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Duality Relations for a Class of a Multiobjective Fractional Programming Problem Involving Support Functions

Vandana[1], Ramu Dubey[2], Deepmala[3], Lakshmi Narayan Mishra[4,5]*, Vishnu Narayan Mishra[6]

[1]Department of Management Studies, Indian Institute of Technology Madras, Chennai, India
[2]Department of Mathematics, Central University of Haryana, Pali, India
[3]Mathematics Discipline, PDPM-Indian Institute of Information Technology, Design and Manufacturing, Jabalpur, India
[4]Department of Mathematics, School of Advanced Sciences, Vellore Institute of Technology, Vellore, India
[5]L. 1627 Awadh Puri Colony Beniganj, Phase-III, Opposite-Industrial Training Institute (I.T.I.), Faizabad, India
[6]Department of Mathematics, Indira Gandhi National Tribal University, Lalpur, India
Email: vdrai1988@gmail.com, rdubeyjiya@gmail.com, dmrai23@gmail.com, *lakshminarayanmishra04@gmail.com, vishnunarayanmishra@gmail.com

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Abstract

In this article, for a differentiable function $H: R^n \times R^n \to R$, we introduce the definition of the higher-order (V,α,β,ρ,d)-invexity. Three duality models for a multiobjective fractional programming problem involving nondifferentiability in terms of support functions have been formulated and usual duality relations have been established under the higher-order (V,α,β,ρ,d)-invex assumptions.

Keywords

Efficient Solution, Support Function, Multiobjective Fractional Programming, Generalized Invexity

1. Introduction

Consider the following nonlinear programming problem **(P)** Minimize $f(x)$ subject to $g(x) \le 0$, where $f: R^n \to R$ and $g: R^n \to R$ are twice differentiable functions. The Mangasarian [1] second-order dualof (P) is **(DP)** Maximize

$$f(u) - y^{\mathrm{T}} g(u) - \frac{1}{2} p^{\mathrm{T}} \nabla^2 \left[f(u) - y^{\mathrm{T}} g(u) \right] p$$

such that $\nabla\left[f(u) - y^{\mathrm{T}} g(u) \right] + \nabla^2 \left[f(u) - y^{\mathrm{T}} g(u) \right] p = 0$

*Corresponding author.

By introducing two differentiable functions $H: R^n \times R^n \to R$ and $K: R^n \times R^n \to R^m$, Mangasarian [1] formulated the following higher-order dual of (P): **(DP)$_1$** Maximize

$$f(u) - y^{\mathrm{T}} g(u) + H(u, p) - y^{\mathrm{T}} K(u, p)$$

such that $\nabla_p H(u, p) - \nabla_p \left[y^{\mathrm{T}} K(u, p) \right] = 0,\ y \geq 0$, where $\nabla_p H(u, p)$ denotes the $n \times 1$ gradient of $H(u, p)$ with respect to *p* and $\nabla_p \left(y^{\mathrm{T}} K(u, p) \right)$ denotes the $n \times 1$, gradient of $y^{\mathrm{T}} K(u, p)$ with respect to *p*.

Further, Egudo [2] studied the following multiobjective fractional programming problem: **(MFPP)** Minimize

$$G(x) = \left(\frac{f_1(x)}{g_1(x)}, \frac{f_2(x)}{g_2(x)}, \cdots, \frac{f_k(x)}{g_k(x)} \right)$$

subject to

$$x \in X^0 = \left\{ x \in X \subset R^n : h_j(x) \leq 0, j \in M \right\},$$

where $f = (f_1, f_2, \cdots, f_k): X \to R^k$, $g = (g_1, g_2, \cdots, g_k): X \to R^k$ and $h = (h_1, h_2, \cdots, h_m): X \to R^m$ are differentiable on *X*. Also, he discussed duality results for Mond-Weir and Schaible type dual programs under generalized convexity.

For the nondifferentiable multiobjective programming problem: **(MPP)** Minimize

$$G(x) = \left(f_1(x) + S(x \mid C_1), f_2(x) + S(x \mid C_2), \cdots, f_k(x) + S(x \mid C_k) \right)$$

subject to $x \in X^0 = \left\{ x \in X \subset R^n : g_j(x) + S(x \mid E_j) \leq 0, j = 1, 2, \cdots, m \right\}$, where $f_i: X \to R\ (i = 1, 2, \cdots, k)$ and $g_j: X \to R\ (j = 1, 2, \cdots, m)$ are differentiable functions. C_i and E_j are compact convex sets in R^n and $S(x \mid C_i)(i = 1, 2, \cdots, k)$ and $S(x \mid E_j)(j = 1, 2, \cdots, m)$ denote the support functions of compact convex sets, various researchers have worked. Gulati and Agarwal [3] introduced the higher-order Wolfe-type dual model of (MPP) and proved duality theorems under higher-order (F, ρ, ρ, d)-type *I*-assumptions.

In last several years, various optimality and duality results have been obtained for multiobjective fractional programming problems. In Chen [4], multiobjective fractional problem and its duality theorems have been considered under higher-order (F, α, ρ, d)-convexity. Later on, Suneja *et al.* [5] discussed higher-order Mond-Weir and Schaible type nondifferentiable dual programs and their duality theorems under higher-order (F, ρ, σ)-type *I*-assumptions. Several researchers have also worked in this directions such as ([6] [7]).

In this paper, we first introduce the definition of higher-order $(V, \alpha, \beta, \rho, d)$-invex with respect to differentiable function $H: R^n \times R^n \to R$. We also construct a nontrivial numerical example which illustrates the existence of such a function. We then formulate three higher-order dual problems corresponding to the multiobjective nondifferentiable fractional programming problem. Further, we establish usual duality relations for these primal-dual pairs under aforesaid assumptions.

2. Preliminaries

Let $X \subseteq R^n$ be an open set and $\phi : X \to R, H : X \times R^n \to R$ be differentiable functions. $\alpha, \beta : X \times X \to R_+ \setminus \{0\}$, $\eta : X \times X \to R^n$, $\rho \in R^n$ and $\theta : X \times X \to R^n$.

Definition 2.1. ϕ is said to be (strictly) higher-order $(V, \alpha, \beta, \rho, \theta)$-invex at u with respect to $H(u, p)$, if there exist $\eta, \alpha, \beta, \rho$ and θ such that, for any $x \in X$ and $p \in R^n$,

$$\alpha(x,u)\left[\phi(x)-\phi(u)\right](>) \geq \eta^{\mathrm{T}}(x,u)\left(\nabla\phi(u)+\nabla_p H(u,p)\right)$$
$$+\beta(x,u)\left[H(u,p)-p^{\mathrm{T}}\nabla_p H(u,p)\right]+\rho\left\|\theta(x,u)\right\|^2.$$

Example 2.1. Let $\phi : R \to R$ be such that $\phi(x) = x^4 + x^2 + 1$.

Let

$$\eta(x,u)=\frac{1}{2}\left(x^2+u^2\right), H(u,p)=-2p(x+1)^2.$$

Also, suppose

$$\alpha(x,u)=1, \beta(x,u)=2, \rho=-1, \left\|\theta(x,u)\right\|=\left(x^2+u^2\right)^{\frac{1}{2}}.$$

Now,

$$\xi=\alpha(x,u)\left[\phi(x)-\phi(u)\right]-\eta^{\mathrm{T}}(x,u)\left(\nabla\phi(u)+\nabla_p H(u,p)\right)$$
$$-\beta(x,u)\left[H(u,p)-p^{\mathrm{T}}\nabla_p H(u,p)\right]-\rho\left\|\theta(x,u)\right\|^2.$$

$$\xi=\left(x^4+x^2-u^4-u^2\right)-\frac{1}{2}\left(x^2+u^2\right)\left[4u^3+2u-2(u+1)^2\right]-\left(x^2+u^2\right)$$

$$\xi = x^4 + x^2 \quad (\text{at } u = 0).$$
$$\geq 0, \forall x \in R.$$

Hence, ϕ is higher-order $(V, \alpha, \beta, \rho, \theta)$-invex at $u = 0$ with respect to $H(u, p)$.

Remark 2.1.

1) If $H(u, p) = 0$, then the Definition 2.1 reduces to (V, ρ)-invex function introduced by Kuk *et al.* [8].

2) If $H(u, p) = 0$ and $\rho = 0$, then the Definition 2.1 becomes that of V-invexity introduced by Jeyakumar and Mond [9].

3) If $H(u,p)=\frac{1}{2}p^{\mathrm{T}}\nabla^2\phi(u)p, \alpha(x,u)=0$ and $\rho = 0$, then above definition yields in η-bonvexity given by Pandey [10].

4) If $\beta = 1$, then the Definition 2.1 reduced in $(V, \alpha, \rho, \theta)$-invex given by Gulati and Geeta [11].

A differentiable function $f = (f_1, f_2, \cdots, f_k) : X \to R^k$ is $(V, \alpha, \beta, \rho, \theta)$-invex if for all $i = 1, 2, \cdots, k$, f_i is $(V, \alpha_i, \beta_i, \rho_i, \theta_i)$-invex.

Definition 2.2. [12]. Let C be a compact convex set in R^n. The support function of C is defined by

$$S(x\,|\,C)=\max\left\{x^{\mathrm{T}}y : y\in C\right\}.$$

3. Problem Formulation

Consider the multiobjective programming problem with support function given as: **(MFP)** Minimize

$$F(x)=\left\{\frac{f_1(x)+S(x\,|\,C_1)}{g_1(x)-S(x\,|\,D_1)},\frac{f_2(x)+S(x\,|\,C_2)}{g_2(x)-S(x\,|\,D_2)},\cdots,\frac{f_k(x)+S(x\,|\,C_k)}{g_k(x)-S(x\,|\,D_k)}\right\}$$

subject to $x\in X^0=\left\{x\in X\subset R^n : h_j(x)+S(x\,|\,E_j)\le 0, j=1,2,\cdots,m\right\}$,
where $f=(f_1,f_2,\cdots,f_k):X\to R^k$, $g=(g_1,g_2,\cdots,g_k):X\to R^k$ and $h=(h_1,h_2,\cdots,h_m):X\to R^m$ are differentiable on *X*, $f_i(.)+S(.\,|\,C_i)\ge 0$ and $g_i(.)-S(.\,|\,D_i)>0$. Let $H_i:X\times R^n\to R$ be differentiable functions, C_i, D_i and E_j are compact convex sets in R^n, for all $i=1,2,\cdots,k$, $j=1,2,\cdots,m$.

