Generalized Irreducible $\alpha$-Matrices and Its Applications

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Abstract

The class of generalized $\alpha$-matrices is presented by Cvetković, L. (2006), and proved to be a subclass of $H$-matrices. In this paper, we present a new class of matrices—generalized irreducible $\alpha$-matrices, and prove that a generalized irreducible $\alpha$-matrix is an $H$-matrix. Furthermore, using the generalized arithmetic-geometric mean inequality, we obtain two new classes of $H$-matrices.

As applications of the obtained results, three regions including all the eigenvalues of a matrix are given.

Keywords

Generalized Irreducible $\alpha$-Matrices, $H$-Matrices, Irreducible, Nonsingular, Eigenvalues

1. Introduction

$H$-matrices play a very important role in Numerical Analysis, in Optimization theory and in other Applied Sciences [1]-[7]. Here we call a matrix $A=(a_{ij})\in C^n$ a $H$-matrix if its comparison matrix $com(A)=(m_{ij})$ defined by

$$m_{ij}=|a_{ij}|, m_{ij}=-|a_{ij}|, i,j \in N = \{1,2,\cdots,n\}, j \neq i$$

is an $M$-matrix, i.e., $(com(A))^{-1} \geq 0$ [4].

One interesting problem involving on $H$-matrices is to identify whether or not a matrix is an $H$-matrix [2] [8]. But it is not easy to do this by its definition. So researchers turned to study some subclasses of $H$-matrices, which are easy to identify [1] [2] [3] [4] [5] [8] [9] [10]. One of the classical subclasses is strictly diagonally dominant matrices (see Definition 1) which was first presented by Lévy only for real matrices [11]. And Minkowski [12] and Desplanques [13] ob-
tained the general complex result.

**Definition 1.** A matrix \( A = (a_{ij}) \in C^{\alpha_{\alpha}} \) is called a strictly diagonally dominant matrix if for any \( i \in N \),

\[
|a_{ii}| > r_i(A) = \sum_{j \neq i} |a_{ij}|
\]

As is well known, a strictly diagonally dominant matrix is nonsingular. This can lead to the following famous Geršgorin’s Theorem.

**Theorem 1.** [12] Let \( A = (a_{ij}) \in C^{\alpha_{\alpha}} \) and \( \sigma(A) \) be the spectrum of \( A \). Then

\[
\sigma(A) \subseteq \bigcup_{i \in N} \Gamma_i(A)
\]

where \( \Gamma_i(A) = \{ z \in C : |z - a_{ii}| \leq r_i(A) \} \).

By considering the irreducibility of a matrix, Taussky [14] [15] extended the notion of a strictly diagonally dominant matrix, and given the following subclass of \( H \)-matrices (see Definition 2). A matrix \( A \) is irreducible if and only if its directed graph \( G(A) \) is strongly connected (for details, see [16] [17]).

**Definition 2.** A matrix \( A = (a_{ij}) \in C^{\alpha_{\alpha}} \) is called an irreducibly diagonally dominant matrix if \( A \) is irreducible, if for any \( i \in N \),

\[
|a_{ii}| \geq r_i(A)
\]

and if strict inequality holds in (1) for at least one \( i \).

**Theorem 2.** ([17], Theorem 1.11) For an irreducibly diagonally dominant matrix \( A \), then \( A \) is nonsingular.

Another one subclass of \( H \)-matrices is provided by Ostrowski (see [14] or Theorem 1.16 of [17]).

**Theorem 3.** [18] For any \( A = (a_{ij}) \in C^{\alpha_{\alpha}} \), and any \( \alpha \in [0,1] \), assume that

\[
|a_{ii}| > (r_i(A))^\alpha (c_i(A))^{1-\alpha}, \text{ for each } i \in N
\]

where \( c_i(A) = r_i(A^T) \). Then \( A \), which is called \( \alpha \)-matrices, is nonsingular and is an \( H \)-matrix.

By the nonsingularity of \( \alpha \)-matrices, one can easily obtain the corresponding eigenvalue localization theorem as below.

