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Approaches to Solve MID_CPLP Problem: Theoretical Framework and Empirical Investigation

Renduchintala Raghavendra Kumar Sharma[1], Pritee Agarwal[2]

[1]Department of Industrial and Management Engineering, Indian Institute of Technology, Kanpur, India
[2]Department of Mathematics and Statistics, Indian Institute of Technology, Kanpur, India
Email: rrks@iitk.ac.in, pritee.iitk@gmail.com, pritee@iitk.ac.in

Abstract

In two-stage warehouse location problem, goods are moved from plants to warehouses at stage-1 (which are larger sized warehouses), and from there to warehouses at stage-2 (which are smaller sized warehouses); and finally to the markets. We aim to minimize the sum of location costs of the warehouses at stage-1 and stage-2; plus the total distribution cost of goods to the markets. In this paper two-stage capacitated warehouse location problem (TSCWLP) is vertically decomposed into the smaller problems, which is attained by relaxing the associated flow balance constraints. This leads to three different versions of Capacitated Plant Location Problem (CPLP) referred as RHS_CPLP, MID_CPLP and LHS_CPLP (Verma and Sharma [1]). In this paper MID_CPLP is reduced to RHS_CPLP and a single constraint 0-1 Knapsack problem by relaxing a difficult constraint. Interesting results and conjectures are given. Later two more valid constraints are added to MID_CPLP which are relaxed further to get additional results.

Keywords

TSCWLP, Location, Transportation, Distribution

1. Introduction and Literature Survey

In country like large geographical area, such as India, distribution of goods such as fertilizer, cement, edible oil, sugar, food grains etc, is close in stages. From large sized plants, goods are moved to larger sized warehouses, from larger sized warehouses to smaller sized warehouses and finally to markets/customers. It is to be decided that where to locate warehouses (among potential locations for large sized warehouses as well as smaller sized

warehouses). It is required to minimize sum total of cost of location of warehouse and cost of distribution of goods. Stage wise location-distribution has been attempted by Geoffrion and Graves [2], Sharma [3], Sharma and Berry [4] and Verma and Sharma [5]. In this paper we attempt a two-stage warehouse location problem and alongside give a novel method to solve MID_CPLP that results due to vertical decomposition developed for solution. Facility location problems form an important class of integer programming problems, with application in the distribution and transportation industries. In any organization, one of the most important strategic decisions is locating facilities viz. factories, plants or warehouses and their suitable stages. While locating a facility, the most prioritized criterion is a good service level. However, achievement of an economic optimality is also a key decisive factor. Based on the number of stages between the producing facility and the market, there are different types of facility location problems like plant location problem, single-stage/two-stage warehouse location problem. A review of the literature on location problems can be found in (Sahin and Sural [6], ReVelle and Eiselt [7], ReVelle, Eiselt and Daskin [8] and Brandeau and Chiu [9]). Multistage warehouse location problems are frequently occurring in real life; see (Geoffrion and Graves [2], Sharma [3] and Sharma [10]). TSCWLP problem has been studied by Verma and Sharma [1]. In Section 2, we give formulation of TSCWLP, few relaxations of MID_CPLP; and also give a few results. In the Section 3 we give empirical investigation and related statistical significance. Finally in Section 4 we conclude.

2. Formulation and Relaxations of TSCWLP

2.1. Problem Formulation of TSCWLP

In this section, we propose the formulation of TSCWLP using the style of Sharma [3], Sharma and Namdeo [11], Sharma and Sharma [12]. Verma and Sharma [5] developed a variety of constraints that link real and 0-1 integer variables. They have also developed some strong constraints based on Sharma and Berry [4].

2.1.1. Index Used
h : set of supply points (plants); $h = 1, \cdots, H$
i : set of potential warehouse points at stage 1; $i = 1, \cdots, I$
j : set of potential warehouse points at stage 2; $j = 1, \cdots, J$
k : set of markets; $k = 1, \cdots, K$

2.1.2. Definition of Constants
D_k : Demand for the commodity at market "k"
d_k : $D_k / \sum D_k$, Demand at market "k" as a fraction of total market demand
S_h : Supply available at plant "h"
s_j : $S_h / \sum D_k$, Supply available at plant 'h' as a fraction of the total market demand
$fws1_i$: Fixed cost of locating a warehouse at "i"
$fws2_j$: Fixed cost of locating a warehouse at "j"
$CAPWS1_i$: Capacity of a stage 1 warehouse "i"
$CAPWS2_j$: Capacity of a stage 2 warehouse "j"
$capws1_i$: $CAPWS1_i / \sum D_k$, Capacity of whs-1 at "i" as a fraction of the total market demand
$capws2_j$: $CAPWS2_i / \sum D_k$ Capacity of whs-2 at "j" as a fraction of the total market demand
$cpws1_{hi}$: Cost of transporting $\sum D_k$ goods from plant "h" to whs-1 "i"
$cws1ws2_{ij}$: Cost of transporting $\sum D_k$ goods from whs-1 "i" to whs2 "j"
$cws2m_{jk}$: Cost of transporting $\sum D_k$ goods from whs-2 "j" to market "k"

2.1.3. Definition of Variables
$XPWS1_{hi}$: Quantity of commodity transported from plant "h" to whs-1 "i"
$xpws1_{hi}$: $XPWS1_{hi} / \sum D_k$, Quantity transported from "h" to "i" as fraction of total demand
$XWS1WS2_{ij}$: Quantity of commodity transported from whs-1 "i" to whs-2 "j"
$Xws1ws2_{ij}$: $XWS1WS2_{ij} / \sum D_k$, Quantity transported from "i" to "j" as fraction of total demand
$XWS2M_{jk}$: Quantity of commodity transported from whs-2 "j" to market "k"
$Xws2m_{jk}$: $XWS2M_{jk} / \sum D_k$, Quantity transported from "j" to "k" as fraction of total demand
$yws1_i$: 1 if stage 1 warehouse is located at "i", 0 otherwise

$yws2_j$: 1 if stage 2 warehouse is located at "j", 0 otherwise

2.1.4. Mathematical Formulation

The cost minimization problem for the TSCWLP can be written as mixed 0-1 integer linear programming problem as given in the formulation below.

