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Nonzero Solutions of Generalized Variational Inequalities*

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Abstract

The existence of nonzero solutions for a class of generalized variational inequalities is studied by fixed point index approach for multivalued mappings in finite dimensional spaces and reflexive Banach spaces. Some new existence theorems of nonzero solutions for this class of generalized variational inequalities are established.

Keywords: Variational Inequality, Fixed Point Index of Multivalued Mappings, Nonzero Solution

1. Introduction

Variational inequality theory with applications are an important part of nonlinear analysis and have been applied intensively to mechanics, differential equation, cybernetics, quantitative economics, optimization theory and nonlinear programming etc. (see [1-4]).

Variational inequalities, generalized variational inequalities and generalized quasivariational inequalities were studied intensively in the last 30 years with topological method, variational method, semi-ordering method, fixed point method, minimax theorem of Ky Fan and KKM technique ([1-4]). In 1998, motivated by the paper [5], Zhu [6] studied a system of variational inequalities involving the linear operators in reflexive Banach spaces by using the coincidence degree theory due to Mawhin [7]. Some existence results of positive solutions for this system of variational inequalities in reflexive Banach spaces were proved.

Let X be a real Banach space, X^* its dual and $(\cdot,\cdot)$ the pair between X^* and X. Suppose that K is a nonempty closed convex subset of X.

Find $u\in K$, $u\neq 0$, and $w\in g(u)$ such that

$$(Au,v-u)\geq(w,v-u),\ \forall v\in K \tag{1}$$

where mapping $A:K\to X^*$ is nonlinear and $g:K\to 2^{X^*}$ is a multi-valued mapping.

The existence of nonzero solutions for variational inequalities is an important topic of variational inequality theory. [8] discussed the variational inequality (1) when A is coercive or monotone and g is set-contractive or upper semi-continuous. [9] considered the variational inequality (1) when A is single-valued continuous and g is set-contractive.

On the other hand, recently, under some different conditions, [10,11] obtained some existence theorems of nonzero solutions for a class of generalized variational inequalities by fixed point index approach for multi-valued mappings in reflexive Banach space.

Based on the importance of studying the existence of nonzero solutions for variational inequalities, and motivated and inspired by recent research works in this field, in this paper, we discuss the existence of nonzero solutions for a class of generalized variational inequalities as follows:

Find $u\in K, u\neq 0$ such that

$$\begin{aligned}&(Au,v-u)+j(v)-j(u)\\&\quad\geq(g(u),v-u)+(f,v-u),\ \forall v\in K\end{aligned} \tag{2}$$

where $A,g:K\to X^*$ are two nonlinear mapping and $f\in X^*$.

A mapping $A:X\to X^*$ is called hemicontinuous at $x_0\in X$ if for each $y\in X$, $A(x_0+t_ny)\underline{w^*}Ax_0$ when $t_n\to+0$. A multivalued mapping $T: D(T)\subset X\to 2^{X^*}$ is said to be locally bounded in v if there exists a neighbourhood V of x for each $x\in X$ such that the set $T(V\cap D(T))$ is bounded in X^*. Suppose that K is a closed convex subset of X with $0\in K$. For such K, the recession cone rcK of K is defined by $rcK=\{w\in X: v+w\in K,\forall v\in K\}$. It is easily seen that the recession cone is indeed a cone and we have that $rcK\neq\varnothing$. For a proper lower semicontinuous convex functional $j:X\to R\cup\{\infty\}$ with $j(0)=0$ and $j(K)\subset R_+=[0,+\infty)$, in the virtue of [12], the limit $\lim\limits_{t\to+\infty}\frac{1}{t}j(tw)=$

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$j_\infty(w)$ exists in $R \cup \{\infty\}$ for every $w \in X$ and j_∞ is also a lower semicontinuous convex functional with $j_\infty(0) = 0$ and with the property that $j(u+v) \le j(u) + j_\infty(v), \forall u, v \in X$.