Definition 3.1. [3]. A point $x^0\in X^0$ is said to be an efficient solution (or Pareto optimal) of (MFP), if there exists no $x\in X^0$ such that for every

$$i=1,2,\cdots,k,\quad \frac{f_i(x)+S(x\,|\,C_i)}{g_i(x)-S(x\,|\,D_i)}\le\frac{f_i(x^0)+S(x^0\,|\,C_i)}{g_i(x^0)-S(x^0\,|\,D_i)}$$

and for some $r=1,2,\cdots,k$,

$$\frac{f_r(x)+S(x\,|\,C_r)}{g_r(x)-S(x\,|\,D_r)}<\frac{f_r(x^0)+S(x^0\,|\,C_r)}{g_r(x^0)-S(x^0\,|\,D_r)}.$$

We now state theorems 3.1-3.2, whose proof follows on the lines [13].

Theorem 3.1. For some *t*, if $f_t(.)+(.)^{\mathrm{T}}z_t$ and $-\left(g_t(.)-(.)^{\mathrm{T}}v_t\right)$ are higher-order $(V,\alpha_t,\beta_t,\rho_t,\theta_t)$-invex at u with respect to $H_t(u,p)$ for same $\eta(x,u)$. Then, the fractional function $\left(\frac{f_t(.)+(.)^{\mathrm{T}}z_t}{g_t(.)-(.)^{\mathrm{T}}v_t}\right)$ is higher-order $(V,\bar{\alpha}_t,\bar{\beta}_t,\bar{\rho}_t,\bar{\theta}_t)$-invex at u with respect to $\bar{H}_t(u,p)$, where

$$\bar{\alpha}_t(x,u)=\left(\frac{g_t(x)-x^{\mathrm{T}}v_t}{g_t(u)-u^{\mathrm{T}}v_t}\right)\alpha_t(x,u),\quad \bar{\beta}_t(x,u)=\beta_t(x,u),$$

$$\bar{\theta}_t(x,u)=\theta_t(x,u)\left(\frac{1}{g_t(u)-u^{\mathrm{T}}v_t}+\frac{f_t(u)+u^{\mathrm{T}}z_t}{\left(g_t(u)-u^{\mathrm{T}}v_t\right)^2}\right)^{\frac{1}{2}},\quad \bar{\rho}_t(x,u)=\rho_t(x,u)$$

and

$$\bar{H}_t(u,p)=\left(\frac{1}{g_t(u)-u^{\mathrm{T}}v_t}+\frac{f_t(u)+u^{\mathrm{T}}z_t}{\left(g_t(u)-u^{\mathrm{T}}v_t\right)^2}\right)H_t(u,p).$$

Theorem 3.2. In Theorem 3.1,if either $-\left(g_t(.)-(.)^{\mathrm{T}}v_t\right)$ is strictly higher-order $(V,\alpha_t,\beta_t,\rho_t,\theta_t)$-invex at u with respect to $H_t(u,p)$ and $\left(f_t(.)-(.)^{\mathrm{T}} z_t\right)>0$ or $\left(f_t(.)-(.)^{\mathrm{T}} z_t\right)$ is strictly higher-order $(V,\alpha_t,\beta_t,\rho_t,\theta_t)$ - invex at u with respect to $H_t(u,p)$, then $\left(\dfrac{f_t(.)+(.)^{\mathrm{T}} z_t}{g_t(.)-(.)^{\mathrm{T}} z_t}\right)$ is strictly higher-order $(V,\bar{\alpha}_t,\bar{\beta}_t,\bar{\rho}_t,\bar{\theta}_t)$-invex at $u\in X$ with respect to $\bar{H}_t(u,p)$.

Theorem 3.3 (Necessary Condition) [14]. Assume that $\bar{x}$ is an efficient solution of (MFP) and the Slater's constraint qualification is satisfied on *X*. Then there exist $\bar{\lambda}_i>0, \bar{\mu}_j\in R^m, \bar{z}_i\in R^n, \bar{v}_i\in R^n$ and $\bar{w}_j\in R^m, i=1,2,\cdots,k, j=1,2,\cdots,m$, such that

$$\sum_{i=1}^{k}\bar{\lambda}_i\nabla\left(\frac{f_i(\bar{x})+\bar{x}^{\mathrm{T}}\bar{z}_i}{g_i(\bar{x})-\bar{x}^{\mathrm{T}}\bar{v}_i}\right)+\sum_{j=1}^{m}\bar{\mu}_j\nabla\left(h_j(\bar{x})+\bar{x}^{\mathrm{T}}\bar{w}_j\right)=0, \tag{1}$$

$$\sum_{j=1}^{m}\bar{\mu}_j\left(h_j(\bar{x})+\bar{x}^{\mathrm{T}}\bar{w}_j\right)=0, \tag{2}$$

$$\bar{x}^{\mathrm{T}}\bar{z}_i=S(\bar{x}\mid C_i), \bar{z}_i\in C_i, i=1,2,\cdots,k, \tag{3}$$

$$\bar{x}^{\mathrm{T}}\bar{v}_i=S(\bar{x}\mid D_i), \bar{v}_i\in D_i, i=1,2,\cdots,k, \tag{4}$$

$$\bar{x}^{\mathrm{T}}\bar{w}_j=S(\bar{x}\mid E_j), \bar{w}_j\in E_j, j=1,2,\cdots,m, \tag{5}$$

$$\bar{\lambda}_i>0, i=1,2,\cdots,k, \bar{\mu}_j\geq 0, j=1,2,\cdots,m. \tag{6}$$

Theorem 3.4. (Sufficient Condition). Let u be a feasible solution of (MFP). Then, there exist $\lambda_i>0, i=1,2,\cdots,k$ and $\mu_j\geq 0, j=1,2,\cdots,m$, such that

$$\sum_{i=1}^{k}\lambda_i\nabla\left(\frac{f_i(u)+u^{\mathrm{T}}z_i}{g_i(u)-u^{\mathrm{T}}v_i}\right)+\sum_{j=1}^{m}\mu_j\nabla\left(h_j(u)+u^{\mathrm{T}}w_j\right)=0, \tag{7}$$

$$\sum_{j=1}^{m}\mu_j\left(h_j(u)+u^{\mathrm{T}}w_j\right)=0, \tag{8}$$

$$u^{\mathrm{T}}z_i=S(u\mid C_i), z_i\in C_i, i=1,2,\cdots,k, \tag{9}$$

$$u^{\mathrm{T}}v_i=S(u\mid D_i), v_i\in D_i, i=1,2,\cdots,k, \tag{10}$$

$$u^{\mathrm{T}}w_j=S(u\mid E_j), w_j\in E_j, j=1,2,\cdots,m, \tag{11}$$

$$\bar{\lambda}_i>0, i=1,2,\cdots,k, \bar{\mu}_j\geq 0, j=1,2,\cdots,m. \tag{12}$$

Let, for $i=1,2,\cdots,k, j=1,2,\cdots,m$,

1) $\left(f_i(.)+(.)^{\mathrm{T}} z_i\right)$ and $-\left(g_i(.)-(.)^{\mathrm{T}} v_i\right)$ be higher-order $(V,\alpha_i^1,\beta_i^1,\rho_i^1,\theta_i^1)$-invex at u with respect to $H_i(u,p)$,

2) $\left(h_j(.)+(.)^{\mathrm{T}} w_j\right)$ be higher-order $(V,\alpha_j^2,\beta_j^2,\rho_j^2,\theta_j^2)$-invex at u with respect to $G_j(u,p)$,

3) $\sum_{i=1}^{k}\lambda_i\bar{\rho}_i^1\left\|\bar{\theta}_i^1(x,u)\right\|^2+\sum_{j=1}^{m}\mu_j\rho_j^2\left\|\theta_j^2(x,u)\right\|^2\geq 0,$

4) $\sum_{i=1}^{k}\lambda_i\left(\nabla_p\bar{H}_i(u,p)\right)+\sum_{j=1}^{m}\mu_j\left(\nabla_p G_j(u,p)\right)=0$,

$$\sum_{i=1}^{k}\lambda_i\left(\bar{H}_i(u,p)-p^{\mathrm{T}}\nabla_p\bar{H}_i(u,p)\right)\geq 0 \text{ and } \sum_{j=1}^{m}\mu_j\left(G_j(u,p)-p^{\mathrm{T}}\nabla_pG_j(u,p)\right)\geq 0,$$

5) $\alpha_i^1(x,u)=\alpha_j^2(x,u)=\beta_i^1(x,u)=\beta_j^2(x,u)=\alpha(x,u)$,

where

$$\bar{\alpha}_i(x,u)=\left(\frac{g_i(x)-x^{\mathrm{T}}v_i}{g_i(u)-u^{\mathrm{T}}v_i}\right)\alpha_i(x,u),\quad \bar{\beta}_i(x,u)=\beta_i(x,u),$$

$$\bar{\theta}_i(x,u)=\theta_i(x,u)\left(\frac{1}{g_i(u)-u^{\mathrm{T}}v_i}+\frac{f_i(u)+u^{\mathrm{T}}z_i}{\left(g_i(u)-u^{\mathrm{T}}v_i\right)^2}\right)^{\frac{1}{2}}$$

and $\bar{\rho}_i(x,u)=\rho_i(x,u)$.

Then, u is an efficient solution of (MFP).