Objective Function:

Minimize

$$Z = \sum_{h,i} xpws1_{hi} * cpws1_{hi} + \sum_{i,j} xws1ws2_{ij} * cws1ws2_{ij} + \sum_{j,k} xws2m_{hi} * cws2m_{jk} \\ + \sum_i fws1_i * yws1_i + \sum_j fws2_j * yws2_j \tag{1}$$

This can be written as:

Minimize $Z = f_1 + f_2 + f_3$, where

$$f_1 \left(xpws1_{hi}, cpws1_{hi}, yws1_i, fws1_i \right) = \min_{xpws1, yws1} \sum_{h,i} xpws1_{hi} * cpws1_{hi} + \frac{1}{2} \sum_i yws1_i * fws1_i$$

$$f_2 \left(xws1ws2_{ij}, cws1ws2_{ij}, yws1_i, yws2_j, fws1_i, fws2_j \right)$$

$$= \min_{xws1ws2, yws1, yws2} \sum_{i,j} xws1ws2_{ij} * cws1ws2_{ij} + \frac{1}{2} \sum_i yws1_i * fws1_i + \frac{1}{2} \sum_j yws2_j * fws2_j$$

$$f_3 \left(xws2m_{jk}, cws2m_{jk}, yws2_j, fws2_j \right) = \min_{xws2m, yws2} \sum_{j,k} xws2m_{jk} * cws2m_{jk} + \frac{1}{2} \sum_j yws2_j * fws2_j$$

Subject to:

$$\sum_{h,i} xpws1_{hi} = 1 \tag{2}$$

$$\sum_{i,j} xws1ws2_{ij} = 1 \tag{2a}$$

$$\sum_{j,k} xws2m_{jk} = 1 \tag{2b}$$

$$xpws1_{hi} \le yws1_i s_h, \ \forall h,i, \tag{3a}$$

$$xws2m_{jk} \le yws2_j d_k, \ \forall j,k, \tag{3b}$$

$$xpws1_{hi} \le yws1_i, \ \forall h,i, \tag{4a}$$

$$xws2m_{jk} \le yws2_j, \ \forall j,k, \tag{4b}$$

$$\sum_h xpws1_{hi} \le yws1_i capws1_i, \ \forall i, \tag{5}$$

$$\sum_j xws1ws2_{ij} \le yws1_i capws1_i, \ \forall i, \tag{5a-i}$$

$$\sum_i xws1ws2_{ij} \le yws2_j capws2_j, \ \forall j, \tag{5a-ii}$$

$$\sum_k xws2m_{jk} \le yws2_j capws2_j, \ \forall j, \tag{5b}$$

$$\sum_i xpws1_{hi} \le s_h, \ \forall h, \tag{6}$$

$$\sum_j xws1ws2_{ij} \le capws1_i, \ \forall i, \tag{6a-i}$$

$$\sum_i xws1ws2_{ij} \le capws2_j, \ \forall j, \tag{6a-ii}$$

$$\sum_j xws2m_{jk} = d_k, \ \forall k, \tag{6b}$$

$$xpws1_{hi} \geq 0, \ \forall h, i, \tag{7}$$

$$xws1ws2_{ij} \geq 0, \ \forall i, j, \tag{7a}$$

$$xws2m_{jk} \geq 0, \ \forall j, k, \tag{7b}$$

$$yws1_i = (0,1), \ \forall i, \tag{8a}$$

$$yws2_j = (0,1), \ \forall j, \tag{8b}$$

$$\sum_i yws1_i \, capws1_i \geq 1 \tag{9a}$$

$$\sum_j yws2_j \, capws2_j \geq 1 \tag{9b}$$

$$\sum_i capws1_i \geq 1 \tag{10a}$$

$$\sum_j capws2_j \geq 1 \tag{10b}$$

$$\sum_h xpws1_{hi} = \sum_k xws1ws2_{ij}, \ \forall i, \tag{11a}$$

$$\sum_i xws1ws2_{ij} = \sum_k xws2m_{jk}, \ \forall j, \tag{11b}$$

Since $\sum d_k = 1$, constraints 2, 2(a) and 2(b) ensure that flow across stages is equal to total demand by all the markets. Constraints 3(a) and 3(b) are strong linking constraints (see Sharma and Berry [4] and Verma and Sharma [5]). Equations 4(a) and 4(b) are weak linking constraints. Equations 5, 5(a-i), 5(a-ii) and 5(b) strong capacity constraints (see Sharma and Berry [4]). Equation 6 ensures the flow from plant is less than supply available at that plant. Equations 6(a-i) and 6(a-ii) ensure that throughput form a warehouse is less than or equal to its capacity. Equation 6(b) ensures that the quantity received at a market is equal to its demand. Equations 7, 7(a) and 7(b) are non-negativity restrictions on real variables. Equations 8(a) and 8(b) are 0-1 restrictions on binary location variables. Equations 9(a) and 9(b) ensure that located capacity is more than or equal to the total demand of the markets. Equations 10(a) and 10(b) ensure that total average capacity is more than market demand at each stage. Equations 11(a) and 11(b) are flow balance constraints (inflow is equal to outflow) at each of the warehouses. By relaxing different constraints, various relaxations can be obtained as Lagrangian relaxation (LR). LR is a relaxation technique, which works by moving hard constraints into the objective to impose a penalty on the objective if they are not satisfied. This is easier to solve than the original problem. An optimal objective value of the Lagrangian relaxed problem, for a given set of multipliers, provides a lower bound (in the case of minimization) for the optimal solution to the original problem. The best lower bound can be derived by updating the multipliers by a dual ascent procedure. An upper bound on the optimal solution of the original problem can be derived by using the information obtained from the LR to construct a feasible solution to the original problem. This is normally done by applying some heuristic. In the next section, we present vertical decomposition approach for solving TSCWLP and variety of LR and LP relaxations.