**Theorem 4.** [17] For any \( A = (a_{ij}) \in C^{\alpha_{\alpha}} \), and any \( \alpha \in [0,1] \), then

\[
\sigma(A) \subseteq \left\{ z \in C : |z - a_{ii}| \leq r_i(A)^\alpha c_i(A)^{1-\alpha} \right\}
\]

For irreducible matrices, Hadjidimos in [19] gave extensions of Theorem 4 by the nonsingularity of the so-called irreducible \( \alpha \)-matrices (see Theorems 5 and 6).

**Definition 3.** A matrix \( A = (a_{ij}) \in C^{\alpha_{\alpha}} \) is called an irreducible \( \alpha \)-matrix if \( A \) is irreducible, if for any \( i \in N \),

\[
|a_{ii}| \geq r_i(A)^\alpha c_i(A)^{1-\alpha}
\]

hold for some \( \alpha \in [0,1] \), with at least one inequality being strict.

**Theorem 5.** ([19], Theorem 2.1) For an irreducible \( \alpha \)-matrix \( A \), then \( A \) is nonsingular.
Theorem 6. [19] For any $A = (a_{ij}) \in C^{\alpha_a}$, and any $\alpha \in [0,1]$, for which (3) holds, then

$$\sigma(A) \subseteq \Gamma^{\alpha_1}(A) \cup \Gamma^{\alpha_2}(A)$$

where

$$\Gamma^{\alpha_1}(A) = \bigcup_{i \in N} \left\{ z \in C : z - a_{ii} \leq r_i(A)^\alpha c_i(A)^{1-\alpha} \right\}$$

$$\Gamma^{\alpha_2}(A) = \bigcup_{i \in N, i \neq i_1} \left\{ z \in C : z - a_{ii} < r_i(A)^\alpha c_i(A)^{1-\alpha} \right\}$$

and $N_1$ is the set of indices for which strict inequality holds in (3).

We remark here that although Hadjijimos in [19] pointed out that irreducible $\alpha_2$-matrices is nonsingular, he didn’t give the relationship between $\alpha_2$-matrices and $H$-matrices. In fact, the class of $\alpha_2$-matrices is a subclass of $H$-matrices, which is showed by the following theorem.

Theorem 7. For an irreducible $\alpha_2$-matrix $A$, then $A$ is an $H$-matrix.

Proof. We let $\text{com}(A) = D - B$, where $D = \text{diag}(a_{11}, a_{22}, \cdots , a_{nn})$, and prove that the spectral radius $\rho(D^-1B)$ of $D^-1B$ is less than 1. In fact, if there exists an eigenvalue $\lambda$ of $D^-1B$ such that $|\lambda| \geq 1$, then $D(\lambda I - D^-1B) = \lambda D - B$, is an irreducible $\alpha_2$-matrix, and hence it is nonsingular.

But this contradicts the fact that $\lambda$ is an eigenvalue of the matrix $D^-1B$. Therefore, $\rho(D^-1B) < 1$.

According to $\left( \text{com}(A) \right)^{-1} = \sum_{j=1}^{n} (D^-1B)^j D^{-1} \geq 0$, the conclusion follows.

Recently, Cvetković in [4] presented a new subclass of $H$-matrices, which is called generalized $\alpha$-matrices defined as below, and given a new eigenvalue localization set by using the nonsingularity of generalized $\alpha$-matrices (see Theorem 9).

Theorem 8. ([4], Theorem 16) If for a matrix $A = (a_{ij}) \in C^{\alpha_2}$, there exists $\alpha \in [0,1]$ and $k \in N$ such that for each subset $S \subseteq N$ of cardinality $k$

$$|a_{ij}| > \left( r_s(A)^\alpha \left( c_s(A)^{1-\alpha} \right) \right) + r_s(A), S \subset N$$

holds, where $r_s(A) = \sum_{j \in S} |a_{ij}|$ and $c_s(A) = r_s(A^T)$, then the matrix $A$, which is called a generalized $\alpha$-matrices, is nonsingular, moreover it is an $H$-matrix.

