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Schur Complement of con-s-k-EP Matrices

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ABSTRACT

Necessary and sufficient conditions for a schur complement of a con-s-k-EP matrix to be con-s-k-EP are determined. Further it is shown that in a con-s-k-EP_r matrix, every secondary sub matrix of rank "r" is con-s-k-EP_r. We have also discussed the way of expressing a matrix of rank r as a product of con-s-k-EP_r matrices. Necessary and sufficient conditions for products of con-s-k-EP_r partitioned matrices to be con-s-k-EP_r are given.

Keywords: con-s-k-EP Matrices; Partitioned Matrices; Schur Complements

1. Introduction

Let $C_{n\times n}$ be the space of n × n complex matrices of order n. Let C_n be the space of all complex n-tuples. For $A \in C_{n\times n}$, let $\overline{A}$, A^T, A^*, A^S, $\overline{A}^S$, $A^\dagger$, $R(A)$, $N(A)$ and $\rho(A)$ denote the conjugate, transpose, conjugate transpose, secondary transpose, conjugate secondary transpose, Moore-Penrose inverse, range space, null space and rank of A, respectively. A solution X of the equation $AXA = A$ is called generalized inverses of A and is denoted by A^-. If $A \in C_{n\times n}$, then the unique solution of the equations $AXA = A$, $XAX = X$, $[AX]^* = AX$, $[XA]^* = XA$[2] is called the moore penrose inverse of A and is denoted by $A^\dagger$.

A matrix A is called con-s-k-EP_r if $\rho(A) = r$ and $N(A) = N(A^TVK)$ or $R(A) = R(KVA^T)$. Throughout this paper let "k" be the fixed product of disjoint transposition in $S_n = \{1, 2, \cdots, n\}$ and K be the associated permute- tion matrix. Let us define the function $k(x) = (x_{k(1)}, x_{k(2)}, \cdots, x_{k(n)})$. A matrix $A = (a_{ij}) \in C_{nxn}$ is s-k symmetric if $a_{ij} = a_{n-k(j)+1,n-k(i)+1}$ for i, j = 1, 2, ⋯, n. A matrix $A \in C_{nxn}$ is said to be con-s-k-EP if it satisfies the condition $Ax = 0 \Leftrightarrow A^s k(x) = 0$ or equivalently $N(A) = N(A^TVK)$. In addition to that A is con-s-k-EP $\Leftrightarrow KVA$ is con-EP or AVK is con-EP and A is con-s-k-EP $\Leftrightarrow$ A^T is con-s-k-EP. Moreover A is said to be con-s-k-EP_r if A is con-s-k-EP and of rank r. For further properties of con-s-k-EP matrices one may refer [1].

In this paper we derive the necessary and sufficient conditions for a schur complement of a con-s-k-EP matrix to be con-s-k-EP. Further it is shown that in a con-s-k-EP_r matrix, every secondary submatrix of rank r is con-s-k-EP_r. We have also discussed the way of expressing a matrix of rank r as a product of con-s-k-EP_r matrices. Necessary and sufficient conditions for products of con-s-k-EP_r partitioned matrices to be con-s-k-EP_r are given. In this sequel, we need the following theorems.

Theorem 1.1 [2]

Let $A, B \in C_{nxn}$, then

1) $N(A) \subseteq N(B) \Leftrightarrow R(B^T) \subseteq R(A^T) \Leftrightarrow B = BA^-A$ for all $A^- \in A\{1\}$

2) $N(A^T) \subseteq N(B^T) \Leftrightarrow R(B) \subseteq R(A) \Leftrightarrow B = AA^-B$ for all $A^- \in A\{1\}$

Theorem 1.2 [3]

Let, $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, then

$$M^\dagger = \begin{bmatrix} A^\dagger + A^\dagger B(M/A)^\dagger CA & -A^\dagger B(M/A)^\dagger \\ -(M/A)^\dagger CA^\dagger & (M/A)^\dagger \end{bmatrix}$$

$\Leftrightarrow N(A) \subseteq N(C), N(A^T) \subseteq N(B^T)$, $N(M/A)^T \subseteq N(C^T)$ and $N(M/A) \subseteq N(B)$.

Also, $M^\dagger = \begin{bmatrix} (M/D)^\dagger & -A^\dagger B(M/A)^\dagger \\ -D^\dagger C(M/D)^\dagger & (M/A)^\dagger \end{bmatrix}$

$\Leftrightarrow N(A) \subseteq N(C)$
$N(A^T) \subseteq N(B^T), N(M/A)^T \subseteq N(C^T)$,
$N(M/A) \subseteq N(B)$ and $\Leftrightarrow N(D) \subseteq N(B)$,
$N(D^T) \subseteq N(C^T), N(M/D)^T \subseteq N(B^T)$,
$N(M/D) \subseteq N(C)$.

When $\rho(M) = \rho(A)$, then $M = \begin{pmatrix} A & B \\ C & CA^-B \end{pmatrix}$ and

$M = \begin{pmatrix} A^TPA^T & A^TPC^T \\ B^TPA^T & B^TPC^T \end{pmatrix}$, where, $P=\left(AA^T+BB^T\right)^- A\left(A^TA+C^TC\right)^-$.

Theorem 1.3 [4]

Let $A,B\in C_{n\times n}$ and $U\in C_{n\times n}$ be any nonsingular matrix, then,

1) $R(A)=R(B)\Leftrightarrow R\left(UAU\right)^T=R\left(UBU\right)^T$

2) $N(A)=N(B)\Leftrightarrow N\left(UAU\right)^T=N\left(UBU\right)^T$

2. Schur Complements of con-s-k-EP Matrices

In this section we consider a 2r × 2r matrix M Partitioned in the form,

$$M=\begin{pmatrix}A & B\\ C & D\end{pmatrix} \tag{2.1}$$

where A, B, C and D are all square matrices. If a partitioned matrix M of the form 2.1 is con-s-k-EP, then in general, the schur complement of C in M, that is (M/C) is not con-s-k-EP. Here, necessary and sufficient conditions for (M/C) to be con-s-k-EP are obtained for the class $\rho(M)=\rho(C)$ and $\rho(M)\neq\rho(C)$, analogous to that of results in [5]. Now we consider the matrix

$$S=\begin{pmatrix}(M/A) & (M/B)\\ (M/C) & (M/B)\end{pmatrix} \tag{2.2}$$

the matrix formed by the Schur complements of M over A, B, C and D respectively. This is also a partitioned matrix. If a partitioned matrix S of the form 2.2 is con-s-k-EP, then in general, Schur complement of (M/C) in S, that is [S/(M/C)] is not con-s-k-EP. Here, the necessary and sufficient conditions for [S/(M/C)] to be con-s-k-EP are obtained for the class $\rho(S)=\rho(M/C)$ and $\rho(S)\neq\rho(M/C)$, analogous to that of results in [5]

As an application, a decomposition of a partitioned matrix into a sum of con-s-k-EP$_r$ matrices is obtained. Further it is shown that in a con-s-k-EP$_r$ matrix, every secondary sub matrix of rank r, is con-s-k-EP$_r$. Throughout this section let k = k_1k_2 with.

$$K=\begin{pmatrix}K_1 & 0\\ 0 & K_2\end{pmatrix} \tag{2.3}$$

where K_1 and K_2 are the permutation matrices relative to k_1 and k_2 and let "V" be the permutation matrix with units in its secondary diagonal of order 2r × 2r partitioned in such a way that

$$V=\begin{pmatrix}0 & v\\ v & 0\end{pmatrix} \tag{2.4}$$

Theorem 2.5

Let S be a matrix of the form 2.2 with $\mathrm{N}(M/C)\subseteq\mathrm{N}(M/A)$ and $\mathrm{N}\left[\mathrm{S}/(M/C)\right]\subseteq\mathrm{N}(M/D)$, then the following are equivalent:

1) S is a con-s-k-EP$_r$ matrix with k = k_1k_2 and $V=\begin{pmatrix}0 & v\\ v & 0\end{pmatrix}$.

