Duality Relations for a Class of a Multiobjective Fractional Programming Problem Involving Support Functions

Vandana¹, Ramu Dubey², Deepmala³, Lakshmi Narayan Mishra⁴,⁵*, Vishnu Narayan Mishra⁶

¹Department of Management Studies, Indian Institute of Technology Madras, Chennai, India
²Department of Mathematics, Central University of Haryana, Pali, India
³Mathematics Discipline, PDPM-Indian Institute of Information Technology, Design and Manufacturing, Jabalpur, India
⁴Department of Mathematics, School of Advanced Sciences, Vellore Institute of Technology, Vellore, India
⁵L. 1627 Awadh Puri Colony Beniganj, Phase-III, Opposite-Industrial Training Institute (I.T.I.), Faizabad, India
⁶Department of Mathematics, Indira Gandhi National Tribal University, Lalpur, India

Abstract

In this article, for a differentiable function \( H : R^n \times R^n \to R \), we introduce the definition of the higher-order \((V, \alpha, \beta, \rho, d)\)-invexity. Three duality models for a multiobjective fractional programming problem involving nondifferentiability in terms of support functions have been formulated and usual duality relations have been established under the higher-order \((V, \alpha, \beta, \rho, d)\)-invex assumptions.

Keywords

Efficient Solution, Support Function, Multiobjective Fractional Programming, Generalized Invexity

1. Introduction

Consider the following nonlinear programming problem \((P)\) Minimize \( f(x) \) subject to \( g(x) \leq 0 \), where \( f : R^n \to R \) and \( g : R^n \to R \) are twice differentiable functions. The Mangasarian [1] second-order dual of \((P)\) is \((DP)\) Maximize

\[
\max \quad f(u) - y^T g(u) - \frac{1}{2} \beta \nabla^2 \left[ f(u) - y^T g(u) \right] \nu
\]

such that

\[
\nu^T \left[ f(u) - y^T g(u) \right] + \nabla^2 \left[ f(u) - y^T g(u) \right] p = 0
\]

*Corresponding author.
By introducing two differentiable functions $H : R^n \times R^n \rightarrow R$ and $K : R^n \times R^n \rightarrow R^n$, Mangasarian [1] formulated the following higher-order dual of (P): 

\[(DP) \text{ Maximize } f(u) - y^T g(u) + H(u, p) - y^T K(u, p)\]

such that $\nabla_p H(u, p) - \nabla_p \left( y^T K(u, p) \right) = 0$, $y \geq 0$, where $\nabla_p H(u, p)$ denotes the $n \times 1$ gradient of $H(u, p)$ with respect to $p$ and $\nabla_p \left( y^T K(u, p) \right)$ denotes the $n \times 1$, gradient of $y^T K(u, p)$ with respect to $p$.

Further, Egudo [2] studied the following multiobjective fractional programming problem: 

\[(MFPP) \text{ Minimize } G(x) = \left( f_1(x), f_2(x), \ldots, f_k(x) \right)\]

subject to $x \in X^0 = \{ x \in X \subset R^n : h_j(x) \leq 0, j \in M \}$, where $f = (f_1, f_2, \ldots, f_k) : X \rightarrow R^k$, $g = (g_1, g_2, \ldots, g_k) : X \rightarrow R^k$ and $h = (h_1, h_2, \ldots, h_m) : X \rightarrow R^m$ are differentiable on $X$. Also, he discussed duality results for Mond-Weir and Schaible type dual programs under generalized convexity.

For the nondifferentiable multiobjective programming problem: 

\[(MPP) \text{ Minimize } G(x) = \left( f_1(x) + S(x | C_1), f_2(x) + S(x | C_2), \ldots, f_k(x) + S(x | C_k) \right)\]

subject to $x \in X^0 = \{ x \in X \subset R^n : g_j(x) + S(x | E_j) \leq 0, j = 1, 2, \ldots, m \}$, where $f_i : X \rightarrow R^k \ (i = 1, 2, \ldots, k)$ and $g_j : X \rightarrow R^k \ (j = 1, 2, \ldots, m)$ are differentiable functions. $C_i$ and $E_j$ are compact convex sets in $R^n$ and $S(x | C_i) \ (i = 1, 2, \ldots, k)$ and $S(x | E_j) \ (j = 1, 2, \ldots, m)$ denote the support functions of compact convex sets, various researchers have worked. Gulati and Agarwal [3] introduced the higher-order Wolfe-type dual model of (MPP) and proved duality theorems under higher-order $(F, \rho, \rho, d)$-type I-assumptions.

In last several years, various optimality and duality results have been obtained for multiobjective fractional programming problems. In Chen [4], multiobjective fractional problem and its duality theorems have been considered under higher-order $(F, \alpha, \rho, d)$-convexity. Later on, Suneja et al. [5] discussed higher-order Mond-Weir and Schaible type nondifferentiable dual programs and their duality theorems under higher-order $(F, \rho, \rho, \sigma)$-type I-assumptions. Several researchers have also worked in this directions such as ([6] [7]).

In this paper, we first introduce the definition of higher-order $(V, \alpha, \beta, \rho, d)$-invex with respect to differentiable function $H : R^n \times R^n \rightarrow R$. We also construct a nontrivial numerical example which illustrates the existence of such a function. We then formulate three higher-order dual problems corresponding to the multiobjective nondifferentiable fractional programming problem. Further, we
establish usual duality relations for these primal-dual pairs under aforesaid assumptions.

2. Preliminaries

Let \( X \subseteq \mathbb{R}^n \) be an open set and \( \phi : X \rightarrow \mathbb{R}, H : X \times \mathbb{R}^n \rightarrow \mathbb{R} \) be differentiable functions. \( \alpha, \beta : X \times X \rightarrow \mathbb{R} \setminus \{0\}, \eta : X \times X \rightarrow \mathbb{R}^n, \rho \in \mathbb{R}^n \) and \( \theta : X \times X \rightarrow \mathbb{R}^n \).

**Definition 2.1.** \( \phi \) is said to be (strictly) higher-order \((V, \alpha, \beta, \rho, \theta)\)-invex at \( u \) with respect to \( H(u, p) \), if there exist \( \eta, \alpha, \beta, \rho \) and \( \theta \) such that, for any \( x \in X \) and \( p \in \mathbb{R}^n \),

\[
\alpha(x,u)\left[\phi(x) - \phi(u)\right] \geq \eta^T(x,u)\left(\nabla\phi(u) + \nabla_{\rho}H(u, p)\right) + \beta(x,u)\left[H(u, p) - p^T\nabla_{\rho}H(u, p)\right] + \rho\left\|\theta(x,u)\right\|^2.
\]

**Example 2.1.** Let \( \phi : \mathbb{R} \rightarrow \mathbb{R} \) be such that \( \phi(x) = x^4 + x^2 + 1 \).

Let

\[
\eta(x,u) = \frac{1}{2}(x^2 + u^2), H(u, p) = -2p(x+1)^2.
\]

Also, suppose

\[
\alpha(x,u) = 1, \beta(x,u) = 2, \rho = -1, \|\theta(x,u)\| = (x^2 + u^2)^{\frac{1}{2}}.
\]

Now,

\[
\xi(x,u) = \alpha(x,u)\left[\phi(x) - \phi(u)\right] - \eta^T(x,u)\left(\nabla\phi(u) + \nabla_{\rho}H(u, p)\right) - \beta(x,u)\left[H(u, p) - p^T\nabla_{\rho}H(u, p)\right] - \rho\left\|\theta(x,u)\right\|^2.
\]

\[
\xi = (x^4 + x^2 - u^4 - u^2) - \frac{1}{2}(x^2 + u^2)\left[4u^2 + 2x - 2(x+1)^2\right] - (x^2 + u^2)
\]

\[
\xi = x^4 + x^2 \quad \text{(at } u = 0).\]

\[\geq 0, \forall x \in \mathbb{R} .\]

Hence, \( \phi \) is higher-order \((V, \alpha, \beta, \rho, \theta)\)-invex at \( u = 0 \) with respect to \( H(u, p) \).

**Remark 2.1.**

1) If \( H(u, p) = 0 \), then the Definition 2.1 reduces to \((V, \rho)\)-invex function introduced by Kuk et al. [8].