2.2. Lagrangian Relaxation of TSCWLP

Proceeding in a manner similar to Verma and Sharma [13], TSCWLP is also vertically decomposed into three sub problems viz. LHS_CPLP, MID_CPLP and RHS_CPLP. In this approach, stages of warehouses are vertically decomposed to get smaller sized problems, which are relatively easier to solve. 1st LR of this formulation is obtained by relaxing flow balance constraints (11(a)) and (11(b)) with suitable multipliers $\lambda 1$ and $\lambda 2$ respectively. In main problem, we note that constraint set (11(a)) connects the "$xpws1_{hi}$" and "$xws1ws2_{ij}$" variables. Similarly, constraint set (11(b)) connects the "$xws1ws2_{ij}$" and "$xws2m_{jk}$" variables. If these two sets of constraints are relaxed, main problem will be separated into three sub-problems. One problem with "$xpws1_{hi}$" and

"$yws1_i$", called as LHS_CPLP, the other with "$xws1ws2_{ij}$", "$yws1_i$" and "$yws2_j$" to be called as MID_CPLP, and the last problem with "$xws2m_{jk}$" and "$yws2_j$" called as RHS_CPLP. Verma and Sharma [13] have already attempted RHS_CPLP and LHS_CPLP. Here different relaxations of MID_CPLP are developed and relationship is obtained for them.

2.2.1. LHS_CPLP

$$Z_{LHS\_CPLP} = f_1\left(xpws1_{hi}, \left(cpws1_{hi} + \lambda 1_i\right), yws1_i, fws1_i\right)$$

$$= \min_{xpws1, yws1} \sum_{h,i} xpws1_{hi} * \left(cpws1_{hi} + \lambda 1_i\right) + \frac{1}{2}\sum_i yws1_i * fws1_i \tag{1(a)}$$

s.t.: (2), (3), (4), (5), (6), (7), (8(a)), (9(a)), (10(a)).

2.2.2. MID_CPLP

$$Z_{MID\_CPLP} = f_2\left(xws1ws2_{ij}, \left(cws1ws2_{ij} - \lambda 1_i + \lambda 2_j\right), yws1_i, yws2_j, fws1_i, fws2_j\right)$$

$$= \min_{xws1ws2, yws1, yws2} \sum_{i,j} xws1ws2_{ij} * \left(cws1ws2_{ij} - \lambda 1_i + \lambda 2_j\right) + \frac{1}{2}\sum_i yws1_i * fws1_i + \frac{1}{2}\sum_j yws2_j * fws2_j \tag{1(b)}$$

s.t.: (2(a)), (5(a-i)), (5(a-ii)), (6(a-i)), (6(a-ii)), (7(a)), (8(a)), (9(a)), (10(a)), (8(b)), (9(b)), (10(b)).

2.2.3. RHS_CPLP

$$Z_{RHS\_CPLP} = f_3\left(xws2m_{jk}, \left(cws2m_{jk} - \lambda 2_j\right), yws2_j, fws2_j\right)$$

$$= \min_{xws2m, yws2} \sum_{j,k} xws2m_{jk} * \left(cws2m_{jk} - \lambda 2_j\right) + \frac{1}{2}\sum_j yws2_j * fws2_j \tag{1(c)}$$

s.t.: (2(b)), (3(b)), (4(b)), (5(b)), (6(b)), (7(b)), (8(b)), (9(b)) and (10(b)).

2.2.4. Original relaxed TSCWLP

$$Z_{ori} = \max_{\lambda 1_i, \lambda 2_j}\left(Z_{LHS\_CPLP} + Z_{MID\_CPLP} + Z_{RHS\_CPLP}\right) \tag{1(d)}$$

s.t.: (2)-(7), (2(a)), (5(a-i)),(5(a-ii)), (6(a-i)), (6(a-ii)), (7(a))-(10(a)), (2(b))-(10(b)).
We focus on MID_CPLP in this chapter.

2.3. A Few Theoretical Results on *MID_CPLP*

We relax constraint (5(a-ii)) by attaching multiplier $\lambda 3_j$ to get,

2.3.1. MID_CPLP ($\lambda 3_j$)

$$Z_{MID\_CPLP} = f_2\left(xws1ws2_{ij}, \left(cws1ws2_{ij} - \lambda 1_i + \lambda 2_j + \lambda 3_j\right), yws1_i, yws2_j, fws1_i, fws2_j\right)$$

$$= \min_{xws1ws2, yws1, yws2} \sum_{i,j} xws1ws2_{ij} * \left(cws1ws2_{ij} - \lambda 1_i + \lambda 2_j + \lambda 3_j\right) \tag{1(e)}$$

$$+ \frac{1}{2}\sum_i yws1_i * fws1_i + \frac{1}{2}\sum_j yws2_j * fws2_j - \sum_j capws2_j * yws2_j * \lambda 3_j$$

s.t.: (2(a)), (5(a-i)), (6(a-i)), (6(a-ii)), (7(a)), (8(a)), (9(a)), (10(a)), (8(b)), (9(b)), (10(b)).
$MID\_CPLP\left(\lambda 3_j\right)$ problem will be separated into two sub-problems: $CPLP\_R\left(\lambda 3_j\right)$ (Verma and Sharma [5]) and one 0-1 knapsack problem (single constraint).
1) CPLP_R ($\lambda 3_j$)

$$Z_{CPLP\_R} = \min_{xws1ws2, yws1, yws2} \sum_{i,j} xws1ws2_{ij} * \left(cws1ws2_{ij} - \lambda 1_i + \lambda 2_j + \lambda 3_j\right) + \frac{1}{2}\sum_i yws1_i * fws1_i \tag{1(f)}$$

s.t.: (2(a)), (5(a-i)), (6(a-i)), (6(a-ii)), (7(a)), (8(a)), (9(a)), (10(a)), (10(b)).
2) 0-1_KSP ($\lambda 3_j$)

$$Z_{0\text{-}1\_KSP} = \frac{1}{2} \sum_j yws2_j * fws2_j - \sum_j capws2_j * yws2_j * \lambda 3_j \tag{1(g)}$$

s.t.: (8(b)), (9(b)).