2) (M/C) is a con-s-k-EP, $\left[S/(M/C)\right]$ is con-s-k_2-EP.

$N(M/C)^T\subseteq N(M/D)^T$ and

$N\left[S/(M/C)\right]^T\subseteq N(M/A)^T$.

3) Both the matrices

$\begin{pmatrix}(M/C) & 0\\ (M/A) & \left[S/(M/C)\right]\end{pmatrix}$ and $\begin{pmatrix}(M/C) & (M/D)\\ 0 & \left[S/(M/C)\right]\end{pmatrix}$

are con-s-k-EP$_r$.

Proof:

Since S is con-s-k-EP$_r$ with k=k_1k_2, KVS is Con-EP and $K=\begin{pmatrix}K_1 & o\\ o & K_2\end{pmatrix}$ where K_1 and K_2 are permutation matrices associated with k_1 and k_2 and $V=\begin{pmatrix}o & v\\ v & o\end{pmatrix}$.

Consider $P=\begin{pmatrix}I & (M/A)(M/C)^-\\ O & I\end{pmatrix}$,

$Q=\begin{pmatrix}I & O\\ (M/D)\left[S/(M/C)\right]^- & I\end{pmatrix}$ and

$L=\begin{pmatrix}O & \left[S/(M/C)\right]\\ (M/C) & O\end{pmatrix}$.

Clearly P and Q are non singular.

Now,

$$KVPQL=\begin{pmatrix}K_1 & O\\ O & K_2\end{pmatrix}\begin{pmatrix}O & v\\ v & O\end{pmatrix}\begin{pmatrix}I & (M/A)(M/C)^-\\ O & I\end{pmatrix}\begin{pmatrix}I & O\\ (M/D)\left[S/(M/C)\right]^- & I\end{pmatrix}\begin{pmatrix}O & \left[S/(M/C)\right]\\ (M/C) & O\end{pmatrix}$$

$$=\begin{pmatrix}O & K_1v\\ K_2v & O\end{pmatrix}\begin{pmatrix}I+(M/A)(M/C)^-(M/D)\left[S/(M/C)\right]^- & (M/A)(M/C)^-\\ (M/D)\left[S/(M/C)\right]^- & I\end{pmatrix}\begin{pmatrix}O & \left[S/(M/C)\right]\\ (M/C) & O\end{pmatrix}$$

$$=\begin{pmatrix}K_1v(M/C) & K_1v(M/D)\left[S/(M/C)\right]^-\left[S/(M/C)\right]\\ K_2v(M/A)(M/C)^-(M/C) & K_2v\left[S/(M/C)\right]+(M/A)(M/C)^-(M/D)\left[S/(M/C)\right]^-\left[S/(M/C)\right]\end{pmatrix}$$

Since, $N(M/C) \subseteq N(M/A)$, by Theorem 1.1 we have $(M/A) = (M/A)(M/C)^{-}(M/C)$,

that is, $K_2 v(M/A) = K_2 v(M/A)(M/C)^{-}(M/C)$.

Since, $N[S/(M/C)] \subseteq N/(M/D)$,

we have by Theorem 1.1

$$(M/D) = (M/D)[S/(M/C)]^{-}[S/(M/C)].$$

That is,

$$K_1 v(M/D) = K_1 v(M/D)[S/(M/C)]^{-}[S/(M/C)].$$

Also,

$$K_2 v[S/(M/C)] + (M/A)(M/C)^{-}(M/D)[S/(M/C)]^{-}[S/(M/C)] = K_2 v(M/B).$$

Since,

$$([S/(M/C)] = (M/B) - (M/A)(M/C)^{-}(M/D)),$$

therefore,

$$KVPQL = \begin{pmatrix} K_1 v(M/C) & K_1 v(M/D) \\ K_2 v(M/A) & K_2 v(M/B) \end{pmatrix} = \begin{pmatrix} O & K_1 v \\ K_2 v & O \end{pmatrix}\begin{pmatrix} (M/A) & (M/B) \\ (M/C) & (M/D) \end{pmatrix} = \begin{pmatrix} K_1 & O \\ O & K_2 \end{pmatrix}\begin{pmatrix} O & v \\ v & O \end{pmatrix}\begin{pmatrix} (M/A) & (M/B) \\ (M/C) & (M/D) \end{pmatrix} = KVS$$

Thus KVS is factorized as $KVS = KVPQL$.

Hence $\rho(KVS) = \rho(L)$ and $N(KVS) = N(L)$.

But S is con-s-k-EP. Therefore, KVS is con-EP (By Theorem 2.11 [1]).

$N(KVS) = N(KVS)^T \Rightarrow N(L) = N(S^T VK)$

Therefore, by using Theorem 1.1 again we get,

$S^T VK = S^T VKL^{-} L$ holds for every L^{-}.

We choose L^{-} as $L^{-} = \begin{pmatrix} O & (M/C)^{-} \\ [S/(M/C)]^{-} & O \end{pmatrix}$

$$S^T VK = \begin{pmatrix} (M/A) & (M/B) \\ (M/C) & (M/D) \end{pmatrix}^T \begin{pmatrix} O & v \\ v & O \end{pmatrix}\begin{pmatrix} K_1 & O \\ O & K_2 \end{pmatrix} = \begin{pmatrix} (M/A)^T & (M/C)^T \\ (M/B)^T & (M/D)^T \end{pmatrix}^T \begin{pmatrix} O & vK_2 \\ vK_1 & O \end{pmatrix} = \begin{pmatrix} (M/C)^T vK_1 & (M/A)^T vK_2 \\ (M/D)^T vK_1 & (M/B)^T vK_2 \end{pmatrix}$$

As the equation (at the bottom of this page).
and since

$$\rho\left[K_1 v(M/C)^T\right] = \rho[K_1 v(M/C)]$$

$$\Rightarrow \rho\left[(M/C)^T vK_1\right] = \rho(M/C)$$

$$\Rightarrow N(M/C) = N\left[(M/C)^T vK_1\right]$$

Hence, (M/C) is con-s-k-EP.

From $(M/D)^T vK_1 = (M/D)^T vK_1 (M/C)^{-}(M/C)$, is follows that

$$N(M/C) \subseteq N\left[(M/D)^T vK_1\right]$$

$$\Rightarrow N\left[(M/C)^T vK_1\right] \subseteq N\left[(M/D)^T vK_1\right]$$

(using (M/C) is con-s-k-EP_r).

