Stochastic Oscillators with Quadratic Nonlinearity Using WHEP and HPM Methods

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ABSTRACT

In this paper, quadratic nonlinear oscillators under stochastic excitation are considered. The Wiener-Hermite expansion with perturbation (WHEP) method and the homotopy perturbation method (HPM) are used and compared. Different approximation orders are considered and statistical moments are computed in the two methods. The two methods show efficiency in estimating the stochastic response of the nonlinear differential equations.

Keywords: Nonlinear Stochastic Differential Equations; Wiener-Hermite Expansion; WHEP Technique; Homotopy Perturbation Method

1. Introduction

Quadratic oscillation arises through many applied models in applied sciences and engineering when studying oscillatory systems [1]. These systems can be exposed to a lot of uncertainties through the external forces, the damping coefficient, the frequency and/or the initial or boundary conditions. These input uncertainties cause the output solution process to be also uncertain. For most of the cases, getting the probability density function (p.d.f.) of the solution process may be impossible. So, developing approximate techniques through which approximate statistical moments can be obtained, is an important and necessary work.

Since Meecham and his co-workers [2] developed a theory of turbulence involving a truncated Wiener-Hermite expansion (WHE) of the velocity field, many authors studied problems concerning turbulence [3-8]. A lot of general applications in fluid mechanics were also studied in [9-11]. Scattering problems attracted the WHE applications through many authors [12-16]. The nonlinear oscillators were considered as an opened area for the applications of WHE as can be found in [17-23]. There are a lot of applications in boundary value problems [24, 25] and generally in different mathematical studies [26-29]. The WHE properties and description of its usage are given in [30].

In HPM technique [31-34], the response of nonlinear differential equations can be obtained analytically as a series solution. The basic idea of the homotopy method is to deform continuously a simple problem (and easy to solve) into the difficult problem under study [35]. The HPM method is a special case of homotopy analysis method (HAM) propounded by Liao in 1992 [36]. The HAM was systematically described in Liao’s book in 2003 [37] and was applied by many authors in [38-41]. The HAM method possesses auxiliary parameters and functions which can control the convergence of the obtained series solution.

The stochastic oscillator with cubic nonlinearity (Duffing oscillator) was considered in [17,42]. The nonlinear term is due to the restoring nonlinear force. In some applications, the restoring force is quadratic and it is required to estimate the response in this case. The main goal of this paper is to consider the quadratic nonlinear oscillator under stochastic excitation. The WHEP and HPM methods are used and compared.

This paper is organized as follows. The problem formulation is outlined in Section 2. The WHEP technique is described and applied to the stochastic quadratic oscillator in Section 3. The HPM is outlined in Section 4 and applied also to the quadratic oscillator. A comparison between the two methods is shown in Section 5.

2. Problem Formulation

In this section, the following quadratic nonlinear oscillatory equation is considered:
\[ \ddot{x}(t; \omega) + 2w\zeta \dot{x} + w^2 x + \epsilon w^2 \dot{x} = F(t; \omega), \quad t \in [0, T] \]  
(1)

under stochastic excitation \( F(t; \omega) \) with deterministic initial conditions

\[ x(0) = x_0, \quad \dot{x}(0) = \dot{x}_0, \]

where

\( w \): frequency of oscillation,

\( \zeta \): damping coefficient,

\( \epsilon \): deterministic nonlinearity scale,

\( \omega \in (\Omega, \sigma, \rho) \): a triple probability space with \( \Omega \) as the sample space, \( \sigma \) is a \( \sigma \)-algebra on events in \( \Omega \) and \( P \) is a probability measure.

### 3. WHEP Technique

The application of the WHE aims at finding a truncated series solution to the solution process of differential equations. The truncated series composes of two major parts; the first is the Gaussian part which consists of the first two terms, while the rest of the series constitute the non-Gaussian part. In nonlinear cases, there exists always difficulties of solving the resultant set of deterministic integro-differential equations got from the applications of a set of comprehensive averages on the stochastic integro-differential equation obtained after the direct application of WHE. Many authors introduced different methods to face these obstacles. Among them, the WHEP technique was introduced in [22] using the perturbation technique to solve perturbed nonlinear problems.