Theorem 9. ([5], Theorem 17) For any $A = (a_{ij}) \in C^{\alpha_2}$, and any $\alpha \in [0,1]$, then

$$\sigma(A) \subseteq \bigcap_{k \in N / S} \bigcup_{i \in N} \Gamma^{\alpha, S}_{i,k}$$

where

$$\Gamma^{\alpha, S}_{i,k} = \left\{ z \in C : |z - a_{ij}| \leq \left( r_s(A)^\alpha \left( c_s(A)^{1-\alpha} \right) \right) + r_s(A) \right\}$$

We now present a new class of matrices—generalized irreducible $\alpha$-matrix, which is different from the class of generalized $\alpha$-matrices and will be proved to be an $H$-matrix in Section 2.
Definition 4. A matrix \( A = (a_{ij}) \in C^{\infty \alpha} \) is called a generalized irreducible \( \alpha \)-matrix if \( A \) is irreducible and if there exists \( \alpha \in [0,1] \) and \( k \in N \) such that for each subset \( S \subseteq N \) of cardinality \( k \)

\[
|a_{ii}| \geq \left( \eta_{S}^{\alpha} (A) \right)^{\alpha} \left( \lambda_{S}^{\alpha} (A) \right)^{1-\alpha} + r_{S}^{\alpha} (A)
\]

holds, with at least one inequality in (5) being strict.

The outline of this paper is given as follows. In Section 2, we prove that a generalized irreducible \( \alpha \)-matrix is nonsingular, and is an \( H \)-matrix. By using its nonsingularity, we also obtain a new eigenvalue localization set. Combining with the generalized arithmetic-geometric mean inequality, we in Section 3 obtain two other subclasses of \( H \)-matrices, consequently, two corresponding eigenvalue localization set. And then the simplifications of the obtained eigenvalue localization sets are given in Section 4.

2. Nonsingularity of Generalized Irreducible \( \alpha \)-Matrices

In this section, we prove that a generalized irreducible \( \alpha \)-matrix is nonsingular, and obtain a new eigenvalue localization set by using its nonsingularity.

Theorem 10. If a matrix \( A = (a_{ij}) \in C^{\infty \alpha} \) is a generalized irreducible \( \alpha \)-matrix, then it is nonsingular, moreover it is an \( H \)-matrix.

Proof. First, Apparent we remark that the case \( k = 1 \) represents the class of irreducibly diagonally dominant matrices, while \( k = n \) represents irreducible \( \alpha_{2} \)-matrices, so in both cases the nonsingularity has already been shown in Theorem 2 and Theorem 5, respectively. So, from now on, we suppose that \( 1 < k < n \).

Suppose on the contrary that \( A \) is singular. Then there exists a nonzero vector \( x = (x_1, x_2, \cdots, x_n)^{T} \) such that \( Ax = 0 \), that is,

\[
-a_{ii}x_i = \sum_{i \neq j, j \neq i} a_{ij}x_j, \text{ for each } i \in N
\]

Taking absolute values in the above equation and using the triangle inequality gives

\[
|a_{ii}|x_i | \leq \sum_{i \neq j, j \neq i} |a_{ij}| |x_j| = \sum_{i \neq j, j \neq i} a_{ij} |x_j| + \sum_{i \neq j, j \neq i} a_{ij} |x_j| \text{ for each } i \in N
\]

Note that for the nonzero vector \( x = (x_1, x_2, \cdots, x_n)^{T} \) there always exists a subset \( S \subset N \) of cardinality \( k \) such that \( |x_i| \geq |x_j| \) and \( |x_i| > 0 \) for each \( i \in S \) and each \( j \in S \). Hence, for each \( i \in S \),

\[
|a_{ii}|x_i | \leq \sum_{i \neq j, j \neq i} a_{ij} |x_j| \leq \sum_{i \neq j, j \neq i} a_{ij} |x_j| + \eta_{S}^{\alpha} (A)x_i
\]

equivalently,

\[
\left( |a_{ii}| + r_{S}^{\alpha} (A) \right)|x_i| \leq \sum_{i \neq j, j \neq i} a_{ij} |x_j|
\]