2) If \( H(u, p) = 0 \) and \( \rho = 0 \), then the Definition 2.1 becomes that of \( V \)-invexity introduced by Jeyakumar and Mond [9].

3) If \( H(u, p) = \frac{1}{2}p^T\nabla^2\phi(u) p, \alpha(x,u) = 0 \) and \( \rho = 0 \), then above definition yields in \( \eta \)-bonvexity given by Pandey [10].

4) If \( \beta = 1 \), then the Definition 2.1 reduced in \((V, \alpha, \rho, \theta)\)-invex given by Gulati and Geeta [11].

A differentiable function \( f = (f_1, f_2, \ldots, f_k) : X \rightarrow \mathbb{R}^k \) is \((V, \alpha, \beta, \rho, \theta)\)-invex if for all \( i = 1, 2, \ldots, k \), \( f_i \) is \((V, \alpha, \beta, \rho, \theta)\)-invex. 

**Definition 2.2.** [12]. Let \( C \) be a compact convex set in \( \mathbb{R}^n \). The support
function of \( C \) is defined by
\[
S(x | C) = \max \{ x^T y : y \in C \}.
\]

### 3. Problem Formulation

Consider the multiobjective programming problem with support function given as:

\[
(MFP) \quad \text{Minimize} \quad \left\{ \begin{array}{l}

g_i(x) - S(x | D_i), \quad i = 1, 2, \ldots, n,
\end{array} \right.
\]

subject to \( x \in X^0 = \{ x \in X \subset \mathbb{R}^n : h_j(x) + S(x | E_j) \leq 0, \ j = 1, 2, \ldots, m \} \),

where \( f = (f_1, f_2, \ldots, f_i) : X \rightarrow \mathbb{R}^k \), \( g = (g_1, g_2, \ldots, g_t) : X \rightarrow \mathbb{R}^t \), and \( h = (h_1, h_2, \ldots, h_m) : X \rightarrow \mathbb{R}^m \) are differentiable on \( X \). \( f_j(\cdot) + S(\cdot | C_j) \geq 0 \) and \( g_j(\cdot) - S(\cdot | D_j) > 0 \). Let \( H_i : \mathbb{R}^k \times \mathbb{R}^t \rightarrow \mathbb{R} \) be differentiable functions, \( C_j, D_i \) and \( E_j \) are compact convex sets in \( \mathbb{R}^n \), for all \( i = 1, 2, \ldots, k, j = 1, 2, \ldots, m \).

**Definition 3.1.** [3]. A point \( x^0 \in X^0 \) is said to be an efficient solution (or Pareto optimal) of (MFP), if there exists no \( x \in X^0 \) such that for every

\[
i = 1, 2, \ldots, k, \quad \frac{f_i(x) + S(x | C_i)}{g_i(x) - S(x | D_i)} \leq \frac{f_i(x^0) + S(x^0 | C_i)}{g_i(x^0) - S(x^0 | D_i)}
\]

and for some \( r = 1, 2, \ldots, k \),

\[
\frac{f_r(x) + S(x | C_r)}{g_r(x) - S(x | D_r)} < \frac{f_r(x^0) + S(x^0 | C_r)}{g_r(x^0) - S(x^0 | D_r)}.
\]

We now state theorems 3.1-3.2, whose proof follows on the lines [13].

**Theorem 3.1.** For some \( t \), if \( f_t(\cdot) + (\cdot)^T z_i \) and \( -g_t(\cdot) - (\cdot)^T v_i \) are higher-order \( (V, \alpha_i, \beta_i, \rho_i, \theta_i) \)-invex at \( u \) with respect to \( H_t(u, p) \) for same \( \eta(x,u) \).

Then, the fractional function \( \left( f_t(\cdot) + (\cdot)^T z_i \right) \) is higher-order \( (V, \alpha_i, \beta_i, \rho_i, \theta_i) \)-invex at \( u \) with respect to \( H_t(u, p) \), where

\[
\alpha_i(x,u) = \left( g_i(x) - x^T v_i \over g_i(u) - u^T v_i \right) \alpha_i(x,u), \quad \beta_i(x,u) = \beta_i(x,u),
\]

\[
\bar{\theta}_i(x,u) = \theta_i(x,u) \left( \frac{1}{g_i(u) - u^T v_i} + \frac{f_i(u) + u^T z_i}{(g_i(u) - u^T v_i)^2} \right)^{1/2}, \quad \rho_i(x,u) = \rho_i(x,u)
\]

and

\[
H_t(u, p) = \left( \frac{1}{g_i(u) - u^T v_i} + \frac{f_i(u) + u^T z_i}{(g_i(u) - u^T v_i)^2} \right) H_t(u, p).
\]

**Theorem 3.2.** In Theorem 3.1, if either \( -(g_t(\cdot) - (\cdot)^T v_i) \) is strictly higher-
order \((V,\alpha,\beta,\rho,\theta)\)-invex at \(u\) with respect to \(H_i(u, p)\) and  
\(\{f_\alpha(\cdot) - (\cdot)^T z\} > 0\) or \(\{f_\beta(\cdot) - (\cdot)^T z\}\) is strictly higher-order \((V,\alpha,\beta,\rho,\theta)\)

-invex at \(u\) with respect to \(H_i(u, p)\), then \(\{f_\beta(\cdot) + (\cdot)^T z\}/g_\alpha(\cdot) - (\cdot)^T z\} is strictly higher-order

\((V,\alpha,\beta,\rho,\theta)\)-invex at \(u\) with respect to \(\bar{H}_i(u, p)\).

**Theorem 3.3** (Necessary Condition) [14]. Assume that \(\bar{x}\) is an efficient solution of (MFP) and the Slater’s constraint qualification is satisfied on \(X\). Then there exist \(\bar{\lambda}_i > 0, \bar{\mu}_j \in R^n, \bar{\tau}_i \in R^n, \bar{\nu}_i \in R^n\) and \(\bar{\omega}_j \in R^n, i = 1, 2, \cdots, k, j = 1, 2, \cdots, m\), such that

\[
\sum_{i=1}^{k} \lambda_i \nabla \left( f_i(\bar{x}) + \bar{\tau}_i \bar{\sigma} \right) + \sum_{j=1}^{m} \mu_j \nabla \left( h_j(\bar{x}) + \bar{\omega}_j \right) = 0,  
\]

\[
\sum_{j=1}^{m} \mu_j \left( h_j(\bar{x}) + \bar{\omega}_j \right) = 0,  
\]

\[
\bar{\tau}_i \bar{\sigma} = S(\bar{x} | C_i), \bar{\tau}_i \in C_i, i = 1, 2, \cdots, k,  
\]

\[
\bar{\nu}_i \bar{\nu} = S(\bar{x} | D_i), \bar{\nu}_i \in D_i, i = 1, 2, \cdots, k,  
\]

\[
\bar{\omega}_j = S(\bar{x} | E_j), \bar{\omega}_j \in E_j, j = 1, 2, \cdots, m,  
\]

\[
\bar{\lambda}_i > 0, i = 1, 2, \cdots, k, \bar{\mu}_j \geq 0, j = 1, 2, \cdots, m.  
\]

**Theorem 3.4.** (Sufficient Condition). Let \(u\) be a feasible solution of (MFP). Then, there exist \(\lambda_i > 0, i = 1, 2, \cdots, k\) and \(\mu_j \geq 0, j = 1, 2, \cdots, m\), such that

\[
\sum_{i=1}^{k} \lambda_i \nabla \left( f_i(u) + u^T \bar{z}_i \right) + \sum_{j=1}^{m} \mu_j \nabla \left( h_j(u) + u^T \bar{w}_j \right) = 0,  
\]

\[
\sum_{j=1}^{m} \mu_j \left( h_j(u) + u^T \bar{w}_j \right) = 0,  
\]

\[
u_j = S(u | Z_j), z_j \in Z_j, i = 1, 2, \cdots, k,  
\]

\[
u_j = S(u | D_j), v_j \in D_j, i = 1, 2, \cdots, k,  
\]

\[
u_j = S(u | E_j), w_j \in E_j, j = 1, 2, \cdots, m,  
\]

\[
\lambda_i > 0, i = 1, 2, \cdots, k, \mu_j \geq 0, j = 1, 2, \cdots, m.  
\]