Suppose that K is a closed convex subset of X and U is an open subset of X with $U_K = U \cap K \ne \varnothing$. The closure and boundary of U_K relative to K are denoted by $\bar{U}_K$ and $\partial(U_K)$ respectively. Assume that $T : \bar{U}_K \to 2^K$ is an upper semicontinuous mapping with nonempty compact convex values and T is also condensing, *i.e.*, $\alpha(T(S)) < \alpha(S)$ where α is the Kuratowski measure of noncompactness on X. If $x \notin T(x)$ for $x \in \partial(U_K)$, then the fixed point index, $i_K(T,U)$, is well defined(see[13]).

Proposition 1[13] *Let K be a nonempty closed convex subset of real Banach space X and U be an open subset of X. Suppose that $T : \bar{U}_K \to 2^K$ is an upper semicontinuous mapping with nonempty compact convex values and $x \notin T(x)$ for $x \in \partial(U_K)$. Then the index, $i_K(T,U)$, has the following properties:*

1) *If $i_K(T,U) \ne 0$, then T has a fixed point;*

2) *For mapping $\widehat{X_0}$ with constant value $\{x_0\}$, if $x_0 \in U_K$, then $i_K(\widehat{X_0}, U) = 1$;*

3) *Let U_1, U_2 be two open subsets of X with $U_1 \cap U_2 = \varnothing$.*

If $x \notin T(x)$ when $x \in \partial((U_1)_K) \cup \partial((U_2)_K)$, then $i_K(T, U_1 \cup U_2) = i_K(T,U_1) + i_K(T,U_2)$;

4) *Let $H : [0,1] \times \bar{U}_K \to 2^K$ be an upper semicontinuous mapping with nonempty compact convex values and $\alpha(H([0,1] \times Q)) < \alpha(Q)$ whenever $\alpha(Q) \ne 0, Q \subset \bar{U}_K$.*

If $x \notin H(t,x)$ for every $t \in [0,1], x \in \partial(U_K)$, then $i_K(H(1,\cdot),U) = i_K(H(0,\cdot),U)$.

For every $q \in X^*$, let $U(q)$ be the set of solutions in K of the following variational inequality

$$(Au, v-u) + j(v) - j(u) \ge (q, v-u) + (f, v-u), \forall v \in K \tag{3}$$

Define a mapping $K_A : X^* \to 2^K$ by

$K_A(q) := U(q), \quad q \in X^*.$

Obviously, $K_A(q) = \varnothing$ if and only if the variational inequality (3) has no solution in K.

2. Nonzero Solutions in R^n

Lemma 1 *Let X be a separable reflexive Banach space. Suppose that $A : X \to X^*$ is a bounded monotone hemicontinuous mapping (i.e., for every bounded subset D of X, $A(D)$ is bounded) and*

$j : K \to (-\infty, +\infty]$ is a proper convex lower semicontinuous functional. Assume that there exists $v_0 \in K$ satisfying

$$\liminf_{\|u\| \to +\infty, u \in K} [(Au, u - v_0) + j(u) - j(v_0)] > 0 \tag{4}$$

Then for any given $f \in X^*$ there exists $u \in X$ such that

$$(Au, v-u) + j(v) - j(u) \ge (f, v-u), \forall v \in X. \tag{5}$$

Proof. Without loss of generality, assume that $f = 0$, otherwise, set $\tilde{j}(v) = j(v) - (f,v)$. Let $K^r = \{x \in X : \|x\| \le r\}$. Because X is a separable reflexive Banach space, for given r, there exists a closed convex sets sequences $K_m, m = 1,2,\ldots,$ satisfying the following conditions:

a) $K_m \subset K_{m+1} \subset K^r, m = 1,2,\ldots;$

b) $K_m \subset X_m$, X_m is m-dimensional subspace of X;

c) $\bigcup_{m=1}^\infty K_m$ is dense in K^r.