Proof. Suppose u is not an efficient solution of (MFP). Then there exists $x\in X^0$ such that

$$\frac{f_i(x)+S(x\,|\,C_i)}{g_i(x)-S(x\,|\,D_i)}\leq\frac{f_i(u)+S(u\,|\,C_i)}{g_i(u)-S(u\,|\,D_i)},\ \text{for all } i=1,2,\cdots,k$$

and

$$\frac{f_r(x)+S(x\,|\,C_r)}{g_r(x)-S(x\,|\,D_r)}<\frac{f_r(u)+S(u\,|\,C_r)}{g_r(u)-S(u\,|\,D_r)},\ \text{for some } r=1,2,\cdots,k,$$

which implies

$$\begin{aligned}\frac{f_i(x)+x^{\mathrm{T}}z_i}{g_i(x)-x^{\mathrm{T}}v_i}&\leq\frac{f_i(x)+S(x\,|\,C_i)}{g_i(x)-S(x\,|\,D_i)}\leq\frac{f_i(u)+S(u\,|\,C_i)}{g_i(u)-S(u\,|\,D_i)}\\&=\frac{f_i(u)+u^{\mathrm{T}}z_i}{g_i(u)-u^{\mathrm{T}}v_i},\ \text{for all } i=1,2,\cdots,k\end{aligned}\tag{13}$$

and

$$\begin{aligned}\frac{f_r(x)+x^{\mathrm{T}}z_r}{g_r(x)-x^{\mathrm{T}}v_r}&\leq\frac{f_r(x)+S(x\,|\,C_r)}{g_r(x)-S(x\,|\,D_r)}<\frac{f_r(u)+S(u\,|\,C_r)}{g_r(u)-S(u\,|\,D_r)}\\&=\frac{f_r(u)+u^{\mathrm{T}}z_r}{g_r(u)-u^{\mathrm{T}}v_r},\ \text{for some } r=1,2,\cdots,k.\end{aligned}\tag{14}$$

Since $\lambda_i>0, i=1,2,\cdots,k$, inequalities (13) and (14) gives

$$\sum_{i=1}^{k}\lambda_i\left(\frac{f_i(x)+x^{\mathrm{T}}z_i}{g_i(x)-x^{\mathrm{T}}v_i}-\frac{f_i(u)+u^{\mathrm{T}}z_i}{g_i(u)-u^{\mathrm{T}}v_i}\right)<0.\tag{15}$$

From Theorem 3.1, for each $i, 1\leq i\leq k$, $\left(\frac{f_i(.)+(.)^{\mathrm{T}}z_i}{g_i(.)-(.)^{\mathrm{T}}v_i}\right)$ is higher-order $\left(V,\bar{\alpha}_i^1,\bar{\beta}_i^1,\bar{\rho}_i^1,\bar{\theta}_i^1\right)$-invex at $u\in X^0$ with respect to $\bar{H}_i(u,p)$, we have

$$\bar{\alpha}_i^1(x,u)\left[\frac{f_i(x)+x^{\mathrm{T}}z_i}{g_i(x)-x^{\mathrm{T}}v_i}-\frac{f_i(u)+u^{\mathrm{T}}z_i}{g_i(u)-u^{\mathrm{T}}v_i}\right]$$

$$\begin{aligned}&\ge \eta^{\mathrm{T}}(x,u)\left[\nabla\left(\frac{f_i(u)+u^{\mathrm{T}}z_i}{g_i(u)-u^{\mathrm{T}}v_i}\right)+\nabla_p\bar{H}_i(u,p)\right]\\&+\bar{\beta}_i^1(x,u)\left[\bar{H}_i(u,p)-p^{\mathrm{T}}\nabla_p\bar{H}_i(u,p)\right]+\bar{\rho}_i^1\left\|\bar{\theta}_i^1(x,u)\right\|^2.\end{aligned}\tag{16}$$

where

$$\bar{\alpha}_i(x,u)=\left(\frac{g_i(x)-x^{\mathrm{T}}v_i}{g_i(u)-u^{\mathrm{T}}v_i}\right)\alpha_i(x,u),\quad \bar{\beta}_i(x,u)=\beta_i(x,u),$$

$$\bar{\theta}_i(x,u)=\theta_i(x,u)\left(\frac{1}{g_i(u)-u^{\mathrm{T}}v_i}+\frac{f_i(u)+u^{\mathrm{T}}z_i}{\left(g_i(u)-u^{\mathrm{T}}v_i\right)^2}\right)^{\frac{1}{2}},\quad \bar{\rho}_i(x,u)=\rho_i(x,u)$$

and $\bar{H}_i(u,p)=\left(\frac{1}{g_i(u)-u^{\mathrm{T}}v_i}+\frac{f_i(u)+u^{\mathrm{T}}z_i}{\left(g_i(u)-u^{\mathrm{T}}v_i\right)^2}\right)H_i(u,p)$.

By hypothesis 2), we get

$$\begin{aligned}&\alpha_j^2(x,u)\left[h_j(x)+x^{\mathrm{T}}w_j-\left(h_j(u)+u^{\mathrm{T}}w_j\right)\right]\\&\ge \eta^{\mathrm{T}}(x,u)\left[\nabla\left(h_j(u)+u^{\mathrm{T}}w_j\right)+\nabla_pG_j(u,p)\right]\\&+\beta_j^2(x,u)\left[G_j(u,p)-p^{\mathrm{T}}\nabla_pG_j(u,p)\right]+\rho_j^2\left\|\theta_j^2(x,u)\right\|^2.\end{aligned}\tag{17}$$

Adding the two inequalities after multiplying (16) by λ_i and (17) by μ_j, we obtain

$$\begin{aligned}&\sum_{i=1}^k\lambda_i\bar{\alpha}_i^1(x,u)\left[\frac{f_i(x)+x^{\mathrm{T}}z_i}{g_i(x)-x^{\mathrm{T}}v_i}-\frac{f_i(u)+u^{\mathrm{T}}z_i}{g_i(u)-u^{\mathrm{T}}v_i}\right]\\&+\sum_{j=1}^m\mu_j\alpha_j^2(x,u)\left[h_j(x)+x^{\mathrm{T}}w_j-\left(h_j(u)+u^{\mathrm{T}}w_j\right)\right]\\&\ge \eta^{\mathrm{T}}(x,u)\sum_{i=1}^k\lambda_i\left[\nabla\left(\frac{f_i(u)+u^{\mathrm{T}}z_i}{g_i(u)-u^{\mathrm{T}}v_i}\right)+\nabla_p\bar{H}_i(u,p)\right]\\&+\eta^{\mathrm{T}}(x,u)\sum_{j=1}^m\mu_j\left[\nabla\left(h_j(u)+u^{\mathrm{T}}w_j\right)+\nabla_pG_j(u,p)\right]\\&+\sum_{i=1}^k\lambda_i\bar{\beta}_i(x,u)\left[\bar{H}_i(u,p)-p^{\mathrm{T}}\nabla_p\bar{H}_i(u,p)\right]\\&+\sum_{j=1}^m\mu_j\beta_j^2(x,u)\left[G_j(u,p)-p^{\mathrm{T}}\nabla_pG_j(u,p)\right]\\&+\sum_{i=1}^k\lambda_i\bar{\rho}_i^1\left\|\bar{\theta}_i^1(x,u)\right\|^2+\sum_{j=1}^m\mu_j\rho_j^2\left\|\theta_j^2(x,u)\right\|^2.\end{aligned}\tag{18}$$

Using hypothesis 3)-4), we get

$$\begin{aligned}&\sum_{i=1}^k\lambda_i\left[\frac{f_i(x)+x^{\mathrm{T}}z_i}{g_i(x)-x^{\mathrm{T}}v_i}-\frac{f_i(u)+u^{\mathrm{T}}z_i}{g_i(u)-u^{\mathrm{T}}v_i}\right]+\sum_{j=1}^m\mu_j\left[h_j(x)+x^{\mathrm{T}}w_j-\left(h_j(u)+u^{\mathrm{T}}w_j\right)\right]\\&\ge \eta^{\mathrm{T}}(x,u)\sum_{i=1}^k\lambda_i\nabla\left(\frac{f_i(u)+u^{\mathrm{T}}z_i}{g_i(u)-u^{\mathrm{T}}v_i}\right)+\eta^{\mathrm{T}}(x,u)\sum_{j=1}^m\mu_j\nabla\left(h_j(u)+u^{\mathrm{T}}w_j\right).\end{aligned}\tag{19}$$

Further, using (7)-(8), therefore

$$\sum_{i=1}^{k}\lambda_i\left[\frac{f_i(x)+x^{\mathrm{T}}z_i}{g_i(x)-x^{\mathrm{T}}v_i}-\frac{f_i(u)+u^{\mathrm{T}}z_i}{g_i(u)-u^{\mathrm{T}}v_i}\right]+\sum_{j=1}^{m}\mu_j\left[h_j(x)+x^{\mathrm{T}}w_j\right]\geq 0. \tag{20}$$

Since x is feasible solution for (MFP), it follows that

$$\sum_{i=1}^{k}\lambda_i\left(\frac{f_i(x)+x^{\mathrm{T}}z_i}{g_i(x)-x^{\mathrm{T}}v_i}\right)\geq\sum_{i=1}^{k}\lambda_i\left(\frac{f_i(u)+u^{\mathrm{T}}z_i}{g_i(u)-u^{\mathrm{T}}v_i}\right).$$

This contradicts (15). Therefore, u is an efficient solution of (MFP).