Therefore $N(M/C)^T \subseteq N(M/D)^T$.

After substituting

$$(M/B) = [S/(M/C)] + (M/A)(M/C)^{-}(M/D)$$

and using

$$(M/A)^T vK_2 = (M/A)^T vK_2 [S/(M/C)]^{-}[S/(M/C)]$$

in

$$(M/B)^T vK_2 = (M/B)^T vK_2 [S/(M/C)]^{-}[S/(M/C)]$$

$$S^T VK = S^T VKL^{-} L \Rightarrow \begin{pmatrix} (M/C)^T vK_1 & (M/A)^T vK_2 \\ (M/D)^T vK_1 & (M/B)^T vK_2 \end{pmatrix}$$

$$= \begin{pmatrix} (M/C)^T vK_1 & (M/A)^T vK_2 \\ (M/D)^T vK_1 & (M/B)^T vK_2 \end{pmatrix}\begin{pmatrix} O & (M/C)^{-} \\ [S/(M/C)]^{-} & O \end{pmatrix}\begin{pmatrix} O & [S/(M/C)]^{-} \\ (M/C) & O \end{pmatrix}$$

$$= \begin{pmatrix} (M/C)^T vK_1 (M/C)^{-}(M/C) & A^T vK_2 [S/(M/C)]^{-}[S/(M/C)] \\ (M/D)^T vK_1 (M/C)^{-}(M/C) & B^T vK_2 [S/(M/C)]^{-}[S/(M/C)] \end{pmatrix}$$

$$\Rightarrow (M/C)^T vK_1 = (M/C)^T vK_1 (M/C)^{-}(M/C)$$

$$\Rightarrow [K_1 v(M/C)]^T = [K_1 v(M/C)]^T (M/C)^{-}(M/C)$$

$$\Rightarrow N(M/C) \subseteq N[K_1 v(M/C)]^T = N(M/C)^T vK_1$$

We get,

$$(M/B)^T vK_2 = (M/B)^T vK_2 \left[S/(M/C)\right]^- \left[S/(M/C)\right]$$

$$\left(\left[S/(M/C)\right] + (M/A)(M/C)^- (M/B)\right)^T vK_2$$

$$= \left[S/(M/C)\right]$$

$$+ \left[(M/A)(M/C)^- (M/B)\right]^T vK_2 \left[S/(M/C)\right]^- \left[S/(M/C)\right]$$

$$\left[S/(M/C)\right]^T vK_2 + \left[(M/A)(M/C)^- (M/B)\right]^T vK_2$$

$$= \left[S/(M/C)\right]^T vK_2 \left[S/(M/C)\right]^- \left[S/(M/C)\right]$$

$$+ \left[(M/A)(M/C)^- (M/B)\right]^T vK_2 (M/C)^- (M/C)$$

$$\left[S/(M/C)\right]^T vK_2$$

$$= \left[S/(M/C)\right]^T vK_2 \left[S/(M/C)\right]^- \left[S/(M/C)\right]$$

$$\Rightarrow N\left[S/(M/C)\right] \subseteq N\left[S/(M/C)\right]^T vK_2$$

By Theorem 1.1
and since

$$\rho\left(K_2 v\left[S/(M/C)\right]^T\right) = \rho\left[S/(M/C)\right]^T = \rho\left[S/(M/C)\right]$$

we get,

$$N\left(K_2 v\left[S/(M/C)\right]^T\right) = N\left[S/(M/C)\right]$$

$$\Rightarrow N\left(\left[S/(M/C)\right] vK_2\right) = N\left[S/(M/C)\right]$$

$\Rightarrow \left[S/(M/C)\right]$ is con-s-k_2-EP_r.

Further

$$(M/A)^T vK_2 = (M/A)^T vK_2 \left[S/(M/C)\right]^- \left[S/(M/C)\right]$$

$$\Rightarrow N\left[S/(M/C)\right] \subseteq N\left((M/A)^T vK_2\right)$$

$$\Rightarrow N\left(K_2 v\left[S/(M/C)\right]^T\right) \subseteq N\left((M/A)^T vK_2\right)$$

$$\Rightarrow N\left(\left[S/(M/C)\right]^T vK_2\right) \subseteq N\left((M/A)^T vK_2\right)$$

$$\Rightarrow N\left[S/(M/C)\right]^T \subseteq N(M/A)^T$$

Thus 2) holds 2) $\Rightarrow$ 1). Since $N(M/C) \subseteq N(M/A)$, $N(M/C)^T \subseteq N(M/D)^T$, $N\left[S/(M/C)\right] \subseteq N(M/D)$ and $N\left[S/(M/C)\right]^T \subseteq N(M/A)^T$ holds, according to the assumption by applying Theorem 1.2, $(KVS)^\dagger$ is given by the formula

$$(KVS) = \begin{pmatrix} \begin{matrix} K_1 v(M/C) + \left(K_1 v(M/C)\right)\left(K_1 v(M/D)\right) \\ \left(K_2 v\left[S/(M/C)\right]^\dagger \left(K_2 v(M/A)\right)\right)\left(K_1 v(M/C)\right)^\dagger \end{matrix} & -\left(K_1 v(M/C)\right)\left(K_1 v(M/D)\right)\left(K_2 v\left[S/(M/C)\right]^\dagger\right) \\ -\left(K_2 v\left[S/(M/C)\right]\right)^\dagger \left(K_2 v(M/A)\right)\left(K_1 v(M/C)\right)^\dagger & K_2 v\left[S/(M/C)\right]^\dagger \end{pmatrix} \quad (2.6)$$

$$= \begin{pmatrix} K_1 v(M/C)^\dagger + (M/C)^\dagger (M/D)\left[S/(M/C)\right]^\dagger (M/A)\left(K_1 v(M/C)\right)^\dagger & -(M/C)^\dagger (M/D)\left[S/(M/C)\right]^\dagger vK_2 \\ -\left[S/(M/C)\right]^\dagger (M/A)\left(K_1 v(M/C)\right)^\dagger & \left[S/(M/C)\right]^\dagger vK_2 \end{pmatrix}$$

$$[KVS][KVS]^\dagger = \begin{pmatrix} \begin{matrix} \left(K_1 v(M/C)\right)\left(K_1 v(M/C)\right)^\dagger \\ +\left(K_1 v(M/C)\right)\left((M/C)^\dagger (M/D)\left[S/(M/C)\right]^\dagger\right) \\ (M/A)\left(K_1 v(M/C)\right)^\dagger - \left(K_1 v(M/D)\right)\left[S/(M/C)\right]^\dagger \\ +(M/A)\left(K_1 v(M/C)\right)^\dagger \end{matrix} & \begin{matrix} -\left(K_1 v(M/C)\right)(M/C)^\dagger \\ (M/D)\left[S/(M/C)\right]^{\dagger\dagger} vK_2 \\ +\left(K_1 v(M/D)\right)\left[S/(M/C)\right]^\dagger vK_2 \end{matrix} \\ \begin{matrix} \left(K_2 v(M/A)\right)\left(K_1 v(M/C)\right)^\dagger \left(K_2 v(M/A)\right)(M/C)^\dagger \\ (M/D)\left[S/(M/C)\right]^\dagger (M/D)\left(K_1 v(M/C)\right)^\dagger \\ -\left(K_2 v(M/B)\right)\left[S/(M/C)\right]^\dagger (M/A)\left(K_1 v(M/C)\right) \end{matrix} & \begin{matrix} -\left(K_2 v(M/A)\right)(M/C)^\dagger \\ (M/D)\left[S/(M/C)\right]^\dagger vK_2 \\ +\left(K_2 v(M/B)\right)\left[S/(M/C)\right] vK_2 \end{matrix} \end{pmatrix}$$