First, we shall verify that for each m, there exists $u_m \in K_m$ such that

$$(Au_m, v-u_m) + j(v) - j(u_m) \ge 0, \forall v \in K_m. \tag{6}$$

Because X_m is a finite dimensional subspace (denoted its inner product by [.,.]), there exists a linear continuous mapping $\pi : X^* \to X_m$ such that $(g,\omega) = [\pi g, \omega]$ for all $\omega \in K_m$. Thus inequality (6) can be written

$$[(-\pi Au + u) - u, v - u)] \le j(v) - j(u), \forall v \in K_m. \tag{7}$$

Define a function $J_m(v) : X_m \to (-\infty, +\infty]$ by

$$J_m(v) = \begin{cases} j(v), & v \in K_m \\ +\infty, & v \in X_m \setminus K_m. \end{cases}$$

Then inequality (7) can be written

$$[(-\pi Au + u) - u, v - u)] \le J_m(v) - J_m(u), \forall v \in K_m \tag{8}$$

which is equivalent to the equality

$$u = P_{J_m}(-\pi Au + u) \tag{9}$$

by [2,3], where P_{J_m} is an approximate mapping of J_m.

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Obviously, $P_{J_m}(-\pi A + I): K_m \to K_m$ is continuous. According to Brouwer's fixed point theorem (see [2,3]), there exists $u_m \in K_m$ satisfying the equality (9), that is, u_m is a solution of the variational inequality (6).

Second, we shall verify that for each r, there exists $u_r \in K^r$ such that

$$(Au_r, v - u_r) + j(v) - j(u_r) \geq 0, \forall v \in K^r. \quad (10)$$

In fact, $K_m \subset K^r$ and A is a bounded mapping, which implies that there constant C such that $\| Au_m \| \leq C$ for $m = 1, 2, \ldots$. Since X is a reflexive and K^r is weakly closed, there exists a subsequence $\{u_\mu\} \subset \{u_m\}$ such that $u_\mu \xrightarrow{w} u_r$ and $u_r \in K^r$. Because $\bigcup_{m=1}^{\infty} K_m$ is dense in K^r, for any given $\varepsilon > 0$, there exists $u_0 \in \bigcup_{m=1}^{\infty} K_m$ such that $\| u_r - u_0 \| \leq \varepsilon$. It then follows from (6) that

$$(Au_\mu, u_\mu - u_0) \leq j(u_0) - j(u_\mu). \quad (11)$$

when μ is sufficiently large. Thus we have

$$\begin{aligned} &\limsup_{\mu} (Au_\mu, u_\mu - u_r) \\ &\leq \limsup_{\mu} (Au_\mu, u_\mu - u_0) + \limsup_{\mu} (Au_\mu, u_0 - u_r) \\ &\leq \limsup_{\mu} (j(u_0) - j(u_\mu)) + C \cdot \varepsilon. \end{aligned}$$

Since j is a lower semicontinuous function and ε is an arbitrary positive number, we have

$$\limsup_{\mu} (Au_\mu, u_\mu - u_r) \leq 0. \quad (12)$$

This together with A being a monotone hemicontinuous mapping implies that

$$\begin{aligned} &\liminf_{\mu} (Au_\mu, u_\mu - v) \\ &\quad \geq (Au_r, u_r - v), \forall v \in K^r. \end{aligned} \quad (13)$$

If $v \in \bigcup_{m=1}^{\infty} K_m$, it then follows from (6) that

$$(Au_\mu, u_\mu - v) \leq j(v) - j(u_\mu) \quad (14)$$

when μ is sufficiently large. It thus follows from (13) that

$$\begin{aligned} (Au_r, u_r - v) &\leq \liminf_{\mu} (Au_\mu, u_\mu - v) \\ &\leq \liminf_{\mu} (j(v) - j(u_\mu)) \\ &\leq j(v) - j(u_r), \forall v \in \bigcup_{m=1}^{\infty} K_m. \end{aligned} \quad (15)$$

Because $\bigcup_{m=1}^{\infty} K_m$ is dense in K^r, the above inequality holds for all $v \in K^r$. therefore u_r is a solution of the variational inequality (10).