4. Duality Model-I

Consider the following dual $(\mathrm{MFD})_1$ of (MFP): $\mathbf{(MFD)_1}$ Maximize

$$\left[\frac{f_1(u)+u^{\mathrm{T}}z_1}{g_1(u)-u^{\mathrm{T}}v_1}+\sum_{j=1}^{m}\mu_j\left(h_j(u)+u^{\mathrm{T}}w_j\right)+\left(\bar{H}_1(u,p)-p^{\mathrm{T}}\nabla_p\bar{H}_1(u,p)\right)\right.$$
$$+\sum_{j=1}^{m}\mu_j\left(G_j(u,p)-p^{\mathrm{T}}\nabla_pG_j(u,p)\right),\cdots,$$
$$\frac{f_k(u)+u^{\mathrm{T}}z_k}{g_k(u)-u^{\mathrm{T}}v_k}+\sum_{j=1}^{m}\mu_j\left(h_j(u)+u^{\mathrm{T}}w_j\right)+\left(\bar{H}_k(u,p)-p^{\mathrm{T}}\nabla_p\bar{H}_k(u,p)\right)$$
$$\left.+\sum_{j=1}^{m}\mu_j\left(G_j(u,p)-p^{\mathrm{T}}\nabla_pG_j(u,p)\right)\right]$$

subject to

$$\begin{aligned}&\sum_{i=1}^{k}\lambda_i\nabla\left(\frac{f_i(u)+u^{\mathrm{T}}z_i}{g_i(u)-u^{\mathrm{T}}v_i}\right)+\sum_{j=1}^{m}\mu_j\nabla\left(h_j(u)+u^{\mathrm{T}}w_j\right)\\&+\sum_{i=1}^{k}\lambda_i\nabla_p\bar{H}_i(u,p)+\sum_{j=1}^{m}\mu_j\nabla_pG_j(u,p)=0,\end{aligned} \tag{21}$$

$$z_i\in C_i,v_i\in D_i,w_j\in E_j,i=1,2,\cdots,k,\ j=1,2,\cdots,m,$$

$$\mu_j\geq 0,\lambda_i>0,\sum_{i=1}^{k}\lambda_i=1,i=1,2,\cdots,k,\ j=1,2,\cdots,m.$$

Let Z^0 be feasible solution for $(\mathrm{MFD})_1$.

Theorem 4.1. (Weak duality theorem). Let $x\in X^0$ and $(u,z,v,\mu,\lambda,w,p)\in Z^0$. Suppose that

1) for any $i=1,2,\cdots,k$, $\left(f_i(.)+(.)^{\mathrm{T}}z_i\right)$ and $-\left(g_i(.)-(.)^{\mathrm{T}}v_i\right)$ are higher-order $\left(V,\alpha_i^1,\beta_i^1,\rho_i^1,\theta_i^1\right)$-invex at u with respect to $H_i(u,p)$,

2) for any $j=1,2,\cdots,m$, $\left(h_j(.)+(.)^{\mathrm{T}}w_j\right)$ is higher-order $\left(V,\alpha_j^2,\beta_j^2,\rho_j^2,\theta_j^2\right)$-invex at u with respect to $G_j(u,p)$,

3) $\sum_{i=1}^{k}\lambda_i\bar{\rho}_i^1\left\|\bar{\theta}_i^1(x,u)\right\|^2+\sum_{j=1}^{m}\mu_j\rho_j^2\left\|\theta_j^2(x,u)\right\|^2\geq 0.$

4) $\bar{\alpha}_i^1(x,u)=\alpha_j^2(x,u)=\beta_i^1(x,u)=\beta_j^2(x,u)=\alpha(x,u),\forall\, i=1,2,\cdots,k,$ $j=1,2,\cdots,m,$

where $\bar{\alpha}_t(x,u)=\left(\frac{g_t(x)-x^{\mathrm{T}}v_t}{g_t(u)-u^{\mathrm{T}}v_t}\right)\alpha_t(x,u)$, $\bar{\beta}_t(x,u)=\beta_t(x,u)$,

$$\bar{\theta}_t(x,u)=\theta_t(x,u)\left(\frac{1}{g_t(u)-u^{\mathrm{T}}v_t}+\frac{f_t(u)+u^{\mathrm{T}}z_t}{\left(g_t(u)-u^{\mathrm{T}}v_t\right)^2}\right)^{\frac{1}{2}},\quad \bar{\rho}_t(x,u)=\rho_t(x,u)\quad \text{and}$$

$$\bar{H}_t(u,p)=\left(\frac{1}{g_t(u)-u^{\mathrm{T}}v_t}+\frac{f_t(u)+u^{\mathrm{T}}z_t}{\left(g_t(u)-u^{\mathrm{T}}v_t\right)^2}\right)H_t(u,p).$$

Then, the following cannot hold

$$\begin{aligned}&\frac{f_i(x)+S(x\mid C_i)}{g_i(x)-S(x\mid D_i)}\\&\le\frac{f_i(u)+u^{\mathrm{T}}z_i}{g_i(u)-u^{\mathrm{T}}v_i}+\sum_{j=1}^{m}\mu_j\left(h_j(u)+u^{\mathrm{T}}w_j\right)+\left(\bar{H}_i(u,p)-p^{\mathrm{T}}\nabla_p\bar{H}_i(u,p)\right)\\&\quad+\sum_{j=1}^{m}\mu_j\left(G_j(u,p)-p^{\mathrm{T}}\nabla_pG_j(u,p)\right),\ \text{for all } i=1,2,\cdots,k\end{aligned}\tag{22}$$

and

$$\begin{aligned}&\frac{f_r(x)+S(x\mid C_r)}{g_r(x)-S(x\mid D_r)}\\&<\frac{f_r(u)+u^{\mathrm{T}}z_r}{g_r(u)-u^{\mathrm{T}}v_r}+\sum_{j=1}^{m}\mu_j\left(h_j(u)+u^{\mathrm{T}}w_j\right)+\left(\bar{H}_r(u,p)-p^{\mathrm{T}}\nabla_p\bar{H}_r(u,p)\right)\\&\quad+\sum_{j=1}^{m}\mu_j\left(G_j(u,p)-p^{\mathrm{T}}\nabla_pG_j(u,p)\right),\ \text{for some } r=1,2,\cdots,k.\end{aligned}\tag{23}$$

Proof. Suppose that (22) and (23) hold, then using $\lambda_i>0$, $\sum_{i=1}^{k}\lambda_i=1$, $x^{\mathrm{T}}z_i\le S(x\mid C_i)$, $x^{\mathrm{T}}v_i\le S(x\mid D_i)$, $i=1,2,\cdots,k$, we have

$$\begin{aligned}\sum_{i=1}^{k}\lambda_i\left(\frac{f_i(x)+x^{\mathrm{T}}z_i}{g_i(x)-x^{\mathrm{T}}v_i}\right)&<\sum_{i=1}^{k}\lambda_i\left(\frac{f_i(u)+u^{\mathrm{T}}z_i}{g_i(u)-u^{\mathrm{T}}v_i}\right)+\sum_{j=1}^{m}\mu_j\left(h_j(u)+u^{\mathrm{T}}w_j\right)\\&\quad+\sum_{i=1}^{k}\lambda_i\left(\bar{H}_i(u,p)-p^{\mathrm{T}}\nabla_p\bar{H}_i(u,p)\right)\\&\quad+\sum_{j=1}^{m}\mu_j\left(G_j(u,p)-p^{\mathrm{T}}\nabla_pG_j(u,p)\right).\end{aligned}\tag{24}$$

From hypothesis 1) and Theorem 3.1, for $i=1,2,\cdots,k$, $\left(\frac{f_i(.)+(.)^{\mathrm{T}}z_i}{g_i(.)-(.)^{\mathrm{T}}v_i}\right)$ is higher-order $\left(V,\bar{\alpha}_i^1,\bar{\beta}_i^1,\bar{\rho}_i^1,\bar{\theta}_i^1\right)$-invex at u with respect to $\bar{H}_i(u,p)$, we get

$$\begin{aligned}&\bar{\alpha}_i^1(x,u)\left[\frac{f_i(x)+x^{\mathrm{T}}z_i}{g_i(x)-x^{\mathrm{T}}v_i}-\frac{f_i(u)+u^{\mathrm{T}}z_i}{g_i(u)-u^{\mathrm{T}}v_i}\right]\\&\ge\eta^{\mathrm{T}}(x,u)\left[\nabla\left(\frac{f_i(u)+u^{\mathrm{T}}z_i}{g_i(u)-u^{\mathrm{T}}v_i}\right)+\nabla_p\bar{H}_i(u,p)\right]\\&\quad+\bar{\beta}_i^1(x,u)\left[\bar{H}_i(u,p)-p^{\mathrm{T}}\nabla_p\bar{H}_i(u,p)\right]+\bar{\rho}_i^1\left\|\bar{\theta}_i^1(x,u)\right\|^2.\end{aligned}\tag{25}$$

For any $j=1,2,\cdots,m$, $\left(h_j(.)+(.)^{\mathrm{T}}w_j\right)$ is higher-order $\left(V,\alpha_j^2,\beta_j^2,\rho_j^2,\theta_j^2\right)$-invex at u with respect to $G_j(u,p)$, we have

$$\begin{aligned}&\alpha_j^2(x,u)\left[h_j(x)+x^{\mathrm{T}}w_j-\left(h_j(u)+u^{\mathrm{T}}w_j\right)\right]\\&\geq\eta^{\mathrm{T}}(x,u)\left[\nabla\left(h_j(u)+u^{\mathrm{T}}w_j\right)+\nabla_pG_j(u,p)\right]\\&\quad+\beta_j^2(x,u)\left[G_j(u,p)-p^{\mathrm{T}}\nabla_pG_j(u,p)\right]+\rho_j^2\left\|\theta_j^2(x,u)\right\|^2.\end{aligned}\tag{26}$$