Furthermore, by (5) in Definition 4, we have
with at least one strict inequality holds above. Using Hölder’s inequality (see Lemma 2.1 in [19]) we get

\[
(r_s^x (A))^{\alpha} (c_s^x (A))^{1-\alpha} |x_i| \leq \left( \sum_{j \in S, j \neq i} |a_{ij}| \right)^\alpha \left( \sum_{j \in S, j \neq i} |x_j|^{\frac{1}{1-\alpha}} \right)^{1-\alpha}, \quad i \in S
\]

that is

\[
(r_s^x (A))^{\alpha} (c_s^x (A))^{1-\alpha} |x_i| \leq \left( \sum_{j \in S, j \neq i} |a_{ij}| \right)^\alpha \left( \sum_{j \in S, j \neq i} |x_j|^{\frac{1}{1-\alpha}} \right)^{1-\alpha}, \quad i \in S
\]

without loss of generality, suppose that for any \( i \in S \), \( r_s^x (A) > 0 \). In fact, if there exists \( i_0 \in S \) such that \( r_s^x (A) = 0 \), i.e., \( a_{ik} = 0 \) for each \( k \in S \), \( k \neq i_0 \), then from (7), we have

\[
\left| a_{i_0 i} - r_s^x (A) \right| |x_{i_0}| \leq 0.
\]

Note that \( |x_i| \neq 0 \) for each \( i \in S \), then

\[
|a_{i_0 j}| \leq r_s^x (A) = r_s^x (A).
\]

Since \( A \) is a generalized irreducible \( \alpha \)-matrix, we have

\[
|a_{i_0 j}| \geq (r_s^x (A))^{\alpha} (c_s^x (A))^{1-\alpha} + r_s^x (A) = r_s^x (A)
\]

hence,

\[
|a_{i_0 j}| = r_s^x (A), \quad i_0 \in S
\]

Furthermore, by (6) and (9), we get that

\[
|a_{i_0 j}| = \sum_{j \in S} |a_{i_0 j}| \geq r_s^x (A)
\]

which implies that there is \( j_0 \in S \) such that \( a_{i_0 j_0} = 0 \) and \( |x_{i_0}| = |x_{j_0}| \neq 0 \).

Because \( A \) is irreducible. Let \( S_j = (S \setminus \{ i \}) \cup \{ j \} \), for \( i \in S, i \neq i_0 \). Note that

\[
r_s^x (A) \geq |a_{i_0 i}| > 0
\]

then we only consider \( S_j \) instead of \( S \).

For every \( i \in S \), \( r_s^x (A) > 0 \). By canceling \( (r_s^x (A))^{\alpha} \) on both sides of (8) and raising both sides of (8) to the power \( \frac{1}{1-\alpha} \), we have

\[
\sum_{i \in S} (c_s^x (A))^{\frac{1}{1-\alpha}} |x_i|^{\frac{1}{1-\alpha}} \leq \left( \sum_{j \in S, j \neq i} |a_{ij}| |x_j|^{\frac{1}{1-\alpha}} \right)^{1-\alpha} \quad i \in S
\]

where strict inequality holds above for at least one \( i \in S \). Summing on all \( i \) in \( S \) in the above inequalities gives

\[
\sum_{i \in S} (c_s^x (A))^{\frac{1}{1-\alpha}} |x_i|^{\frac{1}{1-\alpha}} \leq \sum_{i \in S} \left( \sum_{j \in S, j \neq i} |a_{ij}| |x_j|^{\frac{1}{1-\alpha}} \right)
\]

equivalently
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\[ \sum (c_i^S(A))^{1/2} < \sum \left( \sum_{j \neq i} a_{ij} \right)^{1/2} = \sum (c_i^S(A))^{1/2}. \]

This is a contradiction. Therefore, A is nonsingular.

Moreover, similar to the proof of Theorem 7, we can easily prove that A is an H-matrix.

From Theorem 10, we easily get the corresponding eigenvalue localization set as below.

**Corollary 1.** For any \( A = (a_{ij}) \in C^{n \times n} \), and any \( \alpha \in [0,1] \), then

\[ \sigma(A) \subseteq \bigcap_{k = 1}^{n} \bigcup \Gamma_{i}^{\alpha,a,S,1} \cup \Gamma_{i}^{\alpha,a,S,2} \]

where

\[ \Gamma_{i}^{\alpha,a,S,1} = \left\{ z \in C : |z - a_{ii}| \leq \left( r_{i}^S (A) \right)^{\alpha} \left( c_{i}^S (A) \right)^{1-\alpha} + r_{i}^S (A) \right\}; \]

\[ \Gamma_{i}^{\alpha,a,S,2} = \left\{ z \in C : |z - a_{ii}| < \left( r_{i}^S (A) \right)^{\alpha} \left( c_{i}^S (A) \right)^{1-\alpha} + r_{i}^S (A) \right\}. \]

and \( S = S_1 \setminus S_1 \) with \( S_1 \) is the set of indices for which strict inequality holds in (5).