New we shall verify that the variational inequality (5) has a solution. Taking $v = v_0$ in (10), we have

$$(Au_r, u_r - v_0) + j(u_r) - j(v_0) \leq 0 \quad (16)$$

and so it then follows from condition (4) that there exists constant $C > 0$ such that $\| u_r \| \leq C$. Taking $r > C$ then $\| u_r \| < r$ and so u_r is an inner point of B_r. Thus for any given $\omega \in X$, we have $(1-t)u_r + t\omega \in B_r$ by taking $t \in (0,1)$ small enough. Let $v = (1-t)u_r + t\omega$ in (10), then we obtain

$$t(Au_r, \omega - u_r) + t(j(\omega) - j(u_r)) \geq 0$$

by j being a convex lower semicontinuous function. Thus

$$(Au_r, \omega - u_r) + j(\omega) - j(u_r) \geq 0, \forall \omega \in X.$$

Therefore u_r is a solution of the variational inequality (5).

Theorem 1 *Let K be a nonempty unbounded closed convex set in $X = R^n$ with $0 \in K$. Suppose that $X \to X^*$ is a bounded monotone hemicontinuous mappingwith $(Au, u) \geq 0 (\forall u \in K)$ and $j: K \to (-\infty, +\infty]$ is a bounded proper convex lowersemicontinuous functional with $j(0) = 0$ (i.e., for every bounded subset D of K, $j(D)$ is bounded). Give a continuous mapping $g: K \to X^*$ and $f \in X^*$. Assume*

a) $\lim_{\|u\| \to 0} \frac{(Au, u) + j(u)}{\|u\|} = +\infty$;

b) there exists constant $\alpha \geq 0$ such that

$$\liminf_{\|u\| \to +\infty} \frac{(Au, u) + j(u)}{\| u \|^{\alpha+1}} > \limsup_{\|u\| \to +\infty} \frac{\| g(u) \|}{\| u \|^{\alpha}} (u \in K);$$

c) there exists a point $u_0 \in rcK \setminus \{0\}$ such that $(f, u_0) \neq 0$

Then (2) *has a nonzero solution.*

Proof. It is easy to see from condition (b) and Lemma 1 that the variational inequality (3) has a solution in K for every $q \in X^*$. Define a mapping $K_A g: K \to 2^K$ by

$$(K_A g)(u) := K_A(g(u)), u \in K$$

Then $K_A g$ is an upper semi-continuous mapping with nonempty compact convex values by [10, Lemma 1]. Let $K^R = \{x \in K : \| x \| \leq R\}$. We shall verify that $i_K(K_A g, K^R) = 1$ for large enough R and $i_K(K_A g, K^r) = 0$ for small enough r.

Firstly, define a mapping by $H:[0,1]\times\overline{K^R}\to 2^K$, $H(t,u)=tK_A(g(u))$. It is easily seen that $H(t,u)$ is an upper semicontinuous mapping with nonempty compact convex values. We claim that there exists large enough R such that $u\notin H(t,u)$ for all $t\in(0,1)$, $u\in\partial(K^R)$. Otherwise, there exist two sequences $\{t_n\},\{u_n\},t_n\in[0,1]$, $t_n\neq 0,\|u_n\|\to+\infty$ such that

$$u_n\in H(t_n,u_n)=t_nK_A(g(u_n)) \text{ or } \frac{u_n}{t_n}\in K_A(g(u_n)).$$

Thus

$$\begin{aligned}&(A(\frac{u_n}{t_n}),v-\frac{u_n}{t_n})+j(v)+j(\frac{u_n}{t_n})\\&\ge(g(u_n),v-\frac{u_n}{t_n})+(f,v-\frac{u_n}{t_n}),\ \forall u\in K\end{aligned}\tag{17}$$

Letting $v=0$ and denoting $z_n=\frac{u_n}{\|u_n\|}$ in (17), we obtain from (17) that

$$\begin{aligned}&(\frac{t_n}{\|u_n\|})^{\alpha+1}(A(\frac{u_n}{t_n}),\frac{u_n}{t_n})+(\frac{t_n}{\|u_n\|})^{\alpha+1}j(\frac{u_n}{t_n})\\&\le t_n^{\alpha}(\frac{g(u_n)}{\|u_n\|^{\alpha}},z_n)+(\frac{t_n}{\|u_n\|})^{\alpha}(f,z_n)\end{aligned}\tag{18}$$