Adding the two inequalities after multiplying (25) by λ_i and (26) by μ_j, we obtain

$$\begin{aligned}&\sum_{i=1}^{k}\lambda_i\bar{\alpha}_i^1(x,u)\left[\frac{f_i(x)+x^{\mathrm{T}}z_i}{g_i(x)-x^{\mathrm{T}}v_i}-\frac{f_i(u)+u^{\mathrm{T}}z_i}{g_i(u)-u^{\mathrm{T}}v_i}\right]\\&+\sum_{j=1}^{m}\mu_j\alpha_j^2(x,u)\left[h_j(x)+x^{\mathrm{T}}w_j-\left(h_j(u)+u^{\mathrm{T}}w_j\right)\right]\\&\geq\eta^{\mathrm{T}}(x,u)\sum_{i=1}^{k}\lambda_i\left[\nabla\left(\frac{f_i(u)+u^{\mathrm{T}}z_i}{g_i(u)-u^{\mathrm{T}}v_i}\right)+\nabla_p\bar{H}_i(u,p)\right]\\&\quad+\eta^{\mathrm{T}}(x,u)\sum_{j=1}^{m}\mu_j\left[\nabla\left(h_j(u)+u^{\mathrm{T}}w_j\right)+\nabla_pG_j(u,p)\right]\\&\quad+\sum_{i=1}^{k}\lambda_i\bar{\beta}_i(x,u)\left[\bar{H}_i(u,p)-p^{\mathrm{T}}\nabla_p\bar{H}_i(u,p)\right]\\&\quad+\sum_{j=1}^{m}\mu_j\beta_j^2(x,u)\left[G_j(u,p)-p^{\mathrm{T}}\nabla_pG_j(u,p)\right]\\&\quad+\sum_{i=1}^{k}\lambda_i\bar{\rho}_i^1\left\|\bar{\theta}_i^1(x,u)\right\|^2+\sum_{j=1}^{m}\mu_j\rho_j^2\left\|\theta_j^2(x,u)\right\|^2.\end{aligned}\tag{27}$$

Using hypothesis 3) and (21), we get

$$\begin{aligned}&\sum_{i=1}^{k}\lambda_i\bar{\alpha}_i^1(x,u)\left[\frac{f_i(x)+x^{\mathrm{T}}z_i}{g_i(x)-x^{\mathrm{T}}v_i}-\frac{f_i(u)+u^{\mathrm{T}}z_i}{g_i(u)-u^{\mathrm{T}}v_i}\right]\\&+\sum_{j=1}^{m}\mu_j\alpha_j^2(x,u)\left[h_j(x)+x^{\mathrm{T}}w_j-\left(h_j(u)+u^{\mathrm{T}}w_j\right)\right]\\&\geq\sum_{i=1}^{k}\lambda_i\bar{\beta}_i^1(x,u)\left[\bar{H}_i(u,p)+p^{\mathrm{T}}\nabla_p\bar{H}_i(u,p)\right]\\&\quad+\sum_{j=1}^{m}\mu_j\beta_j^2(x,u)\left[G_j(u,p)-p^{\mathrm{T}}\nabla_pG_j(u,p)\right].\end{aligned}\tag{28}$$

Finally, using hypothesis 4) and x is feasible solution for (MFP), it follows that

$$\begin{aligned}\sum_{i=1}^{k}\lambda_i\left(\frac{f_i(x)+x^{\mathrm{T}}z_i}{g_i(x)-x^{\mathrm{T}}v_i}\right)&\geq\sum_{i=1}^{k}\lambda_i\left(\frac{f_i(u)+u^{\mathrm{T}}z_i}{g_i(u)-u^{\mathrm{T}}v_i}\right)+\sum_{j=1}^{m}\mu_j\left(h_j(u)+u^{\mathrm{T}}w_j\right)\\&\quad+\sum_{i=1}^{k}\lambda_i\left(\bar{H}_i(u,p)-p^{\mathrm{T}}\nabla_p\bar{H}_i(u,p)\right)\\&\quad+\sum_{j=1}^{m}\mu_j\left(G_j(u,p)-p^{\mathrm{T}}\nabla_pG_j(u,p)\right).\end{aligned}$$

This contradicts Equation (24). Hence, the result.

Theorem 4.2. (Strong duality theorem). If $\bar{u}\in X^0$ is an efficient solution of (MFP) and the Slater's constraint qualification holds. Also, if for any
$i=1,2,\cdots,k,\ j=1,2,\cdots,m$,

$$\bar{H}_i\left(\bar{u},0\right)=0, G_j\left(\bar{u},0\right)=0, \nabla_p \bar{H}_i\left(\bar{u},0\right)=0, \nabla_p G_j\left(\bar{u},0\right)=0, \tag{29}$$

then there exist $\bar{\lambda}\in R^k, \bar{\mu}\in R^m, \bar{z}_i\in R^n, \bar{v}_i\in R^n$ and $\bar{w}_j\in R^n, i=1,2,\cdots,k, j=1,2,\cdots,m$, such that $\left(u,\bar{z},\bar{v},\bar{\mu},\bar{\lambda},\bar{w},\bar{p}=0\right)$ is a feasible solution of $(\text{MFD})_1$ and the objective function values of (MFP) and $(\text{MFD})_1$ are equal. Furthermore, if the hypotheses of Theorem 4.1 hold for all feasible solutions of (MFP) and $(\text{MFD})_1$ then, $\left(\bar{u},\bar{z},\bar{v},\bar{\mu},\bar{\lambda},\bar{w},\bar{p}=0\right)$ is an efficient solution of $(\text{MFD})_1$.

Proof. Since $\bar{u}$ is an efficient solution of (MFP) and the Slater's constraint qualification holds, then by Theorem 3.3, there exist $\bar{\lambda}\in R^k, \bar{\mu}\in R^m, \bar{z}_i\in R^n, \bar{v}_i\in R^n$ and $\bar{w}_j\in R^n, i=1,2,\cdots,k, j=1,2,\cdots,m$, such that

$$\sum_{i=1}^{k}\bar{\lambda}_i\nabla\left(\frac{f_i\left(\bar{u}\right)+\bar{u}^{\mathrm{T}}\bar{z}_i}{g_i\left(\bar{u}\right)-\bar{u}^{\mathrm{T}}\bar{v}_i}\right)+\sum_{j=1}^{m}\bar{\mu}_j\nabla\left(h_j\left(\bar{u}\right)+\bar{u}^{\mathrm{T}}\bar{w}_j\right)=0, \tag{30}$$

$$\sum_{j=1}^{m}\bar{\mu}_j\left(h_j\left(\bar{u}\right)+\bar{u}^{\mathrm{T}}\bar{w}_j\right)=0, \tag{31}$$

$$\bar{u}^{\mathrm{T}}\bar{z}_i=S\left(\bar{u}\mid C_i\right), \bar{u}^{\mathrm{T}}\bar{v}_i=S\left(\bar{u}\mid D_i\right), \bar{u}^{\mathrm{T}}\bar{w}_j=S\left(\bar{u}\mid E_j\right), \tag{32}$$

$$\bar{z}_i\in C_i,\ \bar{v}_i\in D_i,\ \bar{w}_j\in E_j, \tag{33}$$

$$\bar{\lambda}_i>0,\ \sum_{i=1}^{k}\bar{\lambda}_i=1,\ \bar{\mu}_j\geq 0,\ i=1,2,\cdots,k,\ j=1,2,\cdots,m. \tag{34}$$

Thus, $\left(\bar{u},\bar{z},\bar{v},\bar{\mu},\bar{\lambda},\bar{w},\bar{p}=0\right)$ is feasible for $(\text{MFD})_1$ and the objective function values of (MFP) and $(\text{MFD})_1$ are equal.

We now show that $\left(\bar{u},\bar{z},\bar{v},\bar{\mu},\bar{\lambda},\bar{w},\bar{p}=0\right)$ is an efficient solution of $(\text{MFD})_1$. If not, then there exists $\left(u',z',v',\mu',\lambda',w',p'=0\right)$ of $(\text{MFD})_1$ such that

$$\begin{aligned}&\frac{f_i\left(\bar{u}\right)+\bar{u}^{\mathrm{T}}\bar{z}_i}{g_i\left(\bar{u}\right)-\bar{u}^{\mathrm{T}}\bar{v}_i}+\sum_{j=1}^{m}\bar{\mu}_j\left(h_j\left(\bar{u}\right)+\bar{u}^{\mathrm{T}}\bar{w}_j\right)\\&\leq\frac{f_i\left(u'\right)+u'^{\mathrm{T}}z_i'}{g_i\left(u'\right)-u'^{\mathrm{T}}v_i'}+\sum_{j=1}^{m}\mu_j'\left(h_j\left(u'\right)+u'^{\mathrm{T}}w_j'\right),\ \text{for all } i=1,2,\cdots,k\end{aligned}$$

and

$$\begin{aligned}&\frac{f_r\left(\bar{u}\right)+\bar{u}^{\mathrm{T}}\bar{z}_r}{g_r\left(\bar{u}\right)-\bar{u}^{\mathrm{T}}\bar{v}_r}+\sum_{j=1}^{m}\bar{\mu}_j\left(h_j\left(\bar{u}\right)+\bar{u}^{\mathrm{T}}\bar{w}_j\right)\\&<\frac{f_r\left(u'\right)+u'^{\mathrm{T}}z_r'}{g_r\left(u'\right)-u'^{\mathrm{T}}v_r'}+\sum_{j=1}^{m}\mu_j'\left(h_j\left(u'\right)+u'^{\mathrm{T}}w_j'\right),\ \text{for some } r=1,2,\cdots,k.\end{aligned}$$

By equation (31), we obtain

$$\frac{f_i\left(\bar{u}\right)+\bar{u}^{\mathrm{T}}\bar{z}_i}{g_i\left(\bar{u}\right)-\bar{u}^{\mathrm{T}}\bar{v}_i}\leq\frac{f_i\left(u'\right)+u'^{\mathrm{T}}z_i'}{g_i\left(u'\right)-u'^{\mathrm{T}}v_i'}+\sum_{j=1}^{m}\mu_j'\left(h_j\left(u'\right)+u'^{\mathrm{T}}w_j'\right), \text{for all } i=1,2,\cdots,k$$

and

$$\frac{f_r\left(\bar{u}\right)+\bar{u}^{\mathrm{T}}\bar{z}_r}{g_r\left(\bar{u}\right)-\bar{u}^{\mathrm{T}}\bar{v}_r}<\frac{f_r\left(u'\right)+u'^{\mathrm{T}}z_r'}{g_r\left(u'\right)-u'^{\mathrm{T}}v_r'}+\sum_{j=1}^{m}\mu_j'\left(h_j\left(u'\right)+u'^{\mathrm{T}}w_j'\right), \text{for some } r=1,2,\cdots,k.$$

This contradicts the Theorem 4.1. This complete the result.