According to Theorem 1.1 the assumptions $N(M/C) \subseteq N(M/A)$ and $N/(M/C)^T \subseteq N/(M/D)^T \Rightarrow [S/(M/C)]$ is invariant for every choice of $(M/C)^-$

Hence

$$K_2v(M/B) = K_2v[S/(M/C)] + (K_2v(M/C))(K_1v(M/C)^\dagger(K_1v(M/D)))$$

Therefore

$$K_2v[S/(M/C)] = K_2v(M/B) - (K_2v(M/A))(K_1v(M/C))^\dagger(K_1v(M/D))$$

$$\Rightarrow (K_2v(M/A))(K_1v(M/C))^\dagger(K_1v(M/D)) = (K_2v(M/B)) - K_2v[S/(M/C)]$$

$$\Rightarrow K_2v(M/B)(M/C)^\dagger(M/D) = K_2v((M/B) - [S/(M/C)])$$

$$\Rightarrow (M/A)(M/C)^\dagger(M/D) = (M/B) - [S/(M/C)]$$

Further using

$$K_2v(M/A) = (K_2v[S/(M/C)])(K_2v[S/(M/C)]^\dagger)(K_2v(M/A))$$

and

$$K_1v(M/D) = (K_1v(M/C))(K_1v(M/C))^\dagger(K_1v(M/D)).$$

That is

$$K_2v(M/A) = K_2v[S/(M/C)][S/(M/C)]^\dagger vK_2K_2v(M/A)$$

$$= KK_2v[S/(M/C)][S/(M/C)]^\dagger(M/A)$$

$$(M/A) = [S/(M/C)][S/(M/C)]^\dagger(M/A)$$

and

$$K_1v(M/D) = K_1v(M/C)(M/C)^\dagger vK_1K_1v(M/D)$$

$$= K_1v(M/C)(M/C)^\dagger(M/D)$$

$$(M/D) = (M/C)(M/C)^\dagger(M/D),$$

$(KVS)(KVS)^\dagger$ reduces to the form,

As the Equation (a) below.

Again using

$$(K_1v(M/D))^\dagger = (K_1v(M/D))(K_2v[S/(M/C)])^\dagger(K_2v[S/(M/C)])$$

and

$$(K_2v(M/A)) = (K_2v(M/A))(K_1v(M/C))^\dagger(K_1v(M/C))$$

that is, $(M/D) = (M/D)[S/(M/C)]^\dagger[S/(M/C)]$

and

$(M/A) = (M/A)(M/C)^\dagger(M/C)$, $(KVS)(KVS)^\dagger$ reduces to the form

As the Equation (b) below.

Since, (M/C) is con-s-k$_1$-EP $\Rightarrow K_1v(M/C)$ is con-EP.

Therefore we have

$$[K_1v(M/C)][K_1v(M/C)]^\dagger = [K_1v(M/C)]^\dagger[K_1v(M/C)]$$

Similarly, since $[S/(M/C)]$ is con-s-k$_2$-EP$_r$. We have,

$$(K_2v(M/C))(K_2v[S/(M/C)])^\dagger = (K_2v[S/(M/C)])^\dagger(K_1v[S/(M/C)])$$

Thus

$$(KVS)(KVS)^\dagger = (KVS)^\dagger(KVS)$$

$$\Rightarrow KVSS^\dagger VK = S^\dagger VKKVS$$

$$\Rightarrow KVSS^\dagger VK = S^\dagger S$$

$$\Rightarrow KVSS^\dagger = S^\dagger SKV$$

$\Rightarrow S$ is con-s-k-EP (by Theorem 2.11 [1]).

Thus 1) holds 2) $\Leftrightarrow$ 3)

$$\begin{pmatrix} K_2v(M/C) & 0 \\ K_2v(M/A) & K_2v[S/(M/C)] \end{pmatrix}$$

is con-EP if and only if $K_1v(M/C)$ and $K_2v[S/(M/C)]$ are con-EP.

Therefore,

$$\begin{pmatrix} K_1 & 0 \\ 0 & K_2 \end{pmatrix}\begin{pmatrix} 0 & v \\ v & 0 \end{pmatrix}\begin{pmatrix} (M/C) & 0 \\ (M/A) & [S/(M/C)] \end{pmatrix}$$

$$(KVS)(KVS)^\dagger = \begin{pmatrix} (K_1v(M/C))(K_1v(M/C))^\dagger & 0 \\ 0 & (K_2v[S/(M/C)])(K_2v[S/(M/C)])^\dagger \end{pmatrix} \quad \text{(a)}$$

$$(KVS)(KVS)^\dagger = \begin{pmatrix} (K_1v(M/C))(K_1v(M/C))^\dagger & 0 \\ 0 & (K_2v[S/(M/C)])(K_2v[S/(M/C)])^\dagger \end{pmatrix} \quad \text{(b)}$$

is con-EP if and only if $K_1 v(M/C)$ and $K_2 v(M/C)$ are con-EP.

$\begin{pmatrix} (M/C) & 0 \\ (M/A) & [S/(M/C)] \end{pmatrix}$ is con-s-k-EP if and only if (M/C) is con-s-k_1-EP and $[S/(M/C)]$ is con-s-k_2-EP.

Further $N(M/C) \subseteq N(M/A)$

and $N[S/(M/C)]^T \subseteq N(M/D)^T$

Also $\begin{pmatrix} K_1 v(M/C) & K_1 v(M/D) \\ 0 & K_2 v[S/(M/C)] \end{pmatrix}$ is con-EP if and only if and $K_2 v[S/(M/C)]$ and con-EP.

Therefore, $\begin{pmatrix} (M/C) & (M/D) \\ 0 & [S/(M/C)] \end{pmatrix}$ is con-s-k-EP if and only if (M/C) is con-s-k_1-EP and $[S/(M/C)]$ is con-s-k_2-EP further $N(M/C)^T \subseteq N(M/D)^T$ and $N[S/(M/C)]^T \subseteq N(M/D)$.

This proves the equivalence of 2) and 3). The proof is complete.

Theorem 2.7

Let S be a matrix of the form (2.2) with $N(M/C)^T \subseteq N(M/D)^T$ and $N[S/(M/C)]^T \subseteq N(M/A)^T$, then the following are equivalent.