Denote $y_n=\frac{u_n}{t_n}\in K$. Then $\|y_n\|\to+\infty$. We can obtain from (18) that

$$\begin{aligned}\frac{(Ay_n,y_n)+j(y_n)}{\|y_n\|^{\alpha+1}}&\le t_n^{\alpha}\frac{\|g(u_n)\|}{\|u_n\|^{\alpha}}+\frac{\|f\|}{\|y_n\|^{\alpha}}\\&\le\frac{\|g(u_n)\|}{\|u_n\|^{\alpha}}+\frac{\|f\|}{\|y_n\|^{\alpha}}.\end{aligned}\tag{19}$$

Hence we have

$$\liminf_{\|u\|\to+\infty}\frac{(Au,u)+j(u)}{\|u\|^{\alpha+1}}\le\limsup_{\|u\|\to+\infty}\frac{\|g(u)\|}{\|u\|^{\alpha}}$$

which contradicts to condition (b). Therefore

$$\begin{aligned}i_K(K_Ag,K^R)&=i_K(H(1,\cdot),K^R)\\&=i_K(H(0,\cdot),K^R)\\&=i_K(\hat{0},K^R)=1\end{aligned}\tag{20}$$

by **Proposition 1**(4) and (2).

Secondly, we shall verify that $i_K(K_Ag,K^r)=0$ for small enough r ($r<1$). In fact, there exist constants $C_1,C_2,M>0$ from the boundedness of j, locally boundedness of A and condition (b) such that for all $u\in K^1$, we have

$$|j(u+u_0)-j(u)|\le C_1,\|g(u)\|\le C_2,$$

$$\begin{aligned}&|(g(u),u_0)|\le C_2\|u_0\|,\ \|Au\|\le M,\\&|(Au,u_0)|\le M\|u_0\|\end{aligned}\tag{21}$$

Since $(f,u_0)\neq 0$, let $(f,u_0)<0$. Take N large enough such that

$$(1-N)(f,u_0)>C_1+(C_2+M)\|u_0\|\tag{22}$$

Define a mapping by $H[0,1]\times\overline{K^r}\to 2^K,H(t,u)=K_A(g(u)-tNf)$. Then H is an upper semi- continuous mapping with nonempty compact convex values. We claim that there exists a small enough r such that $u\notin H(t,u)$ for all $u\in\partial(K^r),t\in[0,1]$. Otherwise, there exist sequences $\{t_n\},\{u_n\},t_n\in[0,1]$,

$u_n\in\partial(K^r),\|u_n\|\to 0$ such that $u_n\in H(t_n,u_n)=K_A(g(u_n)-t_nNf)$. Thus

$$\begin{aligned}&(Au_n,v-u_n)+j(v)-j(u_n)\\&\ge(g(u_n)-Nt_nf,v-u_n)+(f,v-u_n),\forall v\in K\end{aligned}$$

Taking $v=0,z_n=\frac{u_n}{\|u_n\|}$, we have

$$\begin{aligned}&\frac{1}{\|u_n\|}(Au_n,u_n)+\frac{j(u_n)}{\|u_n\|}\\&\le(g(u_n),z_n)+(1-t_nN)(f,z_n)\end{aligned}$$

Since $\frac{(Au_n,u_n)+j(u_n)}{\|u_n\|}\to+\infty$ and

$$\begin{aligned}&(g(u_n),z_n)+(1-t_nN)(f,z_n)\\&\le\|g(u_n)\|+(1+N)\|f\|\\&\le C_2+(1+N)\|f\|,\end{aligned}$$

we obtain a contradiction. Therefore $i_K(K_Ag,K^r)=i_K(H(0,\cdot),K^r)=i_K(H(1,\cdot),K^r)$ by Proposition 1 (4). If $i_K(H(1,\cdot),K^r)\neq 0$, then the mapping $H(1,\cdot):K\to 2^K$ has a fixed point u in K^r by Proposition 1(1), *i.e.*, $u\in H(1,u)=K_A(g(u)-Nf)$.