**3. Applications**

Combining the nonsingularity of generalized (irreducible) \( \alpha \)-matrices with the generalized arithmetic-geometric mean inequality:

\[ a \alpha + (1 - \alpha) b \geq a^\alpha b^{1-\alpha} \]

where \( a, b \geq 0 \) and \( \alpha \in [0,1] \).

We obtain two other subclasses of H-matrices, consequently, two new eigenvalue localization set.

**Theorem 11.** If for a matrix \( A = (a_{ij}) \in C^{n \times n} \), there exists \( \alpha \in [0,1] \) and \( k \in N \) such that for each subset \( S \subseteq N \) of cardinality \( k \)

\[ |a_{ij}| > \alpha r_{i}^S (A) + (1 - \alpha) c_{i}^S (A) + r_{i}^S (A) \]

holds, then A, which is called a generalized sum \( \alpha \)-matrix, is nonsingular, moreover it is an H-matrix.

**Proof.** By the generalized arithmetic-geometric mean inequality, we have

\[ |a_{ij}| > \alpha r_{i}^S (A) + (1 - \alpha) c_{i}^S (A) + r_{i}^S (A) \geq \left( r_{i}^S (A) \right)^{\alpha} \left( c_{i}^S (A) \right)^{1-\alpha} + r_{i}^S (A) \]

This implies that A is generalized \( \alpha \)-matrix. Hence A is nonsingular. Furthermore, similar to the proof of Theorem 7, we can obtain easily that A is an H-matrix.

From Theorem 11, we also get a corresponding eigenvalue localization set.

**Corollary 2.** For any \( A = (a_{ij}) \in C^{n \times n} \), and any \( \alpha \in [0,1] \), then

\[ \sigma(A) \subseteq \bigcap_{k = 1}^{n} \bigcup \bigcup_{i \in S} \Gamma_{i}^{\alpha,a,S} \]
where
\[ y_{i}^{\alpha,S} = \left\{ z \in C : |z - a_{i}| \leq \alpha r_{i}^{S}(A) + (1 - \alpha) c_{i}^{S}(A) + r_{i}^{\gamma}(A) \right\} \]

According to Theorem 10 and the generalized arithmetic-geometric mean inequality, we can obtain easily the following subclass of H-matrices and the corresponding eigenvalue localization set.

**Theorem 12.** If for an irreducible matrix \( A = (a_{ij}) \in C^{\alpha} \), there exists \( \alpha \in [0,1] \) and \( k \in N \) such that for each subset \( S \subseteq N \) of cardinality \( k \)

\[ |a_{i}| \geq \alpha r_{i}^{S}(A) + (1 - \alpha) c_{i}^{S}(A) + r_{i}^{\gamma}(A) \] (11)

holds, with at least one inequality in (11) being strict, then \( A \) is nonsingular, moreover it is an H-matrix.

**Corollary 3.** For any \( A = (a_{ij}) \in C^{\alpha} \), and any \( \alpha \in [0,1] \), then

\[ \sigma(A) \subseteq \bigcap_{k \in N/3} \bigcup_{\{1,2\}} \left( \bigcup_{S_{i} \subseteq S} y_{i}^{\alpha,S_{i}} \right) \bigcup \left( \bigcup_{S_{j} \subseteq S} y_{j}^{\alpha,S_{j}} \right) \]

where
\[ y_{i}^{\alpha,S_{i}} = \left\{ z \in C : |z - a_{i}| \leq \alpha r_{i}^{S}(A) + (1 - \alpha) c_{i}^{S}(A) + r_{i}^{\gamma}(A) \right\} \]
\[ y_{j}^{\alpha,S_{j}} = \left\{ z \in C : |z - a_{j}| < \alpha r_{j}^{S}(A) + (1 - \alpha) c_{j}^{S}(A) + r_{j}^{\gamma}(A) \right\} \]

and \( S_{2} = S \setminus S_{1} \) with \( S_{1} \) is the set of indices for which strict inequality holds in (11).