1) S is con-s-k-EP with $k = k_1 k_2$ where $K = \begin{pmatrix} K_1 & 0 \\ 0 & K_2 \end{pmatrix}$ and $V = \begin{pmatrix} 0 & v \\ v & 0 \end{pmatrix}$

2) (M/C) is con-s-k_1-EP. Further and $[S/(M/C)]$ is con-s-k_2-EP. Further $N(M/C) \subseteq N(M/A)$ and $N[S/(M/C)] \subseteq N(M/D)$

3) Both the matrices $\begin{pmatrix} (M/C) & 0 \\ (M/A) & [S/(M/C)] \end{pmatrix}$ and $\begin{pmatrix} (M/C) & (M/D) \\ 0 & [S/(M/C)] \end{pmatrix}$ are con-s-k-EP.

Proof

This follows from Theorem 2.5 and from the fact that S is con-s-k-EP $\Leftrightarrow$ S^T is con-s-k-EP.

In particular, when $(M/D) = (M/A)^T$, we got the following.

Corollary 2.8

Let $S = \begin{pmatrix} (M/A) & (M/B) \\ (M/C) & (M/A)^T \end{pmatrix}$ with $N(M/C) \subseteq N(M/A)$ and $N[S/(M/C)] \subseteq N(M/A)^T$.

Then the following are equivalent.

1) S is a con-s-k-EP matrix.

2) (M/C) is con-s-k_1-EP and $[S/(M/C)]$ is con-s-k_2-EP.

3) The matrix $\begin{pmatrix} (M/C) & 0 \\ (M/A) & [S/(M/C)] \end{pmatrix}$ is con-s-k- EP.

Remark 2.9

The conditions taken on S in Theorem 2.6 and Theorem 2.7 are essential. This is illustrated in the following example.

Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$

$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, C = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, D = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$

$$M = \begin{bmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \\ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \end{bmatrix}$$

$(M/A) = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}, (M/B) = \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix},$

$(M/C) = \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}, (M/D) = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix},$

$$S = \begin{pmatrix} (M/A) & (M/B) \\ (M/C) & (M/D) \end{pmatrix}$$

$$S = \begin{pmatrix} \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix} \\ \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} \end{pmatrix} \quad K = \begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix}$$

$$V = \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix} \quad KV = \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix}$$

Now $KVS = \begin{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix} \\ \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & 2 \end{pmatrix} \end{pmatrix},$

KVS is symmetric of rank 3

$\Rightarrow$ KVS is con-EP $\Rightarrow$ S is con-s-k-EP.

$[S/(M/C)] = (M/B) - (M/D)(M/C)^{-1}(M/A)$

$(M/A) = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}, \ (M/B) = \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}$

$$(M/D)=\begin{pmatrix}1 & -1\\ 2 & 1\end{pmatrix},$$

$$(M/C)^{-1}=\frac{1}{3}\begin{pmatrix}1 & -2\\ 1 & 1\end{pmatrix}$$

$$[S/(M/C)]=\begin{pmatrix}3 & 3\\ 0 & 3\end{pmatrix}$$

Hence $K_2v[S/(M/C)]=\begin{pmatrix}0 & 3\\ 3 & 3\end{pmatrix}$ is con-EP,

that is $[S/(M/C)]$ is con-s-k$_2$-EP.

Also, $(M/C)=\begin{pmatrix}1 & 2\\ -1 & 1\end{pmatrix}\Rightarrow K_1v(M/C)=\begin{pmatrix}-1 & 1\\ 1 & 2\end{pmatrix}$ is con-EP. $K_1v(M/C)$ is con-EP $\Rightarrow (M/C)$ is con-s-k$_1$-EP.

Moreover $N(M/C)\subseteq N(M/A)$ and $N(M/D)^T\subseteq N(M/C)^T$. But $N[S/(M/D)]\subseteq N(M/D)$ and $N[S/(M/C)]^T\subseteq N(M/A)^T$.

Further

$$KV\begin{pmatrix}(M/C) & 0\\ (M/A) & [S/(M/C)]\end{pmatrix}=\begin{bmatrix}\begin{pmatrix}1 & 2\\ -1 & 1\end{pmatrix} & \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}\\ \begin{pmatrix}1 & -1\\ 2 & 1\end{pmatrix} & \begin{pmatrix}3 & 3\\ 0 & 3\end{pmatrix}\end{bmatrix}$$ is not con-EP.

Therefore,

$\begin{pmatrix}(M/C) & (M/D)\\ 0 & [S/(M/C)]\end{pmatrix}$ is not con-s-k-EP.

Thus the Theorem 2.5 and the Theorem 2.7 as well as the corollary 2.8 fail.

Remarks 2.10

We conclude from Theorem 2.5 and Theorem 2.7 that for a con-s-k-EP matrix of the form 2.2 and $k = k_1k_2$ where $K=\begin{pmatrix}k_1 & 0\\ 0 & k_2\end{pmatrix}$ and $v=\begin{pmatrix}0 & v\\ v & 0\end{pmatrix}$ the following are equivalent.

$$N(M/C)\subseteq N(M/A),\quad N[S/(M/C)]\subseteq N(M/D) \qquad 2.11$$

$$N(M/C)^T\subseteq N(M/D)^T,\quad N[S/(M/C)]^T\subseteq N(M/A)^T \qquad 2.12$$

However this fails if we omit the condition that S is con-s-k-EP.

For example,

Let $M=\begin{pmatrix}A & B\\ C & D\end{pmatrix}$, where

$$A=\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix},\ B=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\ C=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\ D=\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$$

$$M=\begin{bmatrix}\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix} & \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\\ \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} & \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\end{bmatrix}$$

$$A,B,C,D\Rightarrow (M/A)=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},$$

$$(M/B)=\begin{pmatrix}0 & -2\\ -1 & 0\end{pmatrix},\ (M/C)=\begin{pmatrix}-1 & -1\\ 1 & -1\end{pmatrix},$$

$$(M/D)=\begin{pmatrix}1 & 0\\ -1 & 2\end{pmatrix}$$

$$S=\begin{pmatrix}(M/A) & (M/B)\\ (M/C) & (M/D)\end{pmatrix}$$

$$S=\begin{bmatrix}\begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix} & \begin{pmatrix}0 & -2\\ -1 & 0\end{pmatrix}\\ \begin{pmatrix}-1 & -1\\ 1 & -1\end{pmatrix} & \begin{pmatrix}1 & 0\\ -1 & 2\end{pmatrix}\end{bmatrix}$$

$$K=\begin{bmatrix}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} & \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}\\ \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix} & \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\end{bmatrix}\quad V=\begin{bmatrix}\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix} & \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\\ \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} & \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}\end{bmatrix}$$

$$KVS=\begin{bmatrix}\begin{pmatrix}1 & -1\\ -1 & -1\end{pmatrix} & \begin{pmatrix}-1 & 2\\ 1 & 0\end{pmatrix}\\ \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} & \begin{pmatrix}-1 & 0\\ 0 & -2\end{pmatrix}\end{bmatrix}$$ is not con-EP.

Therefore S is not con-s-k-EP.

Here $K_1v(M/C)=\begin{pmatrix}1 & -1\\ -1 & -1\end{pmatrix}$ is con-EP.

$\Rightarrow (M/C)$ is con-s-k-EP.