2.3.2. Relaxation of CPLP_R ($\lambda 3_j$)

Now $CPLP\_R(\lambda 3_j)$ is attempted in literature by three different relaxations. They are R1, R2 and R3 respectively. When integrality restriction on $yws1_i$ is relaxed we obtain (12(a)).

$$0 \le yws1_i \le 1, \ \forall i, \tag{12(a)}$$

1) CPLP_R1 ($\lambda 3_j$)
(1(f)); Subject to: (2(a)), (5(a-i)), (6(a-i)), (6(a-ii)), (7(a)), (10(a)), (10(b)) and (12(a)).
Result 1: R1 is equal to strong LP relaxation of $CPLP\_R(\lambda 3_j)$ (Sharma and Berry [4], Verma and Sharma [5]). Here the restriction on yws1ᵢ is relaxed by (12(a)).
Proof: It is easy to see.
2) CPLP_R2 ($\lambda 3_j$)
In this LR a constraint introducing upper (P_{U1}) and lower (P_{L1}) limits on the number of open plants is added up, Cornuejols, Sridharan and Thizy [14] *i.e.*

$$P_{L1?} \le \sum_i yws1_i \le P_U , \ \forall i, \tag{13(a)}$$

This LR was proposed first by Christofides and Beasley [15] and later used Verma and Sharma [5] among others and is obtained by dualizing (2(a)) and (6(a-ii)). Let $\lambda xws1ws2$, $\lambda xws1ws2_j$ be the Lagrangian multipliers associated with (2(a)) and (6(a-ii)) respectively.

$$Z_{R2} = \min_{xws1ws2, yws1, yws2} \sum_{i,j} \left(cws1ws2_{ij} - \lambda 1_i + \lambda 2_j + \lambda 3_j + \lambda xws1ws2 + \lambda xws1ws2_j \right) * xws1ws2_{ij}$$
$$+ \frac{1}{2} \sum_i yws1_i * fws1_i - \lambda xws1ws2 - \sum_j \lambda xws1ws2_j * capws2_j \tag{1(h)}$$

Subject to: (5(a-i)), (6(a-i)), (7(a)), (10(a)), (10(b)), (8(a)) and (13(a)).
3) CPLP_R3 ($\lambda 3_j$)
(1(h)); Subject to: (5(a-i)), (6(a-i)), (7(a)), (8(a)), (9(a)), (10(a)) and (10(b)). It is also attempted by first by Nauss [16] and later by Verma and Sharma [5]. We have following result due to Verma and Sharma [5].
Result 2: $CPLP\_R1(\lambda 3_j) \le CPLP\_R2(\lambda 3_j) \le CPLP\_R3(\lambda 3_j)$.

If we add the objective function value of the 0-1 knapsack problem in the objective value of *CPLP_R* then we get the objective value of the *MID_CPLP_R*. Hence, the same relaxation along with their relative strength is used for *MID_CPLP* here in form of theorem 2 below.
Result 3: $CPLP\_R1(\lambda 3_j) + OFV\_0-1\_KSP(\lambda 3_j)$
$\le CPLP\_R2(\lambda 3_j) + OFV\_0-1\_KSP(\lambda 3_j) \le CPLP\_R3(\lambda 3_j) + OFV\_0-1\_KSP(\lambda 3_j).$
where OFV stands for objective function value of $0-1\_KSP(\lambda 3_j)$. Hence we have
Theorem 1: For any value of $(\lambda 3_j)$,
$MID\_CPLP\_R1(\lambda 3_j) \le MID\_CPLP\_R2(\lambda 3_j) \le MID\_CPLP\_R3(\lambda 3_j)$.
This theorem provides the relative effectiveness of the bounds that may be obtained for these relaxations of *MID_CPLP*. Verma and Sharma [5] have shown that difference between R2-R1 and R3-R2 is significant. Hence we have the result given below.
Result 4: Difference between $MID\_CPLP\_R2(\lambda 3_j)$ and $MID\_CPLP\_R1(\lambda 3_j)$ &
$MID\_CPLP\_R3(\lambda 3_j)$ and $MID\_CPLP\_R2(\lambda 3_j)$ is significant.
Now we add two new constraints to the *MID_CPLP*. They are as follows.

$$xws1ws2_{ij} \le capws1_i \, yws2_j, \ \forall i, j, \tag{14}$$

$$xws1ws2_{ij} \le capws2_j\, yws1_i,\ \forall i, j, \tag{15}$$

We get the modified and a valid formulation of *MID_CPLP i.e.*, *MO_MID_CPLP* as given below.

2.3.3. *MO_MID_CPLP*

$$Z_{MO\_MID\_CPLP} = f_2\left(xws1ws2_{ij}, \left(cws1ws2_{ij} - \lambda1_i + \lambda2_j\right), yws1_i, yws2_j, fws1_i, fws2_j\right)$$

$$= \min_{xws1ws2, yws1, yws2} \sum_{i,j}\left(cws1ws2_{ij} - \lambda1_i + \lambda2_j\right) * xws1ws2_{ij} + \frac{1}{2}\sum_i yws1_i * fws1_i + \frac{1}{2}\sum_j yws2_j * fws2_j$$

$$\tag{1(i)}$$

s.t.: (2(a)), (5(a-i)), (5(a-ii)), (6(a-i)), (6(a-ii)), (7(a)), (8(a)), (9(a)), (10(a)), (8(b)), (9(b)), (10(b)), (14) and (15).

Now we relax the constraints (5(a-ii)), (14) and (15) by attaching multiplier $\lambda3_j$, $\lambda4_{ij}$ and $\lambda5_{ij}$ respectively to get the following.

1) *MO_MID_CPLP_R* ($\lambda3_j$, $\lambda4_{ij}$, $\lambda5_{ij}$)

$$Z_{MO\_MID\_CPLP\_R} = \min_{xws1ws2, yws1, yws2} \sum_{i,j}\left(cws1ws2_{ij} - \lambda1_i + \lambda2_j + \lambda3_j + \lambda4_{ij} + \lambda5_{ij}\right) * xws1ws2_{ij}$$

$$\tag{1(j)}$$

$$+ \frac{1}{2}\sum_i yws1_i * fws1_i - \sum_{i,j} capws2_j * yws1_i * \lambda5_{i,j}$$

s.t.: (2(a)), (5(a-i)), (6(a-i)), (6(a-ii)), (7(a)), (8(a)), (9(a)), (10(a)), (10(b)).

2) *MO_0-1_KSP* ($\lambda3_j$, $\lambda4_{ij}$, $\lambda5_{ij}$)

$$Z_{MO\_0\text{-}1\_KSP} = \frac{1}{2}\sum_j yws2_j * fws2_j - \sum_j capws2_j * yws2_j * \lambda3_j - \sum_{i,j} capws1_i * yws2_j * \lambda4_{i,j} \tag{1(k)}$$

s.t.: (8(b)), (9(b)).