Let, for \(i = 1, 2, \cdots, k, j = 1, 2, \cdots, m\),

1) \(\{f_\alpha(\cdot) + (\cdot)^T \bar{z}\} \) and \(\{-g_\alpha(\cdot) - (\cdot)^T \bar{v}\}\) be higher-order \((V,\alpha,\beta,\rho,\theta)\)-invex at \(u\) with respect to \(H_i(u, p)\),

2) \(\{h_j(\cdot) + (\cdot)^T \bar{w}_j\}\) be higher-order \((V,\alpha,\beta,\rho,\theta)\)-invex at \(u\) with respect to \(G_j(u, p)\),

3) \(\sum_{i=1}^{k} \lambda_i \bar{\rho}_i \left\| \theta_i(x, u) \right\|^2 + \sum_{j=1}^{m} \mu_j \bar{\rho}_j \left\| \theta_j(x, u) \right\|^2 \geq 0,  
\]

4) \(\sum_{i=1}^{k} \lambda_i (\nabla \bar{H}_i(u, p)) + \sum_{j=1}^{m} \mu_j (\nabla \bar{G}_j(u, p)) = 0,  
\]
\[ \sum_{i=1}^{k} \lambda_i (\vec{H}_i(u,p) - p^T \nabla_p \vec{H}_i(u,p)) \geq 0 \quad \text{and} \quad \sum_{j=1}^{\infty} \mu_j (G_j(u,p) - p^T \nabla_p G_j(u,p)) \geq 0, \]

5) \( \alpha^j_i (x,u) = \beta^j_i (v,u) = \beta^+ j (x,u) = \alpha (x,u) \),

where

\[ \overline{\alpha}_i (x,u) = \begin{pmatrix} g_i(x) - x^T v_i \\ g_i(u) - u^T v_i \end{pmatrix} \alpha_i (x,u), \quad \overline{\beta}_i (x,u) = \beta_i (x,u), \]

\[ \frac{1}{(g_i(u) - u^T v_i)^2} \left( \frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i} \right) \]

and \( \overline{\beta}_i (x,u) = \rho_i (x,u) \).

Then, \( u \) is an efficient solution of (MFP).

Proof. Suppose \( u \) is not an efficient solution of (MFP). Then there exists \( x \in X^0 \) such that

\[
\frac{f_i(x) + S(x | C_i)}{g_i(x) - S(x | D_i)} \leq \frac{f_i(u) + S(u | C_i)}{g_i(u) - S(u | D_i)}, \quad \text{for all } i = 1, 2, \ldots, k
\]

and

\[
\frac{f_i(x) + S(x | C_i)}{g_i(x) - S(x | D_i)} < \frac{f_i(u) + S(u | C_i)}{g_i(u) - S(u | D_i)}, \quad \text{for some } i = 1, 2, \ldots, k,
\]

which implies

\[
\frac{f_i(x) + x^T z_i}{g_i(x) - x^T v_i} \leq \frac{f_i(x) + S(x | C_i)}{g_i(x) - S(x | D_i)} \leq \frac{f_i(u) + S(u | C_i)}{g_i(u) - S(u | D_i)} = \frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i}, \quad \text{for all } i = 1, 2, \ldots, k
\]

and

\[
\frac{f_i(x) + x^T z_i}{g_i(x) - x^T v_i} \leq \frac{f_i(x) + S(x | C_i)}{g_i(x) - S(x | D_i)} < \frac{f_i(u) + S(u | C_i)}{g_i(u) - S(u | D_i)} = \frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i}, \quad \text{for some } r = 1, 2, \ldots, k.
\]

Since \( \lambda_i > 0, i = 1, 2, \ldots, k \), inequalities (13) and (14) gives

\[
\sum_{i=1}^{k} \lambda_i \left[ \frac{f_i(x) + x^T z_i}{g_i(x) - x^T v_i} - \frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i} \right] < 0.
\]

From Theorem 3.1, for each \( i, 1 \leq i \leq k \), \( \left[ \frac{f_i(.) + (.)^T z_i}{g_i(.) - (.)^T v_i} \right] \) is higher-order \( \left( V, \overline{\alpha}_i, \overline{\beta}_i, \overline{\beta}_i \right) \)-invex at \( u \in X^0 \) with respect to \( H_i(u,p) \), we have

\[
\overline{\alpha}_i (x,u) \left[ \frac{f_i(x) + x^T z_i}{g_i(x) - x^T v_i} - \frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i} \right]
\]
\[ \geq \eta^T (x, u) \left[ \nabla \left( \frac{f_j(u) + u^T z_j}{g_j(u) - u^T v_j} \right) + \nabla_p \bar{H}_j(u, p) \right] \]
\[ + \bar{p}^T_l (x, u) \left[ \bar{H}_j(u, p) - p^T \nabla_p \bar{H}_j(u, p) \right] + \bar{p}^T \| \theta^j (x, u) \|_2^2. \]

where

\[
\alpha_i(x, u) = \left( \frac{g_i(x) - x^T v_i}{g_i(u) - u^T v_i} \right) \alpha_i(u), \quad \beta_i(x, u) = \beta_i(u),
\]

\[
\alpha_i(x, u) = \left( \frac{1}{g_i(u) - u^T v_i} + \frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i} \right)^{\frac{1}{2}}, \quad \beta_i(x, u) = \beta_i(u),
\]

and \( \bar{H}_j(u, p) = \left( \frac{1}{g_j(u) - u^T v_j} + \frac{f_j(u) + u^T z_j}{g_j(u) - u^T v_j} \right) H_j(u, p) \).

By hypothesis 2), we get

\[
\alpha^2_j(x, u) \left[ h_j(x) + x^T w_j - (h_j(u) + u^T w_j) \right] \\
\geq \eta^T (x, u) \left[ \nabla \left( h_j(u) + u^T w_j \right) + \nabla G_j(u, p) \right] \\
+ \beta^2_j(x, u) \left[ G_j(u, p) - p^T \nabla G_j(u, p) \right] + \rho^2_j \| \theta^j (x, u) \|_2^2.
\]

Adding the two inequalities after multiplying (16) by \( \lambda_i \) and (17) by \( \mu_j \), we obtain

\[
\sum_{i=1}^{k} \lambda_i \alpha_i(x, u) \left[ f_j(x) + x^T z_j - f_j(u) + u^T z_j \right] \\
+ \sum_{j=1}^{m} \mu_j \alpha^2_j(x, u) \left[ h_j(x) + x^T w_j - (h_j(u) + u^T w_j) \right] \\
\geq \eta^T (x, u) \sum_{i=1}^{k} \lambda_i \left[ \nabla \left( f_j(u) + u^T z_j \right) + \nabla_p \bar{H}_j(u, p) \right] \\
+ \eta^T (x, u) \sum_{j=1}^{m} \mu_j \left[ \nabla \left( h_j(u) + u^T w_j \right) + \nabla G_j(u, p) \right] \\
+ \frac{1}{2} \sum_{i=1}^{k} \lambda_i \| \alpha_i(x, u) \|_2^2 + \sum_{j=1}^{m} \mu_j \| \alpha^2_j(x, u) \|_2^2.
\]

Using hypothesis 3)-4), we get

\[
\sum_{i=1}^{k} \lambda_i \left[ f_j(x) + x^T z_j - f_j(u) + u^T z_j \right] + \sum_{j=1}^{m} \mu_j \left[ h_j(x) + x^T w_j - (h_j(u) + u^T w_j) \right] \\
\geq \eta^T (x, u) \sum_{i=1}^{k} \lambda_i \nabla \left( f_j(u) + u^T z_j \right) + \eta^T (x, u) \sum_{j=1}^{m} \mu_j \nabla \left( h_j(u) + u^T w_j \right).
\]
Further, using (7)-(8), therefore
\[
\sum_{i=1}^{k} \lambda_i \left( f_i(x) + x^T z_i - \frac{f_i(u)}{g_i(x)} - u^T v_i \right) + \sum_{j=1}^{m} \mu_j \left[ h_j(x) + x^T w_j \right] \geq 0. \tag{20}
\]
Since \( x \) is feasible solution for (MFP), it follows that
\[
\sum_{i=1}^{k} \lambda_i \left( f_i(x) + x^T z_i \right) \geq \sum_{j=1}^{m} \lambda_j \left( f_j(u) + u^T z_j \right).
\]
This contradicts (15). Therefore, \( u \) is an efficient solution of (MFP).