$K_1v(M/D)\neq (K_1v(M/D))^T$,

$K_1v(M/D)\neq ((M/D)^T vK_1)^T$,

$(M/D)\neq vK_1A^TvK_1$,

$v(M/C)\subseteq v(M/A)$,

and $v(M/C)^T\subseteq v(M/D)^T$.

Hence $[S/(M/C)]$ is independent of the choice of $(M/C)^-$.

Now

$$\left[S/(M/C)\right]=(M/B)-(M/A)(M/C)^{\dagger}(M/D)$$

$$(M/B)=\begin{pmatrix}0 & -2\\ -1 & 0\end{pmatrix},(M/A)=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},$$

$$(M/D)=\begin{pmatrix}1 & 0\\ -1 & 2\end{pmatrix},(M/C)^{-1}=\frac{1}{2}\begin{pmatrix}-1 & -1\\ -1 & -1\end{pmatrix}$$

$$\left[S/(M/C)\right]=\begin{pmatrix}0 & -1\\ -1 & 1\end{pmatrix}$$

$K_2v\left[S/(M/C)\right]=\begin{pmatrix}-1 & 1\\ 0 & -1\end{pmatrix}$ is not con-EP.

$\Rightarrow\left[S/(M/C)\right]$ is not con-s-k_2-EP.

Also, $N\left[S/(M/C)\right]^T\subseteq N(M/D)^T$. But

$N\left[S/(M/C)\right]\not\subset N(M/D)$.

Thus, 2.12 holds while 2.11 fails.

Remark 2.13

It is clear by Remark 2.10 that for a con-s-k-EP marrix *S*, formula 2.6 gives $(KVS)^{\dagger}$ if and only if either 2.11 or 2.12 holds.

Corollary 2.14

Let *S* be a matrix of the form 2.2 with *K* and *V* are of the forms 2.3 and 2.4 respectively, for which $(KVS)^{\dagger}$ is given by the formula then *S* is con-s-k-EP if and only if both (*M*/*C*) and $\left[S/(M/C)\right]$ and con-s-k-EP.

Proof

This follows from Theorem 2.5 and using Remark 2.13. Now we proceed to prove the most important Theorem.

Theorem 2.15

Let *S* be of the form 2.2 with $\rho(S)=\rho(M/C)=r$. Then *S* is con-s-k-EP$_r$ and *K* and *V* are of the form 2.3 and 2.4 if and only if (*M*/*C*) is con-s-k_1-EP$_r$ and

$$(M/A)(M/C)^{\dagger}vK_1=\left((M/C)^{\dagger}(M/D)vK_2\right)^T.$$

Proof

Let *S* be of the form 2.2 and let k= k_1k_2 with $K=\begin{pmatrix}k_1 & 0\\ 0 & k_2\end{pmatrix}$ and $v=\begin{pmatrix}0 & v\\ v & 0\end{pmatrix}$ then

$$KVS=\begin{pmatrix}K_1v(M/C) & K_1v(M/D)\\ K_2v(M/A) & K_2v(M/B)\end{pmatrix}.$$

Since $\rho(S)=\rho(M/C)=r$,

$\rho(KVS)=\rho(K_1v(M/C))=r$ by [6]

$N(M/C)=N(M/A)$, $N(M/C)^T\subseteq N(M/D)^T$ and

$(KVS/K_1v(M/C))$

$=K_2v\left[S/(M/C)\right]=0\Rightarrow\left[S/(M/C)\right]=0.$

By Theorem 1.1 these relation equivalent to $K_2v(M/A)=K_2v(M/A)(M/C)$,

$K_1v(M/D)=K_1v(M/C)(M/C)^{\dagger}(M/D)$ and

$K_2v(M/B)=K_2v(M/A)(M/C)^{\dagger}(M/D)$

Let us consider the matrices

$$P=\begin{pmatrix}I & (M/A)(M/C)\\ 0 & I\end{pmatrix}$$

$Q=\begin{pmatrix}I & (M/C)^{\dagger}(M/D)\\ 0 & I\end{pmatrix}$ and $L=\begin{pmatrix}0 & 0\\ (M/C) & 0\end{pmatrix}$

$$KVPLQ=\begin{pmatrix}K_1 & 0\\ 0 & K_2\end{pmatrix}\begin{pmatrix}0 & v\\ v & 0\end{pmatrix}\begin{pmatrix}I & (M/A)(M/C)^{\dagger}\\ 0 & I\end{pmatrix}\begin{pmatrix}0 & 0\\ (M/C) & 0\end{pmatrix}\begin{pmatrix}I & (M/C)^{\dagger}(M/D)\\ 0 & I\end{pmatrix}$$

$$=\begin{pmatrix}0 & K_1v\\ K_2v & 0\end{pmatrix}\begin{pmatrix}(M/A)(M/C)^{\dagger}(M/C) & 0\\ (M/C) & 0\end{pmatrix}\begin{pmatrix}I & (M/C)^{\dagger}(M/D)\\ 0 & I\end{pmatrix}$$

$$=\begin{pmatrix}0 & K_1v\\ K_2v & 0\end{pmatrix}\begin{pmatrix}(M/A)(M/C)(M/C)^{\dagger} & (M/A)(M/C)(M/C)^{\dagger}(M/C)^{\dagger}(M/C)\\ (M/C) & (M/C)(M/C)^{\dagger}(M/D)\end{pmatrix}$$

$$=\begin{pmatrix}K_1v(M/C) & K_1v(M/C)(M/C)^{\dagger}(M/D)\\ K_2v(M/A)(M/C)(M/C)^{\dagger} & K_2v(M/A)(M/C)^{\dagger}(M/D)\end{pmatrix}$$

$$=\begin{pmatrix}K_1v(M/C) & K_1v(M/D)\\ K_2v(M/A) & K_2v(M/B)\end{pmatrix}$$

$$=\begin{pmatrix}K_1 & 0\\ 0 & K_2\end{pmatrix}\begin{pmatrix}0 & v\\ v & 0\end{pmatrix}\begin{pmatrix}(M/A) & (M/B)\\ (M/C) & (M/D)\end{pmatrix}$$

$$=KVS$$

Thus KVS can be factorized as $KVS = KVPLQ$. Since $KVP = (KVQ)^T$.

We have $KVP^T VK=Q$. Therefore,

$$KVS = KVPLKVP^T VK$$
$$=(KVP)(LKV)(KVP)^T$$
$$=(KVP)(KVL)(KVP)^T$$

[since $LVK = KVL$].

Since (M/C) is con-s-k_1-EP_r. We have $k_1 v(M/C)$ is con-EP_r.

Therefore $N(L)=N\left(L^T VK\right)$

(Theorem 2.11 of [1])

$$\Rightarrow N(KVL)=N(KVL)^T$$

By Theorem 1.3

$$N\left[(KVP)(KVL)(KVP)^T\right]=N\left[(KVP)(KVL)^T(KVP)^T\right]$$
$$\Rightarrow N(KVS)=N\left[(KVS)^T\right]$$
$$\Rightarrow N(S)=N\left[S^T VK\right]$$

$\Rightarrow S$ is con-s-k-EP (Theorem 2.11 of [1]).