3) Relaxation of *MO_MID_CPLP_R* ($\lambda3_j$, $\lambda4_{ij}$, $\lambda5_{ij}$)

Now *MO_MID_CPLP_R* $\left(\lambda3_j, \lambda4_j, \lambda5_j\right)$ can be attempted by three different relaxations. They are *R*1, *R*2 and *R*3 respectively as before.

Result 5: We get
$$MO\_MID\_CPLP\_R1\left(\lambda3_j^*, \lambda4_{i,j}^*, \lambda5_{i,j}^*\right) \le MO\_MID\_CPLP\_R2\left(\lambda3_j^*, \lambda4_{i,j}^*, \lambda5_{i,j}^*\right)$$
$$\le MO\_MID\_CPLP\_R3\left(\lambda3_j^*, \lambda4_{i,j}^*, \lambda5_{i,j}^*\right).$$

Theorem **2:** For optimal value of $\left(\lambda3_j^*, \lambda4_{i,j}^*, \lambda5_{i,j}^*\right)$,

$$MO\_MID\_CPLP\_R1\left(\lambda3_j^*, \lambda4_{i,j}^*, \lambda5_{i,j}^*\right) \le MO\_MID\_CPLP\_R2\left(\lambda3_j^*, \lambda4_{i,j}^*, \lambda5_{i,j}^*\right)$$
$$\le MO\_MID\_CPLP\_R3\left(\lambda3_j^*, \lambda4_{i,j}^*, \lambda5_{i,j}^*\right).$$

Result 6: Differences between *MO_MID_CPLP_R*2 and *MO_MID_CPLP_R*1 & *MO_MID_CPLP_R*3 and *MO_MID_CPLP_R*2 are statistically significant.

Now since *R*1 is *LP* relaxation, it easily follows that

Theorem 3: $CPLP\_R1\left(\lambda3_j^*\right) \le MO\_MID\_CPLP\_R1\left(\lambda3_j^*, \lambda4_{i,j}^*, \lambda5_{i,j}^*\right).$

In **R2**: we solve single constraint bounded variable LP to get a factor that is added to 0-1 single constraint knapsack problem (which is solved by inspection) (see Christifides and Beasley [15]). A priori it is observed that the process is complicated enough and is NOT amenable to theoretical result (for determining the difference between *MID_CPLP* (*R*2) and *MO_MID_CPLP* (*R*2).

In **R3**: we solve several continuous knapsack problems & their objective function value is fed to a 0-1 knapsack problem (see Nauss [16]). This is also complicated enough; and theoretical result may NOT be possible (for determining the difference between *MID_CPLP* (*R*3) and *MO_MID_CPLP* (*R*3)).

Hence, we may have to determine differences empirically (by coding *R*2 and *R*3 for different Lagrangian Relaxations). However, as warehouse capacities are more constrained, *MO_MID_CPLP* for both *R*2 and *R*3 will give significantly superior lagrangian lower bounds at little computational expense. However, we do have a

theoretical result for R1 (for difference between MID_CPLP and MO_MID_CPLP) which a LP relaxation. Hence we have the following Conjectures.

Conjecture 1: $CPLP\_R2\left(\lambda3_j^*\right) \le MO\_CPLP\_R2\left(\lambda3_j^*, \lambda4_{i,j}^*, \lambda5_{i,j}^*\right).$

Conjecture 2: $CPLP\_R3\left(\lambda3_j^*\right) \le MO\_CPLP\_R3\left(\lambda3_j^*, \lambda4_{i,j}^*, \lambda5_{i,j}^*\right).$

If only (14) is added to MID_CPLP then we get MO1_MID_CPLP. Several similar results are given without much discussion as they are obvious.

2.3.4. *MO1_MID_CPLP*

$$Z_{MO1\_MID\_CPLP} = f_2\left(xws1ws2_{ij}, \left(cws1ws2_{ij} - \lambda1_i + \lambda2_j, yws1_i, yws2_j, fws1_i, fws2_j\right)\right)$$

$$= \min_{xws1ws2, yws1, yws2} \sum_{i,j}\left(cws1ws2_{ij} - \lambda1_i + \lambda2_j\right) * xws1ws2_{ij} + \frac{1}{2}\sum_i yws1_i * fws1_i + \frac{1}{2}\sum_j yws2_j * fws2_j$$

$$(1(m))$$

s.t.: (2(a)), (5(a-i)), (5(a-ii)), (6(a-i)), (6(a-ii)), (7(a)), (8(a)), (9(a)), (10(a)), (8(b)), (9(b)), (10(b)), (14).

Now we relax the constraints (5(a-ii)) and (14) by attaching multiplier $\lambda3_j$ and $\lambda4_{ij}$ respectively to get the following.

1) $MO1\_MID\_CPLP\_R\left(\lambda3_j, \lambda4_{ij}\right)$

$$Z_{MO1\_MID\_CPLP\_R}$$

$$= \min_{xws1ws2, yws1, yws2} \sum_{i,j}\left(cws1ws2_{ij} - \lambda1_i + \lambda2_j + \lambda3_j + \lambda4_{ij}\right) * xws1ws2_{ij} + \frac{1}{2}\sum_i yws1_i * fws1_i \qquad (1(n))$$

s.t.: (2(a)), (5(a-i)), (6(a-i)), (6(a-ii)), (7(a)), (8(a)), (9(a)), (10(a)), (10(b)).

2) $MO1\_0-1\_KSP\left(\lambda3_j, \lambda4_{ij}\right)$

$$Z_{MO1\_0\text{-}1\_KSP} = \frac{1}{2}\sum_j yws2_j * fws2_j - \sum_j capws2_j * yws2_j * \lambda3_j - \sum_{i,j} capws1_i * yws2_j * \lambda4_{i,j} \qquad (1(o))$$

s.t.: (8(b)), (9(b)).

3) **Relaxation of** $MO1\_MID\_CPLP\_R\left(\lambda3_j, \lambda4_{ij}\right)$

Now $MO1\_MID\_CPLP\_R\left(\lambda3_j, \lambda4_{ij}\right)$ can be attempted by three different relaxations. They are R1, R2 and R3 respectively as before.