4. Duality Model-I

Consider the following dual (MFD), of (MFP): (MFD), Maximize
\[
\left[ \frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i} + \sum_{j=1}^{m} \mu_j \left( h_j(u) + u^T w_j \right) + \left( \bar{H}_i(u, p) - p^T \nabla_p \bar{H}_i(u, p) \right) \right]
\]
subject to
\[
\sum_{i=1}^{k} \lambda_i \nabla \left( \frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i} \right) + \sum_{j=1}^{m} \mu_j \nabla \left( h_j(u) + u^T w_j \right) \\
+ \sum_{i=1}^{k} \lambda_i \nabla_p \bar{H}_i(u, p) + \sum_{j=1}^{m} \mu_j \nabla_p G_j(u, p) = 0,
\]
\( z_i \in C_i, v_j \in D_j, w_j \in E_j, i = 1, 2, \ldots, k, j = 1, 2, \ldots, m, \)
\( \mu_j \geq 0, \lambda_i > 0, \sum_{i=1}^{k} \lambda_i = 1, i = 1, 2, \ldots, k, j = 1, 2, \ldots, m. \)

Let \( Z^0 \) be feasible solution for (MFD).

**Theorem 4.1.** (Weak duality theorem). Let \( x \in X^0 \) and \( (u, v, w, p) \in Z^0 \). Suppose that
1) for any \( i = 1, 2, \ldots, k, \) \( \left( f_i(\cdot) + (\cdot)^T z_i \right) \) and \(-\left( g_i(\cdot) - (\cdot)^T v_i \right)\) are higher-order \( (V_1, \alpha_i^1, \beta_i^1, \rho_i^1, \theta_i^1) \)-invex at \( u \) with respect to \( H_i(u, p), \)
2) for any \( j = 1, 2, \ldots, m, \) \( \left( h_j(\cdot) + (\cdot)^T w_j \right) \) is higher-order \( (V, \alpha_j^2, \beta_j^2, \rho_j^2, \theta_j^2) \)-invex at \( u \) with respect to \( G_j(u, p), \)
3) \( \sum_{i=1}^{k} \lambda_i \| \nabla f_i(x, u) \| + \sum_{j=1}^{m} \mu_j \| \nabla G_j(x, u) \| \geq 0. \)
4) \( \overline{\alpha}_i^1 (x, u) = \alpha_i^2 (x, u) = \beta_i^1 (x, u) = \beta_i^2 (x, u) = \alpha(x, u), \forall i = 1, 2, \ldots, k, j = 1, 2, \ldots, m, \)
where \( \overline{\alpha}_i^1 (x, u) = \left( g_i(x) - x^T v_i \right) \alpha_i (x, u), \overline{\beta}_j^1 (x, u) = \beta_j (x, u), \)
\[ \bar{\partial}_i (x,u) = \partial_i (x,u) \left( \frac{1}{g_i(u) - u^\top v_i} + \frac{f_i(u) + u^\top z_i}{(g_i(u) - u^\top v_i)^2} \right)^{\frac{1}{2}} \]

and

\[ \bar{H}_i(u,p) = \left( \frac{1}{g_i(u) - u^\top v_i} + \frac{f_i(u) + u^\top z_i}{(g_i(u) - u^\top v_i)^2} \right) H_i(u,p). \]

Then, the following cannot hold

\[ f_i(x) + S(x \mid C_i) \]
\[ g_i(x) - S(x \mid D_i) \]
\[ \leq \frac{f_i(u) + u^\top z_j}{g_i(u) - u^\top v_j} + \sum_{j=1}^m \mu_j \left( h_j(u) + u^\top w_j \right) \left( \bar{H}_j(u,p) - p^\top \nabla_p \bar{H}_j(u,p) \right) \]
\[ + \sum_{j=1}^m \mu_j \left( \frac{G_j(u,p) - p^\top \nabla_p G_j(u,p)}{\bar{H}_j(u,p)} \right), \quad \text{for all } i = 1, 2, \ldots, k \]

and

\[ f_i(x) + S(x \mid C_i) \]
\[ g_i(x) - S(x \mid D_i) \]
\[ < \frac{f_i(u) + u^\top z_j}{g_i(u) - u^\top v_j} + \sum_{j=1}^m \mu_j \left( h_j(u) + u^\top w_j \right) \left( \bar{H}_j(u,p) - p^\top \nabla_p \bar{H}_j(u,p) \right) \]
\[ + \sum_{j=1}^m \mu_j \left( \frac{G_j(u,p) - p^\top \nabla_p G_j(u,p)}{\bar{H}_j(u,p)} \right), \quad \text{for some } r = 1, 2, \ldots, k. \]

Proof: Suppose that (22) and (23) hold, then using \( \lambda_i > 0, \sum_{i=1}^k \lambda_i = 1, \) \( x^\top z_i \leq S(x \mid C_i), \) \( x^\top v_i \leq S(x \mid D_i), \) \( i = 1, 2, \ldots, k, \) we have

\[ \sum_{i=1}^k \lambda_i \left( f_i(x) + x^\top z_i \right) < \sum_{i=1}^k \lambda_i \left( \frac{f_i(u) + u^\top z_j}{g_i(u) - u^\top v_j} + \sum_{j=1}^m \mu_j \left( h_j(u) + u^\top w_j \right) \left( \bar{H}_j(u,p) - p^\top \nabla_p \bar{H}_j(u,p) \right) \right. \]
\[ + \sum_{j=1}^m \mu_j \left( \frac{G_j(u,p) - p^\top \nabla_p G_j(u,p)}{\bar{H}_j(u,p)} \right), \quad \text{for all } i = 1, 2, \ldots, k. \]

From hypothesis 1) and Theorem 3.1, for \( i = 1, 2, \ldots, k, \) \( \left( f_i(.) + (.)^\top z_i \right) \) is higher-order \( \left( V, \alpha_i, \beta_i, \rho_i, \theta_i \right) \)-invex at \( u \) with respect to \( \bar{H}_i(u,p), \) we get

\[ \alpha_i'(x,u) \left[ f_i(x) + x^\top z_i \right. \frac{f_i(u) + u^\top z_j}{g_i(u) - x^\top v_i} \left. \frac{f_i(u) + u^\top z_j}{g_i(u) - u^\top v_j} \right] \]
\[ \geq \eta_i'(x,u) \left[ \nabla \left( \frac{f_i(u) + u^\top z_j}{g_i(u) - u^\top v_j} \right) + \nabla_p \bar{H}_i(u,p) \right] \]
\[ + \beta_i'(x,u) \left[ \bar{H}_i(u,p) - p^\top \nabla_p \bar{H}_i(u,p) \right] + \bar{\beta}_i'(x,u) \]
\[ \leq \right|^2. \]

For any \( j = 1, 2, \ldots, m, \) \( \left( h_j(.) + (.)^\top w_j \right) \) is higher-order \( \left( V, \alpha_j^\top, \beta_j^\top, \rho_j^\top, \theta_j^\top \right) \)-invex at \( u \) with respect to \( G_j(u,p) \), we have
\[ \alpha_j^i (x,u) \left[ h_j(x) + x^T w_j - (h_j(u) + u^T w_j) \right] \]
\[ \geq \eta^T (x,u) \left[ \nabla (h_j(u) + u^T w_j) + \nabla J_j(u,p) \right] \]
\[ + \beta_j^i (x,u) \left[ G_j(u,p) - p^T \nabla_p G_j(u,p) \right] + \rho_j^i \| \theta_j^i (x,u) \|^2. \]  
\[ (26) \]