Theorem 4.3. (Strict converse duality theorem). Let $\bar{x} \in X^0$ and $\left(\bar{u},\bar{z},\bar{v},\bar{\mu},\bar{\lambda},\bar{w},\bar{p}\right) \in Z^0$. Let

1)
$$\sum_{i=1}^{k}\bar{\lambda}_i\left(\frac{f_i(\bar{x})+\bar{x}^T\bar{z}_i}{g_i(\bar{x})-\bar{x}^T\bar{v}_i}\right) \le \sum_{i=1}^{k}\bar{\lambda}_i\left(\frac{f_i(\bar{u})+\bar{u}^{\mathrm{T}}\bar{z}_i}{g_i(\bar{u})-\bar{u}^{\mathrm{T}}\bar{v}_i}\right)+\sum_{j=1}^{m}\bar{\mu}_j\left(h_j(\bar{u})+\bar{u}^{\mathrm{T}}\bar{w}_j\right)$$
$$+\sum_{i=1}^{k}\bar{\lambda}_i\left(\bar{H}_i(\bar{u},\bar{p})-\bar{p}^{\mathrm{T}}\nabla_p\bar{H}_i(\bar{u},\bar{p})\right)$$
$$+\sum_{j=1}^{m}\bar{\mu}_j\left(G_j(\bar{u},\bar{p})-\bar{p}^{\mathrm{T}}\nabla_pG_j(\bar{u},\bar{p})\right),$$

2) for any $i=1,2,\cdots,k$, $\left(f_i(.)+(.)^{\mathrm{T}}\bar{z}_i\right)$ be strictly higher-order $\left(V,\alpha_i^1,\beta_i^1,\rho_i^1,\theta_i^1\right)$-invex at $\bar{u}$ with respect to $H_i(\bar{u},\bar{p})$ and $-\left(g_i(.)+(.)^{\mathrm{T}}\bar{v}_i\right)$ be higher-order $\left(V,\alpha_i^1,\beta_i^1,\rho_i^1,\theta_i^1\right)$-invex at $\bar{u}$ with respect to $H_i(\bar{u},\bar{p})$,

3) for any $j=1,2,\cdots,m$, $\left(h_j(.)+(.)^{\mathrm{T}}w_j\right)$ be higher-order $\left(V,\alpha_j^2,\beta_j^2,\rho_j^2,\theta_j^2\right)$-invex at $\bar{u}$ with respect to $G_j(\bar{u},\bar{p})$,

4) $\sum_{i=1}^{k}\bar{\lambda}_i\bar{\rho}_i^1\left\|\bar{\theta}_i^1(\bar{x},\bar{u})\right\|^2+\sum_{j=1}^{m}\bar{\mu}_j\rho_j^2\left\|\theta_j^2(\bar{x},\bar{u})\right\|^2 \ge 0.$

5) $\bar{\alpha}_i^1(\bar{x},\bar{u})=\alpha_j^2(\bar{x},\bar{u})=\beta_i^1(\bar{x},\bar{u})=\beta_j^2(\bar{x},\bar{u})=\alpha(\bar{x},\bar{u}), \forall i=1,2,\cdots,k,$ $j=1,2,\cdots,m.$

Then, $\bar{x}=\bar{u}$.

Proof. Using hypothesis 2) and Theorem 3.2, we have

$$\bar{\alpha}_i^1(\bar{x},\bar{u})\left[\frac{f_i(\bar{x})+\bar{x}^{\mathrm{T}}\bar{z}_i}{g_i(\bar{x})-\bar{x}^{\mathrm{T}}\bar{v}_i}-\frac{f_i(\bar{u})+\bar{u}^{\mathrm{T}}\bar{z}_i}{g_i(\bar{u})-\bar{u}^{\mathrm{T}}\bar{v}_i}\right]$$
$$>\eta^{\mathrm{T}}(\bar{x},\bar{u})\left[\nabla\left(\frac{f_i(\bar{u})+\bar{u}^{\mathrm{T}}\bar{z}_i}{g_i(\bar{u})-\bar{u}^{\mathrm{T}}\bar{v}_i}\right)+\nabla_p\bar{H}_i(\bar{u},\bar{p})\right] \tag{35}$$
$$+\bar{\beta}_i^1(\bar{x},\bar{u})\left[\bar{H}_i(\bar{u},\bar{p})-\bar{p}^{\mathrm{T}}\nabla_p\bar{H}_i(\bar{u},\bar{p})\right]+\bar{\rho}_i^1\left\|\bar{\theta}_i^1(\bar{x},\bar{u})\right\|^2.$$

For any $j=1,2,\cdots,m$, $\left(h_j(.)+(.)^{\mathrm{T}}w_j\right)$ is higher-order $\left(V,\alpha_j^2,\beta_j^2,\rho_j^2,\theta_j^2\right)$-invex at u with respect to $G_j(\bar{u},\bar{p})$, we have

$$\alpha_j^2(\bar{x},\bar{u})\left[h_j(\bar{x})+\bar{x}^{\mathrm{T}}\bar{w}_j-\left(h_j(\bar{u})+\bar{u}^{\mathrm{T}}\bar{w}_j\right)\right]$$
$$\ge\eta^{\mathrm{T}}(\bar{x},\bar{u})\left[\nabla\left(h_j(\bar{u})+\bar{u}^{\mathrm{T}}\bar{w}_j\right)+\nabla_pG_j(\bar{u},\bar{p})\right] \tag{36}$$
$$+\beta_j^2(\bar{x},\bar{u})\left[G_j(\bar{u},\bar{p})-\bar{p}^{\mathrm{T}}\nabla_pG_j(\bar{u},\bar{p})\right]+\rho_j^2\left\|\theta_j^2(\bar{x},\bar{u})\right\|^2.$$

Adding the two inequalities after multiplying (35) by $\bar{\lambda}_i$ and (36) by $\bar{\mu}_j$, we obtain

$$\sum_{i=1}^{k}\bar{\lambda}_i\bar{\alpha}_i^1(\bar{x},\bar{u})\left[\frac{f_i(\bar{x})+\bar{x}^{\mathrm{T}}\bar{z}_i}{g_i(\bar{x})-\bar{x}^{\mathrm{T}}\bar{v}_i}-\frac{f_i(\bar{u})+\bar{u}^{\mathrm{T}}\bar{z}_i}{g_i(\bar{u})-\bar{u}^{\mathrm{T}}\bar{v}_i}\right]$$
$$+\sum_{j=1}^{m}\bar{\mu}_j\alpha_j^2(\bar{x},\bar{u})\left[h_j(\bar{x})+\bar{x}^{\mathrm{T}}\bar{w}_j-\left(h_j(\bar{u})+\bar{u}^{\mathrm{T}}\bar{w}_j\right)\right]$$
$$>\eta^{\mathrm{T}}(\bar{x},\bar{u})\sum_{i=1}^{k}\bar{\lambda}_i\left[\nabla\left(\frac{f_i(\bar{u})+\bar{u}^{\mathrm{T}}\bar{z}_i}{g_i(\bar{u})-\bar{u}^{\mathrm{T}}\bar{v}_i}\right)-\nabla_pH_i(\bar{u},\bar{p})\right]$$

$$
\begin{aligned}
&+\eta^{\mathrm{T}}(\bar{x},\bar{u})\sum_{j=1}^{m}\bar{\mu}_j\left[\nabla\left(h_j(\bar{u})+\bar{u}^{\mathrm{T}}\bar{w}_j\right)+\nabla_p G_j(\bar{u},\bar{p})\right]\\
&+\sum_{i=1}^{k}\bar{\lambda}_i\beta_i^1(\bar{x},\bar{u})\left[\bar{H}_i(\bar{u},\bar{p})-\bar{p}^{\mathrm{T}}\nabla_p\bar{H}_i(\bar{u},\bar{p})\right]\\
&+\sum_{j=1}^{m}\bar{\mu}_j\beta_j^2(\bar{x},\bar{u})\left[G_j(\bar{u},\bar{p})-\bar{p}^{\mathrm{T}}\nabla_p G_j(\bar{u},\bar{p})\right]\\
&+\sum_{i=1}^{k}\bar{\lambda}_i\rho_i^1\left\|\theta_i^1(\bar{x},\bar{u})\right\|^2+\sum_{j=1}^{m}\bar{\mu}_j\rho_j^2\left\|\theta_j^2(\bar{x},\bar{u})\right\|^2.
\end{aligned}
\tag{37}
$$

Using hypothesis 3) and (21), we get

$$
\begin{aligned}
&\sum_{i=1}^{k}\bar{\lambda}_i\alpha_i^1(\bar{x},\bar{u})\left[\frac{f_i(\bar{x})+\bar{x}^{\mathrm{T}}\bar{z}_i}{g_i(\bar{x})-\bar{x}^{\mathrm{T}}\bar{v}_i}-\frac{f_i(\bar{u})+\bar{u}^{\mathrm{T}}\bar{z}_i}{g_i(\bar{u})-\bar{u}^{\mathrm{T}}\bar{v}_i}\right]\\
&+\sum_{j=1}^{m}\bar{\mu}_j\alpha_j^2(\bar{x},\bar{u})\left[h_j(\bar{x})+\bar{x}^{\mathrm{T}}\bar{w}_j-\left(h_j(\bar{u})+\bar{u}^{\mathrm{T}}\bar{w}_j\right)\right]\\
&>\sum_{i=1}^{k}\bar{\lambda}_i\beta_i^1(\bar{x},\bar{u})\left[\bar{H}_i(\bar{u},\bar{p})-\bar{p}^{\mathrm{T}}\nabla_p\bar{H}_i(\bar{u},\bar{p})\right]\\
&\quad+\sum_{j=1}^{m}\bar{\mu}_j\beta_j^2(\bar{x},\bar{u})\left[G_j(\bar{u},\bar{p})-\bar{p}^{\mathrm{T}}\nabla_p G_j(\bar{u},\bar{p})\right].
\end{aligned}
\tag{38}
$$