Result 7: We get $\begin{aligned}MO1\_MID\_CPLP\_R1\left(\lambda3_j^*, \lambda4_{i,j}^*\right) \le MO1\_MID\_CPLP\_R2\left(\lambda3_j^*, \lambda4_{i,j}^*\right) \\ \le MO1\_MID\_CPLP\_R3\left(\lambda3_j^*, \lambda4_{i,j}^*\right)\end{aligned}.$

Theorem 4: For optimal value of $\left(\lambda3_j^*, \lambda4_{i,j}^*\right),$

$MO1\_MID\_CPLP\_R1\left(\lambda3_j^*, \lambda4_{i,j}^*\right) \le MO1\_MID\_CPLP\_R2\left(\lambda3_j^*, \lambda4_{i,j}^*\right)$
$\le MO1\_MID\_CPLP\_R3\left(\lambda3_j^*, \lambda4_{i,j}^*\right).$

Result 8: Differences between *MO1_MID_CPLP_R2* and *MO1_MID_CPLP_R1* & *MO1_MID_CPLP_R3* and *MO1_MID_CPLP_R2* are statistically significant.

Now since R1 is LP relaxation, it easily follows that

Theorem 5: $CPLP\_R\left(\lambda3_j^*\right) \le MO1\_MID\_CPLP\_R1\left(\lambda3_j^*, \lambda4_{i,j}^*\right).$

Conjecture 3: $CPLP\_R2\left(\lambda3_j^*\right) \le MO1\_CPLP\_R2\left(\lambda3_j^*, \lambda4_{i,j}^*\right).$

Conjecture 4: $CPLP\_R3\left(\lambda3_j^*\right) \le MO1\_CPLP\_R3\left(\lambda3_j^*, \lambda4_{i,j}^*\right).$

If only (15) is added to MID_CPLP then we get, *MO2_MID_CPLP*.

2.3.5. *MO2_MID_CPLP*

$$Z_{MO2\_MID\_CPLP} = f_2\left(xws1ws2_{ij}, \left(cws1ws2_{ij} - \lambda1_i + \lambda2_j\right), yws1_i, yws2_j, fws1_i, fws2_j\right)$$

$$= \min_{xws1ws2, yws1, yws2} \sum_{i,j} \left(cws1ws2_{ij} - \lambda1_i + \lambda2_j\right) * xws1ws2_{ij} + \frac{1}{2}\sum_i yws1_i * fws1_i \qquad (1(q))$$

$$+ \frac{1}{2}\sum_j yws2_j * fws2_j$$

s.t.: (2(a)), (5(a-i)), (5(a-ii)), (6(a-i)), (6(a-ii)), (7(a)), (8(a)), (9(a)), (10(a)), (8(b)), (9(b)), (10(b)), and (15).

Now we relax the constraints (5(a-ii)) and (15) by attaching multiplier $\lambda3_j$ and $\lambda5_{ij}$ respectively to get the following.

1) $MO2\_MID\_CPLP\_R\left(\lambda3_j, \lambda5_{ij}\right)$

$$Z_{MO2\_MID\_CPLP\_R} = \min_{xws1ws2, yws1, yws2} \sum_{i,j} \left(cws1ws2_{ij} - \lambda1_i + \lambda2_j + \lambda3_j + \lambda5_{ij}\right) * xws1ws2_{ij}$$

$$+ \frac{1}{2}\sum_i yws1_i * fws1_i - \sum_{i,j} capws2_j * yws1_i * \lambda5_{i,j} \qquad (1(r))$$

s.t.: (2(a)), (5(a-i)), (6(a-i)), (6(a-ii)), (7(a)), (8(a)), (9(a)), (10(a)), (10(b)).

2) $MO2\_0-1\_KSP\left(\lambda3_j, \lambda5_{ij}\right)$

$$Z_{MO2\_0\text{-}1\_KSP} = \frac{1}{2}\sum_j yws2_j * fws2_j - \sum_j capws2_j * yws2_j * \lambda3_j \qquad (1(s))$$

s.t.: (8(b)), (9(b)).

3) Relaxation of $MO2\_MID\_CPLP\_R\left(\lambda3_j, \lambda5_{ij}\right)$

Now $MO2\_MID\_CPLP\_R\left(\lambda3_j, \lambda5_{ij}\right)$ is attempted in literature by three different relaxations. They are R1, R2 and R3 respectively as before. When integrality restriction on $yws1_i$ is relaxed we obtain (12(a)).

Result 9: we get, $MO2\_MID\_CPLP\_R1\left(\lambda3_j^*, \lambda5_{i,j}^*\right) \le MO2\_MID\_CPLP\_R2\left(\lambda3_j^*, \lambda5_{i,j}^*\right)$
$$\le MO2\_MID\_CPLP\_R3\left(\lambda3_j^*, \lambda5_{i,j}^*\right)$$.

Theorem 6: For optimal value of $\left(\lambda3_j^*, \lambda5_{i,j}^*\right)$,

$$MO2\_MID\_CPLP\_R1\left(\lambda3_j^*, \lambda5_{i,j}^*\right) \le MO2\_MID\_CPLP\_R2\left(\lambda3_j^*, \lambda5_{i,j}^*\right)$$

$$\le MO2\_MID\_CPLP\_R3\left(\lambda3_j^*, \lambda5_{i,j}^*\right)$$.

Result 10: Differences between *MO2_MID_CPLP_R*2 and *MO2_MID_CPLP_R*1 & *MO2_MID_CPLP_R*3 and *MO2_MID_CPLP_R*2 are statistically significant.

Now since R1 is LP relaxation, it easily follows that

Theorem 7: $CPLP\_R1\left(\lambda3_j^*\right) \le MO\_MID\_CPLP\_R1\left(\lambda3_j^*, \lambda5_{i,j}^*\right)$.

Conjecture 5: $CPLP\_R2\left(\lambda3_j^*\right) \le MO2\_CPLP\_R2\left(\lambda3_j^*, \lambda5_{i,j}^*\right)$..

Conjecture 6: $CPLP\_R3\left(\lambda3_j^*\right) \le MO2\_CPLP\_R3\left(\lambda3_j^*, \lambda5_{i,j}^*\right)$.

By the similar logic we have

Theorem 8: we have, $MO1\_CPLP\_R1\left(\lambda3_j^*, \lambda4_{i,j}^*\right) \le MO\_CPLP\_R1\left(\lambda3_j^*, \lambda4_{i,j}^*, \lambda5_{i,j}^*\right)$.