Adding the two inequalities after multiplying (25) by \( \lambda_i \) and (26) by \( \mu_j \), we obtain
\[ \sum_{i=1}^{k} \lambda_i \xi_i^i (x,u) \left[ f_i(x) + x^T z_i - f_i(u) + u^T z_i \right] \]
\[ + \sum_{j=1}^{m} \mu_j \alpha_j^j (x,u) \left[ h_j(x) + x^T w_j - (h_j(u) + u^T w_j) \right] \]
\[ \geq \eta^T (x,u) \sum_{i=1}^{k} \lambda_i \left[ \nabla \left( f_i(u) + u^T z_i \right) - \nabla \left( G_j(u,p) - p^T \nabla_p G_j(u,p) \right) \right] \]
\[ + \sum_{j=1}^{m} \mu_j \beta_j^j (x,u) \left[ h_j(x) + x^T w_j - (h_j(u) + u^T w_j) \right] \]
\[ + \eta^T (x,u) \sum_{j=1}^{m} \mu_j \left[ \nabla (h_j(u) + u^T w_j) + \nabla J_j(u,p) \right] \]
\[ + \sum_{i=1}^{k} \lambda_i \xi_i^i (x,u) \left[ H_i(u,p) - p^T \nabla_p H_i(u,p) \right] \]
\[ + \sum_{j=1}^{m} \mu_j \beta_j^j (x,u) \left[ G_j(u,p) - p^T \nabla_p G_j(u,p) \right] \]
\[ + \sum_{j=1}^{m} \lambda_j \| \theta_j^j (x,u) \|^2 + \sum_{j=1}^{m} \mu_j \rho_j^j \| \theta_j^j (x,u) \|^2. \]  
\[ (27) \]

Using hypothesis 3) and (21), we get
\[ \sum_{i=1}^{k} \lambda_i \xi_i^i (x,u) \left[ f_i(x) + x^T z_i - f_i(u) + u^T z_i \right] \]
\[ + \sum_{j=1}^{m} \mu_j \alpha_j^j (x,u) \left[ h_j(x) + x^T w_j - (h_j(u) + u^T w_j) \right] \]
\[ \geq \sum_{i=1}^{k} \lambda_i \xi_i^i (x,u) \left[ \nabla \left( f_i(u) + u^T z_i \right) - \nabla \left( H_i(u,p) + p^T \nabla_p H_i(u,p) \right) \right] \]
\[ + \sum_{j=1}^{m} \mu_j \beta_j^j (x,u) \left[ h_j(x) + x^T w_j - (h_j(u) + u^T w_j) \right] \]
\[ \geq \sum_{i=1}^{k} \lambda_i \xi_i^i (x,u) \left[ \nabla \left( f_i(u) + u^T z_i \right) - \nabla \left( H_i(u,p) + p^T \nabla_p H_i(u,p) \right) \right] \]
\[ + \sum_{j=1}^{m} \mu_j \beta_j^j (x,u) \left[ G_j(u,p) - p^T \nabla_p G_j(u,p) \right]. \]  
\[ (28) \]

Finally, using hypothesis 4) and \( x \) is feasible solution for (MFP), it follows that
\[ \sum_{i=1}^{k} \lambda_i \left( f_i(x) + x^T z_i \right) \geq \sum_{i=1}^{k} \lambda_i \left( f_i(u) + u^T z_i \right) + \sum_{j=1}^{m} \mu_j \left( h_j(u) + u^T w_j \right) \]
\[ + \sum_{i=1}^{k} \lambda_i \left( H_i(u,p) - p^T \nabla_p H_i(u,p) \right) \]
\[ + \sum_{j=1}^{m} \mu_j \left( G_j(u,p) - p^T \nabla_p G_j(u,p) \right). \]

This contradicts Equation (24). Hence, the result.

**Theorem 4.2.** (Strong duality theorem). If \( \bar{x} \in X^0 \) is an efficient solution of (MFP) and the Slater’s constraint qualification holds. Also, if for any \( i = 1, 2, \cdots, k, j = 1, 2, \cdots, m \),
\[ H_j(\bar{u}, 0) = 0, G_j(\bar{u}, 0) = 0, \nabla_{u} H_j(\bar{u}, 0) = 0, \nabla_{\bar{u}} G_j(\bar{u}, 0) = 0, \tag{29} \]

then there exist \( \lambda \in R^k, \bar{\mu} \in R^n, \bar{\nu} \in R^m, \bar{\varphi} \in R^r \), and \( \bar{w}_j \in R^n, i = 1, 2, \ldots, k, j = 1, 2, \ldots, m \), such that \( (u, \bar{\varphi}, \bar{\nu}, \bar{\varphi}, \bar{\nu}, \bar{\varphi}, \bar{\nu} = 0) \) is a feasible solution of (MFD)\(_i\) and the objective function values of (MFP) and (MFD)\(_i\) are equal. Furthermore, if the hypotheses of Theorem 4.1 hold for all feasible solutions of (MFP) and (MFD)\(_i\), then \( (u, \bar{\varphi}, \bar{\nu}, \bar{\varphi}, \bar{\nu}, \bar{\varphi}, \bar{\nu} = 0) \) is an efficient solution of (MFD)\(_i\).

**Proof.** Since \( \bar{u} \) is an efficient solution of (MFP) and the Slater’s constraint qualification holds, then by Theorem 3.3, there exist \( \lambda \in R^k, \bar{\mu} \in R^n, \bar{\nu} \in R^m, \bar{\varphi} \in R^r \), and \( \bar{w}_j \in R^n, i = 1, 2, \ldots, k, j = 1, 2, \ldots, m \), such that

\[
\sum_{j=1}^{m} \lambda_i \left( \frac{f_i(\bar{u}) + \bar{\varphi}_i}{g_i(\bar{u})} - \bar{u}^T \bar{w}_j \right) + \sum_{j=1}^{m} \bar{\mu}_j \left( h_j(\bar{u}) + \bar{\nu}_j \bar{w}_j \right) = 0, \tag{30} \]

\[
\sum_{j=1}^{m} \bar{\mu}_j \left( h_j(\bar{u}) + \bar{\nu}_j \bar{w}_j \right) = 0, \tag{31} \]

\[
\bar{u}^T \bar{\varphi}_i = S(\bar{u} | C_i), \bar{\varphi}_i \bar{\varphi}_i = S(\bar{u} | D_i), \bar{w}_j = S(\bar{u} | E_j), \tag{32} \]

\[
\bar{\varphi}_i \in C_i, \bar{\nu}_j \in D_i, \bar{w}_j \in E_j, \tag{33} \]

\[
\lambda_i > 0, \quad \sum_{i=1}^{k} \lambda_i = 1, \quad \bar{\mu}_j \geq 0, \quad i = 1, 2, \ldots, k, \quad j = 1, 2, \ldots, m. \tag{34} \]

Thus, \( (u, \bar{\varphi}, \bar{\nu}, \bar{\varphi}, \bar{\nu}, \bar{\varphi}, \bar{\nu} = 0) \) is feasible for (MFD)\(_i\) and the objective function values of (MFP) and (MFD)\(_i\) are equal.

We now show that \( (u, \bar{\varphi}, \bar{\nu}, \bar{\varphi}, \bar{\nu}, \bar{\varphi}, \bar{\nu} = 0) \) is an efficient solution of (MFD)\(_i\). If not, then there exists \( (u', \bar{\varphi}', \bar{\nu}', \bar{\varphi}', \bar{\nu}', \bar{\varphi}', \bar{\nu}' = 0) \) of (MFD)\(_i\) such that

\[
\frac{f_i(\bar{u}) + \bar{\varphi}_i}{g_i(\bar{u})} - \bar{u}^T \bar{w}_j \leq \frac{f_i(u') + u'^T \bar{\varphi}_i}{g_i(u')} - u'^T \bar{w}_j, \quad \text{for all } i = 1, 2, \ldots, k \]

and

\[
\frac{f_r(\bar{u}) + \bar{\varphi}_i}{g_r(\bar{u})} - \bar{u}^T \bar{w}_j \leq \frac{f_r(u') + u'^T \bar{\varphi}_i}{g_r(u')} - u'^T \bar{w}_j, \quad \text{for some } r = 1, 2, \ldots, k. \]

By equation (31), we obtain

\[
\frac{f_i(\bar{u}) + \bar{\varphi}_i}{g_i(\bar{u})} - \bar{u}^T \bar{w}_j \leq \frac{f_i(u') + u'^T \bar{\varphi}_i}{g_i(u')} - u'^T \bar{w}_j, \quad \text{for all } i = 1, 2, \ldots, k \]

and

\[
\frac{f_r(\bar{u}) + \bar{\varphi}_i}{g_r(\bar{u})} - \bar{u}^T \bar{w}_j \leq \frac{f_r(u') + u'^T \bar{\varphi}_i}{g_r(u')} - u'^T \bar{w}_j, \quad \text{for some } r = 1, 2, \ldots, k. \]
This contradicts Theorem 4.1. This completes the result.