Since $\rho(S)=r$, S is con-s-k-EP_r.

Conversely, let us assume that S is con-s-k-EP_r.

Since S is con-s-k-EP_r, KVS is con-EP_r. Since $KVS = KVPLQ$, one choice of

$$(KVS)^- = Q^{-1}\begin{pmatrix} 0 & 0 \\ (M/C)^\dagger & 0 \end{pmatrix} P^{-1}VK\ KVS \quad \text{is con-EP}$$

$\Rightarrow N(KVS)=N\left[(KVS)^T\right]$ By Theorem 1.1

$$(KVS)^T=(KVS)^T(KVS)^-(KVS).$$

That is,

$$\begin{pmatrix} K_1v(M/C) & K_1v(M/D) \\ K_2v(M/A) & K_2v(M/B) \end{pmatrix}^T$$
$$=\begin{pmatrix} K_1v(M/C) & K_1v(M/D) \\ K_2v(M/A) & K_2v(M/B) \end{pmatrix}^T$$
$$Q^{-1}\begin{pmatrix} 0 & 0 \\ (M/C)^\dagger & 0 \end{pmatrix}$$
$$P^{-1}VK\begin{pmatrix} K_1v(M/C) & K_1v(M/D) \\ K_2v(M/A) & K_2v(M/B) \end{pmatrix}$$

As the equation (at the bottom of this page).

or conversely,

$$\left(K_1v(M/C)\right)^T=\left(K_1v(M/C)\right)^T(M/C)^\dagger(M/C)$$

and $\left(K_2v(M/C)\right)^T=\left(K_1v(M/C)\right)^T(M/C)^\dagger(M/D)$

From $\left(K_1v(M/C)\right)^T=\left(K_1v(M/C)\right)^T(M/C)^\dagger(M/C)$ it follows that

$$N(M/C)=N\left[\left(K_1v(M/C)\right)^T\right]$$
$$\Rightarrow N(M/C)\subseteq N(M/C)^T vK_1 \Rightarrow (M/C)$$

is con-s-k-EP.

Since $\rho(M/C)=r$. (M/C) is con-s-k-EP_r.

From

$$\left(K_2v(M/A)\right)^T=\left(K_1v(M/C)\right)^T(M/C)^\dagger(M/D)$$

it follows that.

Now,

$$K_2v(M/A)(M/C)^\dagger$$
$$=(M/D)^T\left((M/C)^\dagger\right)^T\left(K_1v(M/C)\right)(M/C)^\dagger$$
$$=(M/D)^T\left((M/C)^\dagger\right)^T\left((M/C)^\dagger(M/C)K_1v\right)$$
$$=(M/D)^T\left[(M/C)^\dagger(M/C)(M/C)^\dagger\right]^T(vK_1)^T$$
$$=(M/D)^T\left[(M/C)^\dagger\right]^T(vK_1)^T$$
$$=\left[K_1v(M/C)^\dagger(M/D)\right]^T$$

(By theorem 2.11 [1])

$$K_2v(M/A)(M/C)^\dagger=\left[(M/C)^\dagger(M/D)\right]^T vK_1$$
$$(M/A)(M/C)^\dagger vK_1=K_2v\left[(M/C)^\dagger(M/D)\right]^T$$
$$(M/A)(M/C)^\dagger vK_1=\left[(M/C)^\dagger(M/D)vK_2\right]^T$$

Mark 2.16

When (M/A) is non singlular, $KV(M/A)$ is automatically con-EP_r and (M/A) is con-s-k-EP_r and Theorem 2.15 reduces to the following.

Corollary 2.17

Let S be of the form 2.2 with C non singular and $\rho[S]=\rho(M/C)$. Then S is con-s-k-EP with $K = k_1k_2$

and $v=\begin{pmatrix} 0 & v \\ v & 0 \end{pmatrix} \Leftrightarrow (M/A)(M/C)^\dagger vK_1 =\left[(M/C)^\dagger(M/D)vK_2\right]^T$.

$$\begin{pmatrix} \left(K_1v(M/C)\right)^T & \left(K_1v(M/A)\right)^T \\ \left(K_2v(M/D)\right)^T & \left(K_2v(M/B)\right)^T \end{pmatrix}=\begin{pmatrix} \left(K_1v(M/C)\right)^T(M/C)^\dagger(M/C) & \left(K_1v(M/C)\right)^T(M/C)^\dagger(M/D) \\ \left(K_1v(M/D)\right)^T(M/C)^\dagger(M/C) & \left(K_1v(M/C)\right)^T(M/C)^\dagger(M/D) \end{pmatrix}$$

Remark 2.18

When k(i) = i, we have $K_1 = K_2 = I$, then the Theorem 2.15 reduces to the result for con-s-EP matrices.

When KV = I then Theorem 2.15 reduces to Theorem 3 of [5].

Remark 2.19

Theorem 2.15 fails if we relax the condition on the rank of *S*.

For example, let us consider the matrix *S* and *K* given in Remark 2.10, $\rho[KVS] = \rho[S] = 2$.

But $\rho\left(K_1V\left(M/C\right)\right) = \rho\left(M/C\right) = 1$,

$\rho(KVS) \neq \rho\left(K_1v\left(M/A\right)\right) \Rightarrow \rho(S) \neq \rho\left(M/A\right)$.

KVS is not con-EP

Therefore *S* is not Con-s-k-EP.

However,

$$K_1V\left(M/C\right) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}$$ is con-EP.

Therefore (*M*/*C*) is con-s-k_1-EP and

$$\left(M/C\right)^{-1} = \frac{1}{2}\begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix},$$

$$\left(M/A\right)\left(M/C\right)^{-1} vK_1 = \frac{1}{2}\begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix},$$

$$\left(M/C\right)^{-1}\left(M/D\right)vK_2 = \begin{pmatrix} -1 & 0 \\ -1 & 0 \end{pmatrix}.$$

Thus the theorem fails.

Corollary 2.20

Les *S* be a 2r x 2r matrix of rank r. Thus *S* is con-s-k-EP_r with $K = K_1K_2$, where $\begin{pmatrix} K_1 & 0 \\ 0 & K_2 \end{pmatrix}$ and $V = \begin{pmatrix} v & 0 \\ 0 & v \end{pmatrix} \Leftrightarrow$ every secondary sub matrix of *S* of rank r is con-s-k-EP_r.

Proof

Suppose *S* is con-s-k-EP_r matrix then *KVS* is an con-EP_r matrix by Theorem 2.11 [1]. Let $K_1v\left(M/C\right)$ be any Principal submatrix of *KVS* such that $\rho[KVS] = \rho\left[K_1v\left(M/C\right)\right] = r$, then there exists a permutation matrix *P* such that,

$$\left(KVS\right)^T = P\left(KVS\right)P^T\begin{pmatrix} K_1v\left(M/C\right) & K_1v\left(M/D\right) \\ K_2v\left(M/A\right) & K_2v\left(M/B\right) \end{pmatrix}$$

with $\rho\left[KVS\right] = \rho\left[K_1v\left(M/C\right)\right] = r$. By [4] $\left[KVS\right]^T$ is con-EP_r. Now we conclude from Theorem 2.15 that $\left(K_1v\left(M/C\right)\right)$ is con-EP_r. That is (*M*/*C*) is con-s-k_1-EP_r Since [*M*/*C*] is arbitrary it follows that every secondary submatrix of rank r is con-s-k-EP_r. The converse is obvious.