Theorem 9: we have, $MO2\_CPLP\_R1\left(\lambda3_j^*, \lambda5_{i,j}^*\right) \le MO\_CPLP\_R1\left(\lambda3_j^*, \lambda4_{i,j}^*, \lambda5_{i,j}^*\right)$.

Now we give two more conjectures that are based on similar logic given earlier.

Conjecture 7: $CPLP\_R2\left(\lambda3_j^*, \lambda5_{i,j}^*\right) \le MO\_CPLP\_R2\left(\lambda3_j^*, \lambda4_{i,j}^*, \lambda5_{i,j}^*\right)$.

Conjecture 8: $CPLP\_R3\left(\lambda3_j^*, \lambda4_{i,j}^*\right) \le MO2\_CPLP\_R3\left(\lambda3_j^*, \lambda4_{i,j}^*, \lambda5_{i,j}^*\right)$.

Since the effect of $\lambda4_{ij}$ and $\lambda5_{ij}$ is known only after it is through two combinatorial optimization problems, we believe a priori that it will difficult theoretically to prove the conjectures given in this paper. Hence we propose to carryout empirical investigation to support these conjectures.

3. Empirical Investigation and Statistical Significance

We have created sample problems randomly for the TSCWLP problem of sizes $25 \times 25 \times 25$, $50 \times 50 \times 50$, $75 \times 75 \times 75$, and $100 \times 100 \times 100$: CPLP_R1, CPLP_R2, CPLP_R3 and KSP; MO_CPLP_R1, MO_CPLP_R2, MO_CPLP_R3 and MO_KSP; MO1_CPLP_R1, MO1_CPLP_R2, MO1_CPLP_R3 and MO1_KSP; MO2_CPLP_R1, MO2_CPLP_R2, MO2_CPLP_R3 and MO2_KSP for the different values of θ such as 2, 1.6 and 1.2. For each category of the problem size we created 25 problems for fws1(i): 111-999, fws2(j): 111-999, capws1(i): 0.111-0.999 such that sum capws1(i) ≥ 1, capws2(j): 0.111-0.999 such that sum capws2(j) ≥ 1, cws1ws2(i,j): 1111-9999, thus created 3600 problem instances. We used the CPLEX 12.4.0.0 solver of GAMS 23.8 on an Intel (R) Core (TM) i7 CPU 870@ 2.93GHz and 4 GB RAM, 64 bit operating system to solve the problem instances. For each problem categories, we applied Lagrangian relaxations. Here, we obtained after Lagrangian Relaxation CPLP_R: Objective Function value of CPLP_R, KSP: Objective Function value of 0-1 Knapsack, MID_CPLP_R: Objective Function value of MID_CPLP_R, for each type of relaxations for R1, R2 and R3. Similar details for MO, MO1 and MO2 for R1, R2 and R3 can be written. Here, we only concentrate upon the objective function values of CPLP_R, KSP and MID_CPLP_R by applying Lagrangian Relaxations on the different formulations. After taking maximum objective function values of CPLP_R, KSP and MID_CPLP_R for the different values of θ (denoted by *) for each category. The results of statistical significance are reported in the **Table 1**, **Table 2**, **Table 3** and **Table 4**. In **Table 5**, standard t-critical values are given for n = 25, df = 24. **Table 6** is giving all the information about empirical investigation.

4. Conclusion

Here we attempt problem TSCWLP by vertical decomposition. When the flow balance equations are relaxed, it leads to three versions of CPLP; that is, RHS_CPLP, LHS_CPLP and MID_CPLP. RHS_CPLP has been solved by Cornuejol, Sridharan and Thizy [14] and LHS_CPLP has been solved by Verma and Sharma [5]. We solve MID_CPLP again by LR and give interesting theoretical results; and these were mostly verified by empirical

Table 1. T-stat values for difference in performance of Lagrangian Relaxations on Model(j) - Model(i).

No. of warehouses at stage 1*stage 2	Problem Category		R2(i)		R3(i)	
			CPLP_R2*	(CPLP_R2+KSP_R2)*	CPLP_R3*	(CPLP_R3+KSP_R3)*
100*100 n = 25	R1(j)	CPLP_R1*	3.71***		2.96***	
		(CPLP_R1+KSP_R1)*		2.86***		2.06**
	R2(j)	CPLP_R2*			−2.05**	
		(CPLP_R2+KSP_R2)*				−2.06**
75*75 n = 25	R1(j)	CPLP_R1*	2.88***		2.88***	
		(CPLP_R1+KSP_R1)*		1.33*		1.33*
	R2(j)	CPLP_R2*			−1.30*	
		(CPLP_R2+KSP_R2)*				−1.30*
50*50 n = 25	R1(j)	CPLP_R1*	2.01**		1.12	
		(CPLP_R1+KSP_R1)*		1.58*		1.07
	R2(j)	CPLP_R2*			0.99	
		(CPLP_R2+KSP_R2)*				0.99
25*25 n = 25	R1(j)	CPLP_R1*	1.21		1.02	
		(CPLP_R1+KSP_R1)*		0.91		1.02
	R2(j)	CPLP_R2*			1.00	
		(CPLP_R2+KSP_R2)*				1.03

***Significant at 0.005 level, **Significant at 0.05 level, *Significant at 0.10 level. CPLP_R1*, CPLP_R2*, CPLP_R3*, (CPLP_R1+KSP_R1)*, (CPLP_R2+KSP_R2)* and (CPLP_R3+KSP_R3)* are the maximum values of CPLP_R1, CPLP_R2, CPLP_R3, (CPLP_R1+KSP_R1), (CPLP_R2+KSP_R2) and (CPLP_R3+KSP_R3).

Table 2. T-stat values for difference in performance of Lagrangian Relaxations on Model(j) - Model(i).