**4. Simplifications of Eigenvalue Localization Sets**

The eigenvalue localization sets in Theorem 9 and Corollary 2 are not of much practical use because of the restriction of \( \alpha \). To solve this problem, we in this section use the method provided in [5] [6], and obtain more convenient forms of the two eigenvalue localization sets. First, the sufficient and necessary conditions of generalized \( \alpha \)-matrices and generalized sum \( \alpha \)-matrices are given.

For a matrix \( A = (a_{ij}) \in C^{\alpha} \) with \( n \geq 2 \), and for \( S \subseteq N \) of cardinality \( k \in N \), we partition the set of indices \( S \) into three sets:

\[ R = \{ i \in S : r_{i}^{S}(A) > c_{i}^{S}(A) \} \]
\[ C = \{ i \in S : r_{i}^{S}(A) < c_{i}^{S}(A) \} \]
\[ L = \{ i \in S : r_{i}^{S}(A) = c_{i}^{S}(A) \} \]

where \( r_{i}^{S}(A) = c_{i}^{S}(A) = 0 \).

Consequently, \( R = C = 0 \) if \( k = 1 \). Obviously, \( S = R \cup C \cup L \).

**Lemma 13.** A matrix \( A = (a_{ij}) \in C^{\alpha} \) with \( n \geq 2 \), is a generalized \( \alpha \)-matrix if and only if there exists \( k \in N \), such that for each subset \( S \subseteq N \) of cardinality \( k \) the following two conditions hold:

1) \( |a_{i}| > \min \{ r_{i}^{S}(A), c_{i}^{S}(A) \} + r_{i}^{\gamma}(A), i \in S \);
\[2) \log_{\frac{c_i(A)}{c_j(A)}} \left| a_{ij} - r_i^S(A) \right| > \log_{\frac{c_j(A)}{c_i(A)}} \left| a_{ij} - r_i^S(A) \right|, \]

for each \( i \in R \), for which \( c_i^S(A) \neq 0 \), and for each \( j \in C \), for which \( r_i^S(A) \neq 0 \).

**Proof.** The case \( k = 1 \): The class of generalized \( \alpha \)-matrices reduces to strictly diagonally dominant matrices. And note that the condition (1) changes to

\[\left| a_{ii} \right| > r_i^3(A) = r_i(A), i \in S.\]

This also holds for each \( S \subseteq N \) of cardinality \( k = 1 \), that is, for any \( i \in N \), \( \left| a_{ii} \right| > r_i(A) \), which implies that \( A \) is strictly diagonally dominant.

The case \( k = n \): The class of generalized \( \alpha \)-matrices reduces to \( \alpha_2 \)-matrices. On the other hand, the condition (1) changes to

\[\min \left\{ r_i^3(A), c_i^S(A) \right\} = \min \left\{ r_i(A), c_i(A) \right\} \]

And the condition (2) changes to

\[\log_{\frac{c_i(A)}{c_j(A)}} \left| a_{ij} \right| > \log_{\frac{c_i(A)}{c_j(A)}} \left| a_{ij} - r_i^S(A) \right|, i \in S.\]

Hence by Theorem 5 in [5], \( A \) in this case is an \( \alpha_2 \)-matrix.

The case \( 1 < k < n \): Similar to the proof of Theorem 5 in [5], the conclusion in this case follows easily.

Similar to the proof of Lemma 13, for generalized sum \( \alpha \)-matrices we also obtain easily its sufficient and necessary condition by Theorem 4 in [5].

**Lemma 14.** A matrix \( A = (a_{ij}) \in \mathbb{C}^{\alpha \alpha} \) with \( n \geq 2 \), is a generalized sum \( \alpha \)-matrix if and only if there exists \( k \in N \) such that for each subset \( S \subseteq N \) of cardinality \( k \) the following two conditions hold:

1) \( \left| a_{ii} \right| > \min \left\{ r_i^3(A), c_i^S(A) \right\} + r_i^3(A), i \in S ; \)

2) \( \left| a_{ij} \right| - r_i^S(A) - c_i^S(A) > c_j^S(A) - \left( \left| a_{ij} \right| - r_i^3(A) \right) \)

for each \( i \in R \) and each \( j \in C \).

We now establish two eigenvalue localization sets by Lemmas 13 and 14, which are the equivalent forms of the sets in Theorem 9 and Corollary 2 respectively.