**Theorem 4.3.** (Strict converse duality theorem). Let \( \overline{X} \in X^0 \) and \( (\overline{u}, \overline{v}, \overline{p}, \overline{w}, \overline{p}) \in Z^0 \). Let

\[
\sum_{i=1}^k \left( f_i(\overline{x}) + \overline{x}^T \overline{v} \right) \leq \sum_{i=1}^k \left( f_i(\overline{u}) + \overline{u}^T \overline{v} \right) + \sum_{j=1}^m \overline{p}_j \left( h_j(\overline{u}) + \overline{u}^T \overline{w}_j \right)
\]

1) \( + \sum_{i=1}^k \left( H_i(\overline{u}, \overline{p}) - \overline{p}^T \nabla_p H_i(\overline{u}, \overline{p}) \right) \)

\( + \sum_{j=1}^m \left( G_j(\overline{u}, \overline{p}) - \overline{p}^T \nabla_p G_j(\overline{u}, \overline{p}) \right) \),

2) for any \( i = 1, 2, \cdots, k \), \( f_i(\cdot) + (\cdot)^T \overline{v} \) be strictly higher-order \((V, \alpha_i^j, \beta_i^j, \rho_i^j, \theta_i^j)\)-invex at \( \overline{u} \) with respect to \( H_i(\overline{u}, \overline{p}) \) and \( -g_i(\cdot) + (\cdot)^T \overline{v} \) be higher-order \((V, \alpha_i^j, \beta_i^j, \rho_i^j, \theta_i^j)\)-invex at \( \overline{u} \) with respect to \( H_i(\overline{u}, \overline{p}) \),

3) for any \( j = 1, 2, \cdots, m \), \( h_j(\cdot) + (\cdot)^T \overline{w}_j \) be higher-order \((V, \alpha_j^j, \beta_j^j, \rho_j^j, \theta_j^j)\)-invex at \( \overline{u} \) with respect to \( G_j(\overline{u}, \overline{p}) \),

4) \( \sum_{i=1}^k \sum_{j=1}^m \alpha_i^j(\overline{x}, \overline{u}) \| H_i(\overline{u}, \overline{p}) \| + \sum_{j=1}^m \sum_{i=1}^k \beta_j^j(\overline{x}, \overline{u}) \| G_j(\overline{u}, \overline{p}) \| \geq 0. \)

5) \( \alpha_j^j(\overline{x}, \overline{u}) = \alpha_j^j(\overline{x}, \overline{u}) = \beta_j^j(\overline{x}, \overline{u}) = \beta_j^j(\overline{x}, \overline{u}) = \alpha(\overline{x}, \overline{u}), \forall i = 1, 2, \cdots, k, \)

\( j = 1, 2, \cdots, m. \)

Then, \( \overline{x} = \overline{u} \).

**Proof.** Using hypothesis 2) and Theorem 3.2, we have

\[
\alpha_i^j(\overline{x}, \overline{u}) \left[ \frac{f_i(\overline{x}) + \overline{x}^T \overline{v}}{g_i(\overline{x}) - \overline{x}^T \overline{v}} - \frac{f_i(\overline{u}) + \overline{u}^T \overline{v}}{g_i(\overline{u}) - \overline{u}^T \overline{v}} \right] > \eta i(\overline{x}, \overline{u}) \left[ \nabla \left( \frac{f_i(\overline{u}) + \overline{u}^T \overline{v}}{g_i(\overline{u}) - \overline{u}^T \overline{v}} \right) + \nabla_p H_i(\overline{u}, \overline{p}) \right] + \overline{p}_i \| H_i(\overline{u}, \overline{p}) \| \tag{35}
\]

For any \( j = 1, 2, \cdots, m \), \( h_j(\cdot) + (\cdot)^T \overline{w}_j \) is higher-order \((V, \alpha_j^j, \beta_j^j, \rho_j^j, \theta_j^j)\)-invex at \( u \) with respect to \( G_j(\overline{u}, \overline{p}) \), we have

\[
\alpha_j^j(\overline{x}, \overline{u}) \left[ h_j(\overline{x}) + \overline{x}^T \overline{w}_j - h_j(\overline{u}) + \overline{u}^T \overline{w}_j \right] \geq \eta i(\overline{x}, \overline{u}) \left[ \nabla \left( h_j(\overline{u}) + \overline{u}^T \overline{w}_j \right) + \nabla_p G_j(\overline{u}, \overline{p}) \right] + \beta_j^j(\overline{x}, \overline{u}) \| G_j(\overline{u}, \overline{p}) \| \tag{36}
\]

Adding the two inequalities after multiplying (35) by \( \lambda_i \) and (36) by \( \mu_j \), we obtain

\[
\sum_{i=1}^k \sum_{j=1}^m \alpha_i^j(\overline{x}, \overline{u}) \left[ f_i(\overline{x}) + \overline{x}^T \overline{v} - f_i(\overline{u}) + \overline{u}^T \overline{v} \right] + \sum_{i=1}^k \sum_{j=1}^m \beta_j^j(\overline{x}, \overline{u}) \left[ h_j(\overline{x}) + \overline{x}^T \overline{w}_j - h_j(\overline{u}) + \overline{u}^T \overline{w}_j \right] > \eta \| \overline{p} \| \sum_{i=1}^k \sum_{j=1}^m \alpha_i^j(\overline{x}, \overline{u}) \left[ \nabla \left( \frac{f_i(\overline{u}) + \overline{u}^T \overline{v}}{g_i(\overline{u}) - \overline{u}^T \overline{v}} \right) - \nabla_p H_i(\overline{u}, \overline{p}) \right]
\]
Using hypothesis 3) and (21), we get
\begin{align}
\sum_{i=1}^{k} l_i \xi_i (\bar{x}, \bar{u}) + \eta^T (\bar{x}, \bar{u}) \sum_{j=1}^{m} \mu_j \left[ \nabla \left( h_j (\bar{u}) + u^T \bar{w}_j \right) + \nabla \mu_j \left( \bar{u}, \bar{p} \right) \right] \\
+ \sum_{i=1}^{k} \lambda_i \beta^T_i (\bar{x}, \bar{u}) \left[ \nabla \left( \bar{H}_j (\bar{u}, \bar{p}) - \bar{p}^T \nabla \mu_j (\bar{u}, \bar{p}) \right) \right] \\
\sum_{j=1}^{m} \mu_j \beta_j (\bar{x}, \bar{u}) \left[ G_j (\bar{u}, \bar{p}) - \bar{p}^T \nabla \mu_j (\bar{u}, \bar{p}) \right] \\
\sum_{i=1}^{k} \lambda_i \beta^T_i (\bar{x}, \bar{u}) \left[ \nabla \left( \bar{H}_j (\bar{u}, \bar{p}) - \bar{p}^T \nabla \mu_j (\bar{u}, \bar{p}) \right) \right] \\
+ \sum_{i=1}^{k} \lambda_i \beta^T_i (\bar{x}, \bar{u}) \left[ G_j (\bar{u}, \bar{p}) - \bar{p}^T \nabla \mu_j (\bar{u}, \bar{p}) \right].
\end{align}

(37)

