alpha_n = \beta_j \) for \( n = j \) and we take these scaling vectors as Set I, Set II and Set III defined in above for univariate case. The cardinal splines \( f_m = f_i \) if \( i = m \) in these cases for \( i, m = -1, 0, \ldots, 4 \). Based on (18), the points of complete bicubic fractal splines are generated and plotted in Figures 2-4. The effect of change in scaling factors from 0.9 to −0.9 on complete bicubic spline can be seen from Figures 2-3. The difference in the shape of complete bicubic spline for an unequal scaling factors can be observed by comparing Figure 4 with Figures 2-3.

Next, we take scaling vectors in \( t \)-direction as Set I and in \( s \)-direction as Set III and the corresponding complete bicubic spline generated in Figure 5. It has similarity with both Figure 2 and Figure 4 due to self-similarity relation in \( t \) and \( s \) directions respectively. For Figure 6, we chose scaling vectors in \( t \)-direction as Set III and in \( s \)-direction as Set II. The distinct deviation in \( s \)-direction of complete bicubic spline is present in this case as in Figure 3. Finally, we chose \( \alpha_n = \beta_j = 0 \) \( \forall n, j \). Since \( f_m = S_m \) and \( \tilde{f}_i = \tilde{S}_i \) in this case, we retrieve the classical complete bicubic spline \( S_f \) in Figure 7. An infinite number of complete bicubic splines can be constructed interpolation the same data by choosing different sets of scaling vectors. Hence, the presence of scaling vectors in bicubic fractal splines provides an additional advantage over classical bicubic splines in smooth surface modeling. Since bicubic fractal splines are invariant in all scales, it can also be applied to bivariate image compression and zooming problems in image processing.

5. Conclusions

We introduced bases for complete cubic fractal splines through cardinal fractal splines in the present work. These cardinal fractal splines constructed through moments
Figure 2. Complete bicubic fractal spline with scale vectors Set I in both directions.

Figure 3. Complete bicubic fractal spline with scale vectors Set II in both directions.

Figure 4. Complete bicubic fractal spline with scale vectors Set III in both directions.

Figure 5. Complete bicubic fractal spline with scale vectors Set I, Set III.

Figure 6. Complete bicubic fractal spline with scale vectors Set III, Set II.

Figure 7. Classical complete bicubic spline with zero scale vector.
as in the classical case. Using tensor product of cardinal cubic fractal splines, bicubic fractal splines introduced over rectangular domains with rectangular partition. These bicubic fractal splines are invariant in all scales due to underlying fixed point equation. $L_\infty$-norm of the error of complete cubic fractal spline with respect to the original function $f \in C^p(\Omega)$, $p = 2, 3$ or $4$ has been deduced. The presence of scaling factors can be exploited in bivariate optimization problems with prescribed interpolation conditions. The effect of equal and non-equal scaling factors in complete bicubic splines is explained. The present work may play important role in smooth surface modelling in computer graphics and image processing applications.

6. Acknowledgements

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7. References

Appendix

In this section, the defining equations for the construction of a cubic spline \( F_{IF} \) in Subsection 2.2 are given. Using Property 3) of Definition 2.1 and (5), the polynomials associated with \( f_m' \) is affine. So,

\[
f_m'\left(L_n(t)\right) = \alpha_n f_m'(t) + \frac{c_n(t-t_0)}{t_N-t_0} + d_n^{n},
\]

\( n = 1, 2, \ldots, N \).

By (1) and (25), \( c_n = M_n - M_{n+1} - \alpha_n (M_N - M_n) \) and \( d_n = M_{n+1} - \alpha_n M_n \). Substituting \( c_n \) and \( d_n \) in (25) and integrating it twice, we will have two constants of integration. Solving these constants by (1), the cubic fractal spline in terms of moments can be written as

\[
f_m(L_n(t)) = a_n^2 \left\{ \alpha_n f_m(t) + \frac{(M_m - \alpha_n M_{m,n})(t-t_0)}{6(t_N-t_0)} + \frac{(M_{m,n-1} - \alpha_n M_{m,0})(t_N-t)}{6(t_N-t_0)} \right. \\
- \left. \frac{(M_{m,n-1} - \alpha_n M_{m,0})(t_N-t_0)(t_N-t)}{6} \right\},
\]

\( n = 1, 2, \ldots, N \).

(26)

Set, \( h_n = t_n - t_{n-1}; n = 1, 2, \ldots, N \). Now, use the condition that \( f_m''(t) \) is continuous at the knots \( t_1, t_2, \ldots, t_{N-1} \) to give the following result.

\[
-a_n \alpha_n h_n f_m'(t_0) - \frac{2a_n \alpha_n h_n}{6} M_m,0 \\
+ \frac{h_n + h_{n+1}}{6} M_m,0 + \frac{h_n}{6} M_{m,n} + \frac{h_{n+1}}{6} M_{m,n+1} \\
- \frac{2a_n \alpha_n h_n + \alpha_n \alpha_n h_n}{6} M_{m,N} + a_n \alpha_n f_m'(t_N)
\]

(27)

At the initial point \( t_0 \) of the interval \( I \), we have the following relation for \( f_m'(t_0) \).

\[
6(1 - \alpha_n) f_m'(t_0) + 2(1 - \alpha_n) h_n M_m,0 + h_n M_{m,1} \\
- \alpha_n h_n M_{m,N} = \frac{6}{h_n} \left\{ x_t - x_0 - \alpha_n a_n \left( x_N - x_0 \right) \right\}
\]

(28)

Similarly, at the final point \( t_N \) of the interval \( I \) for \( f_m''(t_N) \), we have

\[
-\alpha_n h_N M_m,0 + h_N M_{m,N-1} + 2(1 - \alpha_n) h_N M_{m,N} \\
-6(1 - \alpha_n) f_m'(t_N) = \frac{-6}{h_N} \left\{ x_N - x_{N-1} - \alpha_n a_n \left( x_N - x_0 \right) \right\}
\]

(29)

The moments \( M_m,n; n = 0, 1, \ldots, N \), \( f_m'(t_0) \) and \( f_m''(t_N) \) are evaluated from the system of Equations (27)-(29). The existence of these parameters is guaranteed by the uniqueness of the attractor from the fixed point theorem.