Finally, using hypothesis 4) and $\bar{x}$ is feasible solution for (MFP), it follows that

$$
\begin{aligned}
\sum_{i=1}^{k}\bar{\lambda}_i\left(\frac{f_i(\bar{x})+\bar{x}^{\mathrm{T}}\bar{z}_i}{g_i(\bar{x})-\bar{x}^{\mathrm{T}}\bar{v}_i}\right)&>\sum_{i=1}^{k}\bar{\lambda}_i\left(\frac{f_i(\bar{u})+\bar{u}^{\mathrm{T}}\bar{z}_i}{g_i(\bar{u})-\bar{u}^{\mathrm{T}}\bar{v}_i}\right)+\sum_{j=1}^{m}\bar{\mu}_j\left(h_j(\bar{u})+\bar{u}^{\mathrm{T}}\bar{w}_j\right)\\
&\quad+\sum_{i=1}^{k}\bar{\lambda}_i\left(\bar{H}_i(\bar{u},\bar{p})-\bar{p}^{\mathrm{T}}\nabla_p\bar{H}_i(\bar{u},\bar{p})\right)\\
&\quad+\sum_{j=1}^{m}\bar{\mu}_j\left(G_j(\bar{u},\bar{p})-\bar{p}^{\mathrm{T}}\nabla_p G_j(\bar{u},\bar{p})\right).
\end{aligned}
$$

This contradicts the hypothesis 1). Hence, the result.

5. Duality Model-II

Consider the following dual $(\mathrm{MFD})_2$ of (MFP): **$(\mathrm{MFD})_2$** Maximize

$$
\left[\frac{f_1(u)+u^{\mathrm{T}}z_1}{g_1(u)-u^{\mathrm{T}}v_1}+\sum_{j=1}^{m}\mu_j\left(h_j(u)+u^{\mathrm{T}}w_j\right),\cdots,\frac{f_k(u)+u^{\mathrm{T}}z_k}{g_k(u)-u^{\mathrm{T}}v_k}+\sum_{j=1}^{m}\mu_j\left(h_j(u)+u^{\mathrm{T}}w_j\right)\right]
$$

subject to

$$
\begin{aligned}
&\sum_{i=1}^{k}\lambda_i\nabla\left(\frac{f_i(u)+u^{\mathrm{T}}z_i}{g_i(u)-u^{\mathrm{T}}v_i}\right)+\sum_{j=1}^{m}\mu_j\nabla\left(h_j(u)+u^{\mathrm{T}}w_j\right)\\
&+\sum_{i=1}^{k}\lambda_i\nabla_p H_i(u,p)+\sum_{j=1}^{m}\mu_j\nabla_p G_j(u,p)=0,
\end{aligned}
\tag{39}
$$

$$
\sum_{i=1}^{k}\lambda_i\left(H_i(u,p)-p^{\mathrm{T}}\nabla_p H_i(u,p)\right)+\sum_{j=1}^{m}\mu_j\left(G_j(u,p)-p^{\mathrm{T}}\nabla_p G_j(u,p)\right)\geq 0, \tag{40}
$$

$$
z_i\in C_i,\ v_i\in D_i,\ w_j\in E_j,\ i=1,2,\cdots,k,\ j=1,2,\cdots,m, \tag{41}
$$

$$\mu_j \geq 0, \lambda_i > 0, \sum_{i=1}^{k} \lambda_i = 1,\ i = 1,2,\cdots,k,\ j = 1,2,\cdots,m. \quad (42)$$

Let P^0 be the feasible solution for (MFD)$_2$.

Theorem 5.1. (Weak duality theorem). Let $x \in X^0$ and $(u,z,v,y,\lambda,w,p) \in P^0$. Let for $i = 1,2,\cdots,k,\ j = 1,2,\cdots,m$,

1) $\left(\dfrac{f_i(.)+(.)^{\mathrm{T}} z_i}{g_i(.)-(.)^{\mathrm{T}} v_i}\right)$ be higher-order $\left(V,\alpha_i^1,\beta_i^1,\rho_i^1,\theta_i^1\right)$-invex at u with respect to $H_i(u,p)$,

2) $\left(h_j(.)+(.)^{\mathrm{T}} w_j\right)$ be higher-order $\left(V,\alpha_j^2,\beta_j^2,\rho_j^2,\theta_j^2\right)$-invex at u with respect to $G_j(u,p)$,

3) $\sum_{i=1}^{k}\lambda_i\rho_i^1\left\|\theta_i^1(x,u)\right\|^2+\sum_{j=1}^{m}\mu_j\rho_j^2\left\|\theta_j^2(x,u)\right\|^2 \geq 0.$

4) $\alpha_i^1(x,u)=\alpha_j^2(x,u)=\beta(x,u)=\beta_j^2(x,u)=\alpha(x,u).$

Then the following cannot hold

$$\frac{f_i(x)+S(x\,|\,C_i)}{g_i(x)-S(x\,|\,D_i)} \leq \frac{f_i(u)+u^{\mathrm{T}} z_i}{g_i(u)-u^{\mathrm{T}} v_i}+\sum_{j=1}^{m}\mu_j\left(h_j(u)+u^{\mathrm{T}} w_j\right), \forall\, i=1,2,\cdots,k \quad (43)$$

and

$$\frac{f_r(x)+S(x\,|\,C_r)}{g_r(x)-S(x\,|\,D_r)} < \frac{f_r(u)+u^{\mathrm{T}} z_r}{g_r(u)-u^{\mathrm{T}} v_r}+\sum_{j=1}^{m}\mu_j\left(h_j(u)+u^{\mathrm{T}} w_j\right), \text{ for some } r=1,2,\cdots,k. \quad (44)$$

Proof. The proof follows on the lines of Theorem 4.1.

Theorem 5.2 (Strong duality theorem). If $\bar{u} \in X^0$ is an efficient solution of (MFP) and the Slater's constraint qualification hold. Also, if for any $i = 1,2,\cdots,k,\ j = 1,2,\cdots,m$,

$$H_i(\bar{u},0)=0, G_j(\bar{u},0)=0, \nabla_p H_i(\bar{u},0)=0, \nabla_p G_j(\bar{u},0)=0, \quad (45)$$

then there exist $\bar{\lambda} \in R^k, \bar{\mu} \in R^m, \bar{z}_i \in R^n, \bar{v}_i \in R^n$ and $\bar{w}_j \in R^n, i=1,2,\cdots,k,\ j=1,2,\cdots,m$, such that $\left(u,\bar{z},\bar{v},\bar{\mu},\bar{\lambda},\bar{w},\bar{p}=0\right)$ is a feasible solution of (MFD)$_2$ and the objective function values of (MFP) and (MFD)$_2$ are equal. Furthermore, if the conditions of Theorem 5.1 hold for all feasible solu- tions of (MFP) and (MFD)$_2$ then, $\left(u,\bar{z},\bar{v},\bar{\mu},\bar{\lambda},\bar{w},\bar{p}=0\right)$ is an efficient solution of (MFD)$_2$.

Proof. The proof follows on the lines of Theorem 4.2.

Theorem 5.3. (Strict converse duality theorem). Let $\bar{x} \in X^0$ and $\left(\bar{u},\bar{z},\bar{v},\bar{\mu},\bar{\lambda},\bar{w},\bar{p}\right) \in P^0$. Let $i = 1,2,\cdots,k,\ j = 1,2,\cdots,m$,

1) $\sum_{i=1}^{k}\bar{\lambda}_i\left(\dfrac{f_i(\bar{x})+\bar{x}^{\mathrm{T}}\bar{z}_i}{g_i(\bar{x})-\bar{x}^{\mathrm{T}}\bar{v}_i}\right) \leq \sum_{i=1}^{k}\bar{\lambda}_i\left(\dfrac{f_i(\bar{u})+\bar{u}^{\mathrm{T}}\bar{z}_i}{g_i(\bar{u})-\bar{u}^{\mathrm{T}}\bar{v}_i}\right)+\sum_{j=1}^{m}\bar{\mu}_j\left(h_j(\bar{u})+\bar{u}^{\mathrm{T}}\bar{w}_j\right),$

2) $\left(\dfrac{f_i(.)+(.)^{\mathrm{T}}\bar{z}_i}{g_i(.)-(.)^{\mathrm{T}}\bar{v}_i}\right)$ be strictly higher-order $\left(V,\alpha_i^1,\beta_i^1,\rho_i^1,\theta_i^1\right)$-invex at $\bar{u}$ with respect to $H_i(\bar{u},\bar{p})$,

3) $\left(h_j(.)+(.)^{\mathrm{T}} w_j\right)$ be higher-order $\left(V,\alpha_j^2,\beta_j^2,\rho_j^2,\theta_j^2\right)$-invex at $\bar{u}$ with respect to $G_j(\bar{u},\bar{p})$,

4) $\sum_{i=1}^{k}\bar{\lambda}_i\rho_i^1\left\|\theta_i^1(\bar{x},\bar{u})\right\|^2+\sum_{j=1}^{m}\bar{\mu}_j\rho_j^2\left\|\theta_j^2(\bar{x},\bar{u})\right\|^2\geq 0.$

5) $\alpha_i^1(\bar{x},\bar{u})=\alpha_j^2(\bar{x},\bar{u})=\beta_i^1(\bar{x},\bar{u})=\beta_j^2(\bar{x},\bar{u})=\alpha(\bar{x},\bar{u}).$

Then, $\bar{x}=\bar{u}$.

Proof. The proof follows on the lines of Theorem 4.3.