Finally, using hypothesis 4) and \( \bar{x} \) is feasible solution for (MFP), it follows that
\begin{align}
\sum_{i=1}^{k} l_i \xi_i (\bar{x}, \bar{u}) + \eta^T (\bar{x}, \bar{u}) \sum_{j=1}^{m} \mu_j \left[ \nabla \left( h_j (\bar{u}) + u^T \bar{w}_j \right) \right] \\
+ \sum_{i=1}^{k} \lambda_i \beta^T_i (\bar{x}, \bar{u}) \left[ \nabla \left( \bar{H}_j (\bar{u}, \bar{p}) - \bar{p}^T \nabla \mu_j (\bar{u}, \bar{p}) \right) \right] \\
\sum_{j=1}^{m} \mu_j \beta_j (\bar{x}, \bar{u}) \left[ G_j (\bar{u}, \bar{p}) - \bar{p}^T \nabla \mu_j (\bar{u}, \bar{p}) \right] \\
\sum_{i=1}^{k} \lambda_i \beta^T_i (\bar{x}, \bar{u}) \left[ \nabla \left( \bar{H}_j (\bar{u}, \bar{p}) - \bar{p}^T \nabla \mu_j (\bar{u}, \bar{p}) \right) \right] \\
+ \sum_{i=1}^{k} \lambda_i \beta^T_i (\bar{x}, \bar{u}) \left[ G_j (\bar{u}, \bar{p}) - \bar{p}^T \nabla \mu_j (\bar{u}, \bar{p}) \right].
\end{align}

(38)

This contradicts the hypothesis 1). Hence, the result.

5. Duality Model-II

Consider the following dual (MFD), of (MFP): (MFD), Maximize
\[
\begin{bmatrix}
 f_1(u) + u^T z_1 \\
 g_1(u) - u^T v_1 \\
 \vdots \\
 f_m(u) + u^T z_m \\
 g_m(u) - u^T v_m
\end{bmatrix}
+ \sum_{j=1}^{m} \mu_j \left( h_j(u) + u^T w_j \right) \]
subject to
\[
\sum_{i=1}^{k} \lambda_i \nabla \left( f_i(u) + u^T z_i \right) + \sum_{j=1}^{m} \mu_j \nabla \left( h_j(u) + u^T w_j \right) \\
+ \sum_{i=1}^{k} \lambda_i \nabla \mu_j (H_i(u, p) + \sum_{j=1}^{m} \mu_j \nabla \mu_j (G_j(u, p)) = 0,
\]
\[
\sum_{i=1}^{k} \lambda_i \left( H_i(u, p) - p^T \nabla \mu_j (H_i(u, p)) \right) + \sum_{j=1}^{m} \mu_j \left( G_j(u, p) - p^T \nabla \mu_j (G_j(u, p)) \right) \geq 0,
\]
\[
z_i \in C_i, v_i \in D_i, w_j \in E_j, i = 1, 2, \ldots, k, j = 1, 2, \ldots, m,
\]

(41)
\[ \mu_j \geq 0, \lambda_j > 0, \sum_{j=1}^{k} \lambda_j = 1, \quad i = 1, 2, \ldots, k, \quad j = 1, 2, \ldots, m. \]  

(42)

Let \( P^0 \) be the feasible solution for \((MFD)_2\).

**Theorem 5.1.** (Weak duality theorem). Let \( x \in X^0 \) and \((u, z, v, y, \lambda, w, p) \in P^0 \). Let for \( i = 1, 2, \ldots, k, \quad j = 1, 2, \ldots, m \),

1. \[ \frac{f_i(.)+^t z_i}{g_i(.)-^t v_i} \] be higher-order \((V, \alpha_i', \beta_i', \rho_i', \theta_i')\)-invex at \( u \) with respect to \( H_i(u, p) \),

2. \( h_j(.)+^t w_j \) be higher-order \((V, \alpha_j^2, \beta_j^2, \rho_j^2, \theta_j^2)\)-invex at \( u \) with respect to \( G_j(u, p) \),

3. \[ \sum_{i=1}^{k} \lambda_i \rho_i \left[ \theta_i^j(x, u) \right]^2 + \sum_{j=1}^{m} \mu_j \rho_j^2 \left[ \theta_j^j(x, u) \right]^2 \geq 0. \]

4. \( \alpha_i'(x, u) = \alpha_j^2(x, u) = \beta(x, u) = \beta_j^2(x, u) = \alpha(x, u) \).

Then the following cannot hold

\[ \frac{f_i(x)+S(x|C_i)}{g_i(x)-S(x|D_i)} \leq \frac{f_i(u)+^t z_i}{g_i(u)-^t v_i} + \sum_{j=1}^{m} \mu_j \left( h_j(u)+^t w_j \right), \quad \forall i = 1, 2, \ldots, k. \]  

(43)

and

\[ \frac{f_i(x)+S(x|C_i)}{g_i(x)-S(x|D_i)} < \frac{f_i(u)+^t z_i}{g_i(u)-^t v_i} + \sum_{j=1}^{m} \mu_j \left( h_j(u)+^t w_j \right), \text{ for some } r = 1, 2, \ldots, k. \]  

(44)

**Proof.** The proof follows on the lines of Theorem 4.1.

**Theorem 5.2.** (Strong duality theorem). If \( \overline{u} \in X^0 \) is an efficient solution of \((MFP)\) and the Slater’s constraint qualification hold. Also, if for any \( i = 1, 2, \ldots, k, \quad j = 1, 2, \ldots, m \),

\[ H_i(\overline{u}, 0) = 0, \quad G_j(\overline{u}, 0) = 0, \quad \nabla_p H_i(\overline{u}, 0) = 0, \quad \nabla_p G_j(\overline{u}, 0) = 0, \]  

(45)

then there exist \( \overline{x} \in R^k, \overline{\mu} \in R^m, \overline{z} \in R^n, \overline{v} \in R^n \) and \( \overline{w}_j \in R^n, i = 1, 2, \ldots, k, \quad j = 1, 2, \ldots, m \), such that \((u, \overline{z}, \overline{v}, \overline{\mu}, \overline{x}, \overline{w}, \overline{p} = 0)\) is a feasible solution of \((MFD)_1\) and the objective function values of \((MFP)\) and \((MFD)_2\) are equal. Furthermore, if the conditions of Theorem 5.1 hold for all feasible solutions of \((MFP)\) and \((MFD)_2\) then, \((u, \overline{z}, \overline{v}, \overline{\mu}, \overline{x}, \overline{w}, \overline{p} = 0)\) is an efficient solution of \((MFD)_2\).

**Proof.** The proof follows on the lines of Theorem 4.2.

**Theorem 5.3.** (Strict converse duality theorem). Let \( \overline{x} \in X^0 \) and \((\overline{u}, \overline{z}, \overline{v}, \overline{\mu}, \overline{x}, \overline{w}, \overline{p}) \in P^0 \). Let \( i = 1, 2, \ldots, k, \quad j = 1, 2, \ldots, m \),

1. \[ \sum_{i=1}^{k} \overline{\mu}_i \left( f_i(\overline{x})+^t \overline{x}^t \overline{z}_i \right) \leq \sum_{i=1}^{k} \overline{\mu}_i \left( f_i(\overline{u})+^t \overline{u}^t \overline{z}_i \right) + \sum_{j=1}^{m} \overline{\mu}_j \left( h_j(\overline{u})+^t \overline{u}^t \overline{w}_j \right), \]

2. \[ \frac{f_i(.)+^t \overline{z}_i}{g_i(.)-^t \overline{v}_i} \] be strictly higher-order \((V, \alpha_i', \beta_i', \rho_i', \theta_i')\)-invex at \( \overline{u} \) with respect to \( H_i(\overline{u}, \overline{p}) \),
3) \( h_i(.) + (.)^T w_j \) be higher-order \( (V, \alpha^i, \beta^i, \rho^i, \theta^i) \) -invex at \( \bar{u} \) with respect to \( G_j(\bar{p}, \bar{p}) \),

4) \( \sum_{i=1}^k \lambda_i \rho^i \| \theta^i \| (x^i, y) \|^2 + \sum_{j=1}^m \mu_j \rho^j \| \theta^j \| (x, y) \|^2 \geq 0. \)

5) \( \alpha^i_j (x, y) = \alpha^i_j (x, y) = \beta^i_j (x, y) = \beta^i_j (x, y) = \alpha (x, y) \).

Then, \( x = \bar{u} \).

Proof: The proof follows on the lines of Theorem 4.3.