**Corollary 4.** For any \( A = (a_{ij}) \in \mathbb{C}^{\alpha \alpha} \), then

\[\sigma(A) \subseteq \Gamma^{4,S}(A) \cup \hat{\Gamma}^{4,S}(A),\]

where

\[\Gamma^{4,S}(A) = \bigcap_{k \in N/5} \bigcup_{i \in S} \left\{ z \in \mathbb{C} : \left| z - a_{ii} \right| \leq \min \left( r_i^3(A), c_i^S(A) \right) + r_i^3(A) \right\} ;\]

\[\hat{\Gamma}^{4,S}(A) = \bigcap_{k \in N/5} \bigcup_{i \in S} \bigcup_{j \in C} \hat{\Gamma}^{4,S}(A) ;\]

and
Proof. For any $\lambda \in \sigma(A)$, $\lambda I - A$ is singular. Note that the moduli of every off-diagonal entry of $\lambda I - A$ is the same as $A$. Hence, for each $S \subseteq N$, the sets $R \subseteq N$ and $C \subseteq N$ for the matrix $\lambda I - A$ remain the same. If $\lambda \not\in \Gamma_{4,S}^i(A) \cup \hat{\Gamma}_{4,S}^i(A)$, then $\lambda I - A$ satisfies the conditions (1) and (2) of Lemma 13, hence $\lambda I - A$ is a generalized $\alpha$-matrix, which implies that $\lambda I - A$ is nonsingular. This is a contradiction. Hence, $\lambda = \Gamma_{4,S}^i(A) \cup \hat{\Gamma}_{4,S}^i(A)$.

Combining with Lemma 14 and similar to the proof of Corollary 4, we have the following result.

Corollary 5. For any $A = (a_{ij}) \in C^{m \times n}$, then

$$\sigma(A) \subseteq \Gamma_{4,S}^i(A) \cup \hat{\Gamma}_{4,S}^i(A),$$

where $\Gamma_{4,S}^i(A)$ is defined as Corollary 4,

$$\hat{\Gamma}_{4,S}^i(A) = \bigcap_{k \in N \setminus S} \bigcup_{j \in C \setminus S} \bigcup_{j \in C \setminus S} \hat{\Gamma}_{4,j}^i(A).$$

and

$$\hat{\Gamma}_{4,j}^i(A) = \left\{ \begin{array}{l}
\left\{ z \in C : \left[ z - a_{ij} - r_j^i(A) \right] c^j_i(A) - r_j^i(A) \right\} \\
\left\{ z \in C : \left[ z - a_{ij} - r_j^i(A) \right] r_j^i(A) - c^j_i(A) \right\} \\
\leq c^j_i(A) r_j^i(A) - c^j_i(A) r_j^i(A) \right\}
\end{array} \right.$$

Remark 1. Obviously, the forms of the sets in Corollaries 4 and 5, which are without the restriction of $a$, are easier to be determined than those in Theorem 9 and Corollary 2. In addition, similar to the proof of Lemma 3.5 in [6], we can prove that the set in Corollary 4 is tighter than that in Corollary 5, i.e.,

$$\left( \Gamma_{4,S}^i(A) \cup \hat{\Gamma}_{4,S}^i(A) \right) \subseteq \left( \Gamma_{4,S}^i(A) \cup \hat{\Gamma}_{4,S}^i(A) \right)$$

However, $\Gamma_{4,S}^i(A) \cup \hat{\Gamma}_{4,S}^i(A)$ is determined more difficulty than $\Gamma_{4,S}^i(A) \cup \hat{\Gamma}_{4,S}^i(A)$, because it is difficult to compute exactly $\log \frac{r_j^i(A)}{c^j_i(A) c^j_i(A)}$ in some cases.

5. Conclusion

In this paper, we present a new class of matrices-generalized irreducible $\alpha$-matrices, and prove that a generalized irreducible $\alpha$-matrix is an $H$-matrix. Furthermore, using the generalized arithmetic-geometric mean inequality, we obtain two new classes of $H$-matrices. As applications of the obtained results, three regions including all the eigenvalues of a matrix are given.

Acknowledgements

This work is supported by Applied Basic Research Project of Yunnan Province
(No. 2018FB001), CAS “Light of West China” Program and Program for Excellent Young Talents, Yunnan University.

Conflicts of Interest
The authors declare no conflicts of interest regarding the publication of this paper.

References