Thus

$$\begin{aligned}&(Au,v-u)+j(v)-j(u)\\&\ge(g(u)-Nf,v-u)+(f,v-u),\forall v\in K\end{aligned}$$

Taking $v=u+u_0$, we have

$$\begin{aligned}&(Au,u_0)+j(u+u_0)-j(u)\\&\ge(g(u),u_0)+(1-N)(f,u_0)\end{aligned}\tag{23}$$

Hence

$$\begin{aligned}&(1-N)(f,u_0)\\&\le(Au,u_0)+j(u+u_0)-j(u)-(g(u),u_0)\\&\le M\|u_0\|+C_1+C_2\|u_0\|=(C_2+M)\|u_0\|+C_1\end{aligned}$$

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by (21) and (23). That contradicts to (22). Therefore, $i_K(H(1,\cdot),K^r)=0$ and then $i_K(K_A g,K^r)=0$.

It follows from **Proposition 1**(3) that $i_K(K_A g, K^R\setminus K^r)=1$. Therefore there exists a fixed point $u\in K^R\setminus\overline{K^r}$ which is a nonzero solution of (2).

3. Nonzero Solutions in Reflexive Banach Spaces

Theorem 2 *Let X be a reflexive Banach space and $K\subset X$ a nonempty unbounded closed convex set with $0\in K$. Suppose that $A:X\to X^*$ is a bounded monotone hemicontinuous mapping with $(Au,u)\ge 0$ for $u\in K$ and $j:K\to(-\infty,+\infty]$ is a bounded convex lower semicontinuous functional with $j(0)=0$. Assume that $g:K\to X^*$ is continuous from the weak topology on X to the strong topology on X^*. Give $f\in X^*$. The following conditions are assumed to be satisfied*

(a) $(f,u_0)\ne 0$ for some $u_0\in rcK\setminus\{0\}$;

(b) there constant $\alpha\ge 0$ such that $\liminf_{\|u\|\to+\infty}\frac{(Au,u)+j(u)}{\|u\|^{\alpha+1}}>\limsup_{\|u\|\to+\infty}\frac{\|g(u)\|}{\|u\|^{\alpha}}(u\in K)$;

(c) $\liminf_{u_s \xrightarrow{w} 0} j(u_s)>0$.

Then (2) has a nonzero solution.

Proof. It is easily seen that $\lim_{\|u\|\to 0}\frac{A(u,u)+j(u)}{\|u\|}=+\infty$ by the condition (c). Let $F\subset X$ be a finite dimensional subspace containing u_0. We shall show that all conditions in Theorem 1 are satisfied on space F.

Denote $K_F=K\cap F$ which is a nonempty unbounded closed convex set. Let $j_F:F\to X$ be an injective mapping and $j_F^*:X^*\to F^*$ its dual mapping. Denote $A_F=j_F^*(A|F):F\to F^*, g_F=j_F^*(g|K_F):K_F\to F^*$. We know that $A_F=j_F^*Aj_F$, $g_F=j_F^*gj_F$. Then, A_F,g_F are hemicontinuous and continuous respectively.

For $x_1,x_2\in K_F$, we have

$$\begin{aligned}&(A_F(x_1)-A_F(x_2),x_1-x_2)\\&=(j_F^*A(x_1)-j_F^*A(x_2),x_1-x_2)\\&=(Ax_1-Ax_2,j_F(x_1-x_2))\\&=(Ax_1-Ax_2,x_1-x_2)\ge 0\end{aligned}$$

by the monotony of A. This means that A_F is monotone. On the other hand, $j_F^*f\in F^*$ and $(j_F^*f,u_0)=(f,j_Fu_0)=(f,u_0)\ne 0$. Similarly, we have

$$\liminf_{\|u\|\to+\infty}\frac{(A_Fu,u)+j(u)}{\|u\|^{\alpha+1}}>\limsup_{\|u\|\to+\infty}\frac{\|g_F(u)\|}{\|u\|^{\alpha}}\quad(u\in K_F).$$

Therefore all conditions in Theorem 1 are satisfied on space F and so there exists $u_F\in K_F, u_F\ne 0$ such that

$$\begin{aligned}&(A_F(u_F),v-u_F)+j(v)-j(u_F)\\&\ge(g_F(u_F),v-u_F)+(j_F^*f,v-u_F),\forall v\in K_F\end{aligned}$$

It yields that

$$\begin{aligned}&(A(u_F),v-u_F)+j(v)-j(u_F)\\&\ge(g(u_F),v-u_F)+(f,v-u_F),\forall v\in K_F\end{aligned}$$