6. Duality Model-III

Consider the following dual (MFD)_3 of (MFP): (MFD)_3

\[
\begin{aligned}
\text{Maximize} & \quad \left[ f_i(u) + u^T z_i + (\bar{H}_i(u, p) - p^T \nabla \bar{H}_i(u, p)) \right] \\
\text{subject to} & \quad \sum_{i=1}^k \lambda_i \nabla \left( f_i(u) + u^T z_i \right) + \sum_{j=1}^m \mu_j \nabla \left( h_j(u) + u^T w_j \right) \\
& \quad + \sum_{i=1}^k \lambda_i \nabla \bar{H}_i(u, p) + \sum_{j=1}^m \mu_j \nabla G_j(u, p) = 0,
\end{aligned}
\]

(46)

(47)

(48)

(49)

Let \( S^0 \) be feasible solution of (MFD)_3.

Theorem 6.1. (Weak duality theorem). Let \( x \in X^0 \) and \( (u, z, v, \mu, \lambda, w, p) \in S^0 \). Let \( i = 1, \ldots, k, j = 1, 2, \ldots, m \),

1) \( f_i(.) + (.)^T z_i \) and \(-g_i(.) - (.)^T v_i\) be higher-order \( (V, \alpha^i, \beta^i, \rho^i, \theta^i) \) -invex at \( u \) with respect to \( H_i(u, p) \),

2) \( h_j(.) + (.)^T w_j \) be higher-order \( (V, \alpha^j, \beta^j, \rho^j, \theta^j) \) -invex at \( u \) with respect to \( G_j(u, p) \),

3) \( \sum_{i=1}^k \lambda_i \rho^i \| \theta^i \| (x, y) \|^2 + \sum_{j=1}^m \mu_j \rho^j \| \theta^j \| (x, y) \|^2 \geq 0. \)

4) \( \bar{\alpha}_j (x, y) = \alpha^j (x, y) = \beta^j (x, y) = \beta^j (x, y) = \alpha (x, y) \),

where

\[
\bar{\alpha}_j (x, y) = \left( \frac{g_i(x) - x^T v_i}{g_i(u) - u^T v_i} \right) \alpha_i(u) = \bar{\beta}_j (x, y) = \beta^j (x, y) = \beta^j (x, y) = \alpha (x, y),
\]

\[
\bar{\theta}_j (x, y) = \theta_j (x, y) \left( \frac{1}{g_i(u) - u^T v_i} + \frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i} \right)^{1/2}, \quad \bar{\rho}_j (x, y) = \rho_j (x, y)
\]
and  
\[
H_i(u, p) = \left( \frac{1}{g_i(u) - u^T v_i} + \frac{f_i(u) + u^T z_i}{(g_i(u) - u^T v_i)^2} \right) H_i(u, p).
\]

Then, the following cannot hold  
\[
f_i(x) + S(x | C_i) \\
g_i(x) - S(x | D_i)
\leq \frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i} + (\overline{H}_i(u, p) - p^T \nabla_p \overline{H}_i(u, p)), \text{ for all } i = 1, 2, \cdots, k
\]

and  
\[
\frac{f_i(x) + S(x | C_i)}{g_i(x) - S(x | D_i)} < \frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i} + (\overline{H}_i(u, p) - p^T \nabla_p \overline{H}_i(u, p)), \text{ for some } r = 1, 2, \cdots, k.
\]

**Proof:** The proof follows on the lines of Theorem 4.1.

**Theorem 6.2.** (Strong duality theorem). If \( \overline{u} \in X^0 \) is an efficient solution of (MFP) and let the Slater’s constraint qualification be satisfied. Also, if for any  
\( i = 1, 2, \cdots, k, j = 1, 2, \cdots, m, \)
\( \overline{H}_i(\overline{u}, 0) = 0, G_j(\overline{u}, 0) = 0, \nabla_{p_i} \overline{H}_i(\overline{u}, 0) = 0, \nabla_{p_j} G_j(\overline{u}, 0) = 0, \)  
then there exist  \( \lambda \in R^k, \mu \in R^n, \overline{\alpha}_i \in R^n, \overline{\nu}_i \in R^n \)  
and  \( \overline{w}_j \in R^n, i = 1, 2, \cdots, k, j = 1, 2, \cdots, m, \)  
such that  \( (u, \overline{\alpha}, \overline{\nu}, \overline{\lambda}, \overline{\mu}, \overline{w}, \overline{p}) = 0 \)  
is a feasible solution of (MFD)\(_3\) and the objective function values of (MFP) and (MFD)\(_3\) are equal. Furthermore, if the conditions of Theorem 6.1 hold for all feasible solutions of (MFP) and (MFD)\(_3\) then,  
\( (u, \overline{\alpha}, \overline{\nu}, \overline{\lambda}, \overline{\mu}, \overline{w}, \overline{p}) = 0 \)  
is an efficient solution of (MFD)\(_3\).

**Proof:** The proof follows on the lines of Theorem 4.2.

**Theorem 6.3.** (Strict converse duality theorem). Let  \( \overline{u} \in X^0 \) and  
\( (\overline{u}, \overline{\alpha}, \overline{\nu}, \overline{\lambda}, \overline{\mu}, \overline{w}, \overline{p}) \)  
be feasible for (MFD)\(_3\). Suppose that:

1)  
\[
\sum_{i=1}^{k} \lambda_i \left( f_i(\overline{u}) + \overline{\alpha}^T \overline{z}_i \right) \leq \sum_{i=1}^{k} \lambda_i \left( f_i(\overline{u}) + \overline{\mu}^T \overline{z}_i \right) + \sum_{i=1}^{k} \lambda_i \left( \overline{H}_i(\overline{u}, \overline{w}) - \overline{p}^T \nabla_p \overline{H}_i(\overline{u}, \overline{w}) \right),
\]

2) for any  \( i = 1, 2, \cdots, k, \)  
\( \left( f_i(\cdot) + \cdot^T \overline{z}_i \right) \)  
be strictly higher-order \( (V, \alpha^T, \beta^T, \rho^T, \theta^T) \)-invex at  \( \overline{u} \)  
with respect to  \( \overline{H}_i(\overline{u}, \overline{p}) \)  
and  
\( -\left( g_i(\cdot) + \cdot^T \overline{v}_i \right) \)  
be higher-order \( (V, \alpha^T, \beta^T, \rho^T, \theta^T) \)-invex at  \( \overline{u} \)  
with respect to  \( \overline{H}_i(\overline{u}, \overline{p}) \),

3) for any  \( j = 1, 2, \cdots, m, \)  
\( h_j(\cdot) + \cdot^T \overline{w}_j \)  
is higher-order \( (V, \alpha^T, \beta^T, \rho^T, \theta^T) \)-invex at  \( \overline{u} \)  
with respect to  \( G_j(\overline{u}, \overline{p}) \),

4)  
\[
\sum_{i=1}^{k} \lambda_i \rho_i \left\| \overline{H}_i(\overline{u}, \overline{w}) \right\| + \sum_{j=1}^{m} \rho_j \left\| \theta_j(\overline{u}, \overline{w}) \right\| \geq 0.
\]

5)  
\( \overline{\alpha}_i^T(\overline{u}, \overline{w}) = \alpha_i^T(\overline{u}, \overline{w}) = \beta_i^T(\overline{u}, \overline{w}) = \beta_i^T(\overline{u}, \overline{w}) = \alpha_i(\overline{u}, \overline{w}), \forall i = 1, 2, \cdots, k, \)
\( j = 1, 2, \cdots, m. \)
Then, \( x = \bar{x} \).

**Proof.** The proof follows on the lines of Theorem 4.3.

### 7. Conclusion

In this paper, we consider a class of non differentiable multiobjective fractional programming (MFP) with higher-order terms in which each numerator and denominator of the objective function contains the support function of a compact convex set. Furthermore, various duality models for higher-order have been formulated for (MFP) and appropriate duality relations have been obtained under higher-order \( (V, \alpha, \beta, \rho, d) \)-invexity assumptions.

### Acknowledgements

The second author is grateful to the Ministry of Human Resource and Development, India for financial support, to carry this work.

### References