The conditions under which a partitioned matrix is decomposed into complementary sum and *S* of con-s-k-EP matrices are given. S_1 and S_2 and called complementary summands of *S* if

$S = S_1 + S_2$ and $\rho\left[S\right] = \rho\left[S_1\right] + \rho\left[S_2\right]$.

Theorem 2.21

Let *S* be of the form 2.2 with

$\rho\left(S\right) = \rho\left(M/C\right) + \rho\left[S/\left(M/C\right)\right]$,

where $\left[S/\left(M/C\right)\right] = \left(M/B\right) - \left(M/A\right)((M/C)^{\dagger}\left(M/D\right)$

and *K* is of the form 2.3 and *V* is of the form 2.4. If (*M*/*C*) is con-s-k_1-EP and $\left[S/\left(M/C\right)\right]$ is con-s-k_2-EP matrices such that

$\left(M/A\right)\left(M/C\right)^{\dagger} vK_1 = \left(\left(M/C\right)^{\dagger}\left(M/D\right)vK_2\right)^T$ and

$\left(M/D\right)\left[S/\left(M/C\right)\right]^{\dagger} vK_2 = \left(\left[S/\left(M/C\right)\right]^{\dagger}\left(M/C\right)vK_1\right)^T$

then *S* can be decomposed into complementary summands of con-s-k-EP matrices.

Proof

Let us consider the matrices,

$$S_1 = \begin{pmatrix} \left(M/C\right) & \left(M/C\right)\left(M/C\right)^{\dagger}\left(M/D\right) \\ \left(M/A\right)\left(M/C\right)^{\dagger}\left(M/C\right) & \left(M/A\right)\left(M/C\right)^{\dagger}\left(M/D\right) \end{pmatrix}$$

and

$$S_2 = \begin{pmatrix} 0 & \left(I - \left(M/C\right)\left(M/C\right)^{\dagger}\right)\left(M/D\right) \\ \left(M/A\right)\left(I - \left(M/C\right)^{\dagger}\left(M/C\right)\right) & \left[S/\left(M/C\right)\right] \end{pmatrix}.$$

Taking into account that

$\left(M/C\right) \subseteq N\left(\left(M/A\right)\left(M/C\right)^{\dagger}\left(M/A\right)\right)$

$N\left(M/C\right)vK_1 \subseteq N\left(\left(M/A\right)\left(M/C\right)^{\dagger}\left(M/C\right)\right)^T vK_1$ and

$$\begin{aligned}\left[S_1/\left(M/C\right)\right] &= \left(M/A\right)\left(M/C\right)^{\dagger}\left(M/D\right) - \left(\left(M/A\right)\left(M/C\right)^{\dagger}\left(M/D\right)\right)\left(M/C\right)^{-}\left(\left(M/C\right)\left(M/C\right)^{\dagger}\left(M/D\right)\right)_1 \\ &= \left(M/A\right)\left(M/C\right)^{\dagger}\left(M/D\right) - \left(\left(M/A\right)\left(M/C\right)^{\dagger}\right)\left(\left(M/C\right)\left(M/C\right)^{-}\left(M/C\right)\right)\left(M/C\right)^{\dagger}\left(M/D\right)_1 \\ &= \left(M/A\right)\left(M/C\right)^{\dagger}\left(M/D\right) - \left(M/A\right)\left(\left(M/C\right)^{\dagger}\left(M/C\right)\left(M/C\right)^{\dagger}\right)\left(M/D\right) \\ &= \left(M/A\right)\left(M/C\right)^{\dagger}\left(M/D\right) - \left(M/A\right)\left(M/C\right)^{\dagger}\left(M/D\right) \\ &= 0\end{aligned}$$

We obtain by [6] that

$\rho(S_1)=\rho(M/C)$. Since (M/C) is con-s-k$_1$-EP and

$$\left((M/A)(M/C)^{\dagger}(M/C)\right)(M/C)^{\dagger}vK_1$$
$$=(M/A)(M/C)^{\dagger}(M/C)(M/C)^{\dagger}vK_1$$
$$=(M/A)(M/C)^{\dagger}vK_1$$
$$=\left((M/C)^{\dagger}(M/D)vK_1\right)^T$$
$$=\left((M/C)^{\dagger}\left((M/C)(M/C)^{\dagger}(M/D)\right)vK_2\right)^T$$

We have by Theorem 2.15, that is S_1 is con-s-k$_1$-EP.

Since $\rho(S)=\rho(M/C)+\rho\left[S/(M/C)\right]$,

Theorem 1 of [6], gives

$$N\left[S/(M/C)\right]=N\left(\left[I-(M/C)(M/C)^{\dagger}\right](M/D)\right),$$

$$N\left[S/(M/C)\right]^T=N\left((M/C)\left[I-(M/C)^{\dagger}(M/C)\right]\right)^T$$

and

$$\left[I-(M/C)(M/C)^{\dagger}\right](M/D)\left[S/(M/C)\right]^{\dagger}$$
$$\subset\left[I-(M/C)^{\dagger}(M/C)\right]=0$$

Therefore, $\left[S_2/\left[S/(M/C)\right]\right]=0.$

Thus by [7] we get $\rho(S_2)=\rho\left[S/(M/C)\right]$. Thus $\rho(S)=\rho(S_1)+\rho(S_2)$.

Further using

$$=(M/C)(M/C)K_1v=K_1v(M/C)^{\dagger}(M/C)$$

We obtain,

$$\left[I-(M/C)(M/C)^{\dagger}\right](M/D)\left[S/(M/C)\right]^{\dagger}vK_2$$
$$=\left[I-(M/C)(M/C)^{\dagger}\right]\left[\left[S/(M/C)\right]^{\dagger}(M/A)vK_1\right]^T=\left[\left[\left[S/(M/C)\right]^{\dagger}(v/A)vK_1\right]\left[I-(M/C)(M/C)^{\dagger}\right]^T\right]^T$$
$$=\left[\left[S/(M/C)\right]^{\dagger}(v/A)\left[\left[I-(M/C)(M/C)^{\dagger}\right]K_1v\right]^T\right]^T=\left[\left[S/(M/C)\right]^{\dagger}(v/A)\left[K_1v-(M/C)(M/C)^{\dagger}K_1v\right]^T\right]^T$$
$$=\left[\left[S/(M/C)\right]^{\dagger}(M/A)\left[K_1v-K_1v((M/C)^{\dagger}(M/C)\right]^T\right]^T=\left[\left[S/(M/C)\right]^{\dagger}(M/A)\left[K_1v-I-(M/C)^{\dagger}(M/C)\right]^T\right]^T$$
$$=\left[\left[S/(M/C)\right]^{\dagger}(M/A)\ \ I-\left[(M/C)^{\dagger}(M/C)\right]^T vK_1\right]^T$$

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