Taking $v=0$, we get $(Au_F,u_F)+j(u_F)\le(g(u_F),u_F)+(f,u_F)$. Hence

$$\frac{(Au_F,u_F)+j(u_F)}{\|u_F\|^{\alpha+1}}\le\frac{\|g(u_F)\|}{\|u_F\|^{\alpha}}+\frac{\|f\|}{\|u_F\|^{\alpha}}.$$

This together with condition (b) implies that there exists a constant $M>0$ such that $\|u_F\|\le M$ for all finite dimensional subspace F containing u_0. Since X is reflexive and K is weakly closed, with a similar argument to that in the proof of Theorem 2 in [10] (also see [8]), we shall show that there exists $u'\in K$ such that for every finite dimensional subspace F containing u_0, u' is in the weak closure of the set $V_F=\bigcup_{F\subset F_1}\{u_{F_1}\}$ where F_1 is a finite dimensional subspace in X.

In fact, since V_F is bounded, we know that $\overline{(V_F)}^w$ (the weak closure of the set V_F) is weakly compact. On the other hand, let $F^1,F^2,\cdots,F^m$ be finite dimensional subspaces containing u_0. Define $F^{(m)}:=span\{F^1,F^2,\cdots,F^m\}$. Then $F^{(m)}$ containing u_0 is a finite dimensional subspace. Hence, $\bigcap_{i=1}^m V_{F^i}=\bigcap_{i=1}^m(\bigcup_{F^i\subset F_1}\{u_{F_1}\})=\bigcup_{F^{(m)}\subset F_1}\{u_{F_1}\}\ne\varnothing$, then $\bigcap_F\overline{(V_F)}^w\ne\varnothing$. That is to say, there exists $u'\in K$ such that for every finite dimensional subspace F containing u_0, u' is in the weak closure of the set $V_F=\cup_{F\subset F_1}\{u_{F_1}\}$.

Now let $v\in K$ and F' a finite dimensional subspace of X which contains u_0 and v. Since u' belongs to the weak closure of the set $V_{F'} = \bigcup_{F' \subset F_1} \{u_{F_1}\}$. We may find a sequence $\{u_{F_\alpha}\}$ in $V_{F'}$ such that $u_{F_\alpha} \xrightarrow{w} u'$. However, u_{F_α} satisfies the following inequality

$$\begin{aligned}(Au_{F_\alpha}, v-u_{F_\alpha}) + j(v) - j(u_{F_\alpha}) \\ \geq (g(u_{F_\alpha}), v-u_{F_\alpha}) + (f, v-u_{F_\alpha})\end{aligned} \quad (24)$$

The monotony of A implies that

$$\begin{aligned}(Av, v-u_{F_\alpha}) + j(v) - j(u_{F_\alpha}) \\ \geq (g(u_{F_\alpha}), v-u_{F_\alpha}) + (f, v-u_{F_\alpha})\end{aligned}$$

Letting $u_{F_\alpha} \xrightarrow{w} u'$ yields that

$$\begin{aligned}(Av, v-u') + j(v) - j(u') \\ \geq (g(u'), v-u') + (f, v-u'), \forall v \in K\end{aligned}$$

Thus

$$\begin{aligned}(Au', v-u') + j(v) - j(u') \\ \geq (g(u'), v-u') + (f, v-u'), \forall v \in K\end{aligned}$$

by Minty's Theorem [2,3]. We claim that $u' \neq 0$. Otherwise, $u_{F_\alpha} \xrightarrow{w} 0$. Taking $v = 0$ in (24) yields that

$$\begin{aligned}j(u_{F_\alpha}) \leq -(Au_{F_\alpha}, u_{F_\alpha}) + (g(u_{F_\alpha}), u_{F_\alpha}) + (f, u_{F_\alpha}) \\ \leq (g(u_{F_\alpha}), u_{F_\alpha}) + (f, u_{F_\alpha})\end{aligned}$$

The right side of the above inequality tends to 0, which contradicts to the condition (c). Therefore u' is a nonzero solution of (2).

4. References

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