No. of warehouses at stage 1* stage 2	Problem Category		MO_R2(i)		MO_R3(i)	
		CPLP_R2*	(CPLP_R2+KSP_R2)*	CPLP_R3*	(CPLP_R3+KSP_R3)*
100*100 n = 25	MO_R1(j)	CPLP_R1*	−5.64***		−11.67***	
		(CPLP_R1+KSP_R1)*		−5.61***		−11.63***
	MO_R2(j)	CPLP_R2*			−3.51***	
		(CPLP_R2+KSP_R2)*				−3.50***
75*75 n = 25	MO_R1(j)	CPLP_R1*	−5.15***		−12.22***	
		(CPLP_R1+KSP_R1)*		−2.25**		−4.46***
	MO_R2(j)	CPLP_R2*			−2.73**	
		(CPLP_R2+KSP_R2)*				−0.04
50*50 n = 25	MO_R1(j)	CPLP_R1*	−1.69*		0.47	
		(CPLP_R1+KSP_R1)*		−1.59*		0.21
	MO_R2(j)	CPLP_R2*			0.91	
		(CPLP_R2+KSP_R2)*				0.57
25*25 n = 25	MO_R1(j)	CPLP_R1*	0.45		0.93	
		(CPLP_R1+KSP_R1)*		0.47		0.93
	MO_R2(j)	CPLP_R2*			0.98	
		(CPLP_R2+KSP_R2)*				0.98

***Significant at 0.005 level, **Significant at 0.05 level, *Significant at 0.10 level. CPLP_R1*, CPLP_R2*, CPLP_R3*, (CPLP_R1+KSP_R1)*, (CPLP_R2+KSP_R2)* and (CPLP_R3+KSP_R3)* are the maximum values of CPLP_R1, CPLP_R2, CPLP_R3, (CPLP_R1+KSP_R1), (CPLP_R2+KSP_R2) and (CPLP_R3+KSP_R3).

Table 3. T-stat values for difference in performance of Lagrangian Relaxations on Model(j) - Model(i).

No. of warehouses at stage 1*stage 2	Problem Category		MO1_R2(i)		MO1_R3(i)	
		CPLP_R2*	(CPLP_R2+KSP_R2)*	CPLP_R3*	(CPLP_R3+KSP_R3)*
100*100 n = 25	MO1_R1(j)	CPLP_R1*	−7.86***		−16.26***	
		(CPLP_R1+KSP_R1)*		−7.53***		−16.25***
	MO1_R2(j)	CPLP_R2*			5.65***	
		(CPLP_R2+KSP_R2)*				5.59***
75*75 n = 25	MO1_R1(j)	CPLP_R1*	−8.20***		−14.47***	
		(CPLP_R1+KSP_R1)*		−3.02***		−14.74***
	MO1_R2(j)	CPLP_R2*			−2.78**	
		(CPLP_R2+KSP_R2)*				0.62
50*50 n = 25	MO1_R1(j)	CPLP_R1*	−2.89***		0.31	
		(CPLP_R1+KSP_R1)*		−2.88***		0.31
	MO1_R2(j)	CPLP_R2*			0.92	
		(CPLP_R2+KSP_R2)*				0.92
25*25 n = 25	MO1_R1(j)	CPLP_R1*	0.16		0.90	
		(CPLP_R1+KSP_R1)*		0.17		0.90
	MO1_R2(j)	CPLP_R2*			0.99	
		(CPLP_R2+KSP_R2)*				0.98

***Significant at 0.005 level, **Significant at 0.05 level, *Significant at 0.10 level. CPLP_R1*, CPLP_R2*, CPLP_R3*, (CPLP_R1+KSP_R1)*, (CPLP_R2+KSP_R2)* and (CPLP_R3+KSP_R3)* are the maximum values of CPLP_R1, CPLP_R2, CPLP_R3, (CPLP_R1+KSP_R1), (CPLP_R2+KSP_R2) and (CPLP_R3+KSP_R3).

Table 4. T-stat values for difference in performance of Lagrangian Relaxations on Model(j) - Model(i).

No. of warehouses at stage 1*stage 2	Problem Category		MO2_R2(i)		MO2_R3(i)	
			CPLP_R2*	CPLP_R2*+KSP_R2*	CPLP_R3*	CPLP_R3*+KSP_R3*
100*100	MO2_R1(j)	CPLP_R1*	−1.78**		−7.60***	
		CPLP_R1*+KSP_R1*		−1.02		−3.61***
	MO2_R2(j)	CPLP_R2*			−3.35***	
		CPLP_R2*+KSP_R2*				0.59
75*75	MO2_R1(j)	CPLP_R1*	−0.96		−7.70***	
		CPLP_R1*+KSP_R1*		0.66		0.17
	MO2_R2(j)	CPLP_R2*			−2.48**	
		CPLP_R2*+KSP_R2*				−2.48**
50*50	MO2_R1(j)	CPLP_R1*	0.45		0.84	
		CPLP_R1*+KSP_R1*		0.45		0.84
	MO2_R2(j)	CPLP_R2*			0.89	
		CPLP_R2*+KSP_R2*				0.89
25*25	MO2_R1(j)	CPLP_R1*	0.98		0.98	
		CPLP_R1*+KSP_R1*		0.68		0.98
	MO2_R2(j)	CPLP_R2*			0.98	
		CPLP_R2*+KSP_R2*				1.01

***Significant at 0.005 level, **Significant at 0.05 level, *Significant at 0.10 level. CPLP_R1*, CPLP_R2*, CPLP_R3*, (CPLP_R1+KSP_R1)*, (CPLP_R2+KSP_R2)* and (CPLP_R3+KSP_R3)* are the maximum values of CPLP_R1, CPLP_R2, CPLP_R3, (CPLP_R1+KSP_R1), (CPLP_R2+KSP_R2) and (CPLP_R3+KSP_R3).

Table 5. n = 25; df = 24; t table of critical values.

$\alpha = 0.005$ 99%	$\alpha = 0.05$ 90%	$\alpha = 0.1$ 80%
2.797	1.711	1.318

Table 6. Results, theorems, conjectures.

Table No.	Supported Empirical Investigation
1.	Result 1, 2, 3, 4
	Theorem 1
2.	Result 5, 6
	Theorem 2, 3
	Conjecture 1, 2
3.	Result 7, 8
	Theorem 4, 5
	Conjecture 3, 4
4.	Result 9, 10
	Theorem 6, 7
	Conjecture 5, 6
Miscellaneous	Theorem 8, 9
	Conjecture 7, 8

investigation. Theoretical results say that R1 < R2 < R3 in general; but in some cases this was not observed (as found by Verma and Sharma [5]).

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