6. Duality Model-III

Consider the following dual $(\mathrm{MFD})_3$ of (MFP): $\mathbf{(MFD)_3}$ Maximize

$$\left[\frac{f_1(u)+u^{\mathrm{T}}z_1}{g_1(u)-u^{\mathrm{T}}v_1}+\left(\bar{H}_1(u,p)-p^{\mathrm{T}}\nabla_p\bar{H}_1(u,p)\right),\cdots,\right.$$
$$\left.\frac{f_k(u)+u^{\mathrm{T}}z_k}{g_k(u)-u^{\mathrm{T}}v_k}+\left(\bar{H}_k(u,p)-p^{\mathrm{T}}\nabla_p\bar{H}_k(u,p)\right)\right]$$

subject to

$$\sum_{i=1}^{k}\lambda_i\nabla\left(\frac{f_i(u)+u^{\mathrm{T}}z_i}{g_i(u)-u^{\mathrm{T}}v_i}\right)+\sum_{j=1}^{m}\mu_j\nabla\left(h_j(u)+u^{\mathrm{T}}w_j\right)$$
$$+\sum_{i=1}^{k}\lambda_i\nabla_p\bar{H}_i(u,p)+\sum_{j=1}^{m}\mu_j\nabla_pG_j(u,p)=0, \tag{46}$$

$$\sum_{j=1}^{m}\mu_j\left[h_j(u)+u^{\mathrm{T}}w_j+G_j(u,p)-p^{\mathrm{T}}\nabla_pG_j(u,p)\right]\geq 0, \tag{47}$$

$$z_i\in C_i, v_i\in D_i, w_j\in E_j, i=1,2,\cdots,k,\ j=1,2,\cdots,m, \tag{48}$$

$$\mu_j\geq 0, \lambda_i>0, \sum_{i=1}^{k}\lambda_i=1, i=1,2,\cdots,k,\ j=1,2,\cdots,m. \tag{49}$$

Let S^0 be feasible solution of $(\mathrm{MFD})_3$.

Theorem 6.1. (Weak duality theorem). Let $x\in X^0$ and $(u,z,v,\mu,\lambda,w,p)\in S^0$. Let $i=1,2,\cdots,k,\ j=1,2,\cdots,m$,

1) $\left(f_i(.)+(.)^{\mathrm{T}}z_i\right)$ and $-\left(g_i(.)-(.)^{\mathrm{T}}v_i\right)$ be higher-order $\left(V,\alpha_i^1,\beta_i^1,\rho_i^1,\theta_i^1\right)$-invex at u with respect to $H_i(u,p)$,

2) $\left(h_j(.)+(.)^{\mathrm{T}}w_j\right)$ be higher-order $\left(V,\alpha_j^2,\beta_j^2,\rho_j^2,\theta_j^2\right)$-invex at u with respect to $G_j(u,p)$,

3) $\sum_{i=1}^{k}\lambda_i\bar{\rho}_i^1\left\|\bar{\theta}_i^1(x,u)\right\|^2+\sum_{j=1}^{m}\mu_j\rho_j^2\left\|\theta_j^2(x,u)\right\|^2\geq 0.$

4) $\bar{\alpha}_i^1(x,u)=\alpha_j^2(x,u)=\beta_i^1(x,u)=\beta_j^2(x,u)=\alpha(x,u)$,

where

$$\bar{\alpha}_t(x,u)=\left(\frac{g_t(x)-x^{\mathrm{T}}v_t}{g_t(u)-u^{\mathrm{T}}v_t}\right)\alpha_t(x,u),\quad \bar{\beta}_t(x,u)=\beta_t(x,u),$$

$$\bar{\theta}_t(x,u)=\theta_t(x,u)\left(\frac{1}{g_t(u)-u^{\mathrm{T}}v_t}+\frac{f_t(u)+u^{\mathrm{T}}z_t}{\left(g_t(u)-u^{\mathrm{T}}v_t\right)^2}\right)^{\frac{1}{2}},\quad \bar{\rho}_t(x,u)=\rho_t(x,u)$$

and

$$\bar{H}_t(u,p)=\left(\frac{1}{g_t(u)-u^{\mathrm{T}}v_t}+\frac{f_t(u)+u^{\mathrm{T}}z_t}{\left(g_t(u)-u^{\mathrm{T}}v_t\right)^2}\right)H_t(u,p).$$

Then, the following cannot hold

$$\begin{aligned}&\frac{f_i(x)+S(x\mid C_i)}{g_i(x)-S(x\mid D_i)}\\&\le\frac{f_i(u)+u^{\mathrm{T}}z_i}{g_i(u)-u^{\mathrm{T}}v_i}+\left(\bar{H}_i(u,p)-p^{\mathrm{T}}\nabla_p\bar{H}_i(u,p)\right),\text{ for all } i=1,2,\cdots,k\end{aligned}\tag{50}$$

and

$$\begin{aligned}&\frac{f_r(x)+S(x\mid C_r)}{g_r(x)-S(x\mid D_r)}\\&<\frac{f_r(u)+u^{\mathrm{T}}z_r}{g_r(u)-u^{\mathrm{T}}v_r}+\left(\bar{H}_r(u,p)-p^{T}\nabla_p\bar{H}_r(u,p)\right),\text{ for some } r=1,2,\cdots,k.\end{aligned}\tag{51}$$

Proof. The proof follows on the lines of Theorem 4.1.

Theorem 6.2. (Strong duality theorem). If $\bar{u}\in X^0$ is an efficient solution of (MFP) and let the Slater's constraint qualification be satisfied. Also, if for any $i=1,2,\cdots,k,\ j=1,2,\cdots,m$,

$$\bar{H}_i(\bar{u},0)=0,G_j(\bar{u},0)=0,\nabla_p\bar{H}_i(\bar{u},0)=0,\nabla_pG_j(\bar{u},0)=0,\tag{52}$$

then there exist $\bar{\lambda}\in R^k,\bar{\mu}\in R^m,\bar{z}_i\in R^n,\bar{v}_i\in R^n$ and $\bar{w}_j\in R^n, i=1,2,\cdots,k,\ j=1,2,\cdots,m$, such that $\left(u,\bar{z},\bar{v},\bar{\mu},\bar{\lambda},\bar{w},\bar{p}=0\right)$ is a feasible solution of $(\mathrm{MFD})_3$ and the objective function values of (MFP) and $(\mathrm{MFD})_3$ are equal. Furthermore, if the conditions of Theorem 6.1 hold for all feasible solutions of (MFP) and $(\mathrm{MFD})_3$ then, $\left(u,\bar{z},\bar{v},\bar{\mu},\bar{\lambda},\bar{w},\bar{p}=0\right)$ is an efficient solution of $(\mathrm{MFD})_3$.

Proof. The proof follows on the lines of Theorem 4.2.

Theorem 6.3. (Strict converse duality theorem). Let $\bar{x}\in X^0$ and $\left(\bar{u},\bar{z},\bar{v},\bar{\mu},\bar{\lambda},\bar{w},\bar{p}\right)$ be feasible for $(\mathrm{MFD})_3$. Suppose that:

1)

$$\sum_{i=1}^k\bar{\lambda}_i\left(\frac{f_i(\bar{x})+\bar{x}^{\mathrm{T}}\bar{z}_i}{g_i(\bar{x})-\bar{x}^{\mathrm{T}}\bar{v}_i}\right)\le\sum_{i=1}^k\bar{\lambda}_i\left(\frac{f_i(\bar{u})+\bar{u}^{\mathrm{T}}\bar{z}_i}{g_i(\bar{u})-\bar{u}^{\mathrm{T}}\bar{v}_i}\right)+\sum_{i=1}^k\bar{\lambda}_i\left(\bar{H}_i(\bar{x},\bar{u})-\bar{p}^T\nabla_p\bar{H}(\bar{x},\bar{u})\right),$$

2) for any $i=1,2,\cdots,k$, $\left(f_i(.)+(.)^{\mathrm{T}}\bar{z}_i\right)$ be strictly higher-order $\left(V,\alpha_i^1,\beta_i^1,\rho_i^1,\theta_i^1\right)$-invex at $\bar{u}$ with respect to $H_i(\bar{u},\bar{p})$ and $-\left(g_i(.)+(.)^{\mathrm{T}}\bar{v}_i\right)$ be higher-order $\left(V,\alpha_i^1,\beta_i^1,\rho_i^1,\theta_i^1\right)$-invex at $\bar{u}$ with respect to $H_i(\bar{u},\bar{p})$,

3) for any $j=1,2,\cdots,m$, $\left(h_j(.)+(.)^{\mathrm{T}}w_j\right)$ is higher-order $\left(V,\alpha_j^2,\beta_j^2,\rho_j^2,\theta_j^2\right)$-invex at $\bar{u}$ with respect to $G_j(\bar{u},\bar{p})$,

4) $\sum_{i=1}^k\bar{\lambda}_i\bar{\rho}_i^1\left\|\bar{\theta}_i^1(\bar{x},\bar{u})\right\|^2+\sum_{j=1}^m\bar{\mu}_j\rho_j^2\left\|\theta_j^2(\bar{x},\bar{u})\right\|^2\ge 0.$

5) $\bar{\alpha}_i^1(\bar{x},\bar{u})=\alpha_j^2(\bar{x},\bar{u})=\beta_i^1(\bar{x},\bar{u})=\beta_j^2(\bar{x},\bar{u})=\alpha(\bar{x},\bar{u}),\forall i=1,2,\cdots,k,$ $j=1,2,\cdots,m.$

Then, $\bar{x} = \bar{u}$.

Proof. The proof follows on the lines of Theorem 4.3.

7. Conclusion

In this paper, we consider a class of non differentiable multiobjective fractional programming (MFP) with higher-order terms in which each numerator and denominator of the objective function contains the support function of a compact convex set. Furthermore, various duality models for higher-order have been formulated for (MFP) and appropriate duality relations have been obtained under higher-order $(V, \alpha, \beta, \rho, d)$-invexity assumptions.

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