The WHE method utilizes the Wiener-Hermite polynomials which are the elements of a complete set of statistically orthogonal random functions [30]. The Wiener-Hermite polynomial \( H^{(i)}(t_1, t_2, \cdots, t_i) \) satisfies the following recurrence relation:

\[
H^{(0)} = 1, \quad H^{(1)}(t) = n(t), \quad H^{(2)}(t_1, t_2) = H^{(1)}(t_1) \cdot H^{(1)}(t_2) - \delta(t_1 - t_2),
\]

\[
H^{(3)}(t_1, t_2, t_3) = H^{(2)}(t_1, t_2) \cdot H^{(1)}(t_3) - H^{(2)}(t_1, t_3) \cdot H^{(1)}(t_2) - H^{(2)}(t_2, t_3) \cdot \delta(t_1 - t_4),
\]

\[
H^{(4)}(t_1, t_2, t_3, t_4) = H^{(3)}(t_1, t_2, t_3) \cdot H^{(1)}(t_4) - H^{(3)}(t_1, t_4) \cdot H^{(2)}(t_2, t_3) - H^{(3)}(t_2, t_3) \cdot H^{(2)}(t_1, t_4) - H^{(3)}(t_2, t_4) \cdot H^{(2)}(t_1, t_3) \cdot \delta(t_1 - t_4),
\]

in which \( n(t) \) is the white noise with the following statistical properties

\[
E(n(t)) = 0, \quad E(n(t_1) \cdot n(t_2)) = \delta(t_1 - t_2),
\]

(4)

where \( \delta(\cdot) \) is the Dirac delta function and \( E \) denotes the ensemble average operator.

The Wiener-Hermite set is a statistically orthogonal set, i.e.,

\[
G(t; \omega) = G^{(0)}(t) + \int_{-\infty}^{\infty} G^{(1)}(t; \tau) H^{(1)}(\tau) d\tau + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G^{(2)}(t; \tau, \tau') H^{(2)}(\tau, \tau') d\tau d\tau' + \cdots
\]

(7)

where the first two terms are the Gaussian part of \( G(t; \omega) \). The rest of the terms in the expansion represent the non-Gaussian part of \( G(t; \omega) \). The average of \( G(t; \omega) \) is

\[
\mu_G = EG(t; \omega) = G^{(0)}(t)
\]

(8)

The covariance of \( G(t; \omega) \) is

\[
\text{Cov}(G(t; \omega), G(\tau; \omega)) = E(G(t; \omega) - \mu_G(t))(G(\tau; \omega) - \mu_G(\tau))
\]

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G^{(1)}(t; \tau) G^{(1)}(\tau, \tau') d\tau d\tau' + 2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G^{(2)}(t; \tau, \tau') G^{(2)}(\tau, \tau', \tau'') d\tau d\tau' d\tau'' + \cdots
\]

(9)

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The variance of \( G(t; \omega) \) is

\[
\text{Var} G(t; \omega) = E \left( G(t; \omega) - \mu_G(t) \right)^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[ G^{(0)}(t; t_1) \right]^2 dt_1 + 2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[ G^{(2)}(t; t_1, t_2) \right]^2 dt_1 dt_2 + \cdots.
\]

The WHE method can be elementary used in solving stochastic differential equations by expanding the solution process as well as the stochastic input processes via the WHE. The resultant equation is more complex than the original one due to being a stochastic integro-differential equation. Taking a set of ensemble averages together with using the statistical properties of the WH polynomials, a set of deterministic integro-differential equations are obtained in the deterministic kernels \( G^{(i)}(t, \omega), i = 0, 1, 2, \ldots \). To obtain an approximate solutions for these deterministic kernels, one can use perturbation theory in the case of having a perturbed system depending on, say, \( \varepsilon \). Expanding the kernels as a power series of \( \varepsilon \), another set of simpler iterative equations in the kernel series components are obtained. This is the main algorithm of the WHEP technique. The technique was successfully applied to several nonlinear stochastic equations; see [20,22,23,25].

The WHEP technique can be applied on linear or nonlinear perturbed systems described by ordinary or partial differential equations. The solution can be modified in the sense that additional parts of the Wiener-Hermite expansion can always be taken into considerations and the required order of approximations can always be made. It can be even run through a package if it is coded in some sort of symbolic languages.

**Case-Study**

The quadratic nonlinear oscillatory problem, Equation (1) under stochastic excitation \( F(t; \omega) \) with deterministic initial conditions is solved using WHEP technique. The solution process takes the following form:

\[
x(t; \omega) = x^{(0)}(t) + \int_{-\infty}^{\infty} x^{(1)}(t; t_1) H^{(1)}(t_1) dt_1 + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^{(2)}(t; t_1, t_2) H^{(2)}(t_1, t_2) dt_1 dt_2 + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^{(3)}(t; t_1, t_2, t_3) H^{(3)}(t_1, t_2, t_3) dt_1 dt_2 dt_3 + \cdots
\]

Applying the WHEP technique, the following equations in the deterministic kernels are obtained:

\[
L x^{(0)}(t) + \varepsilon w^2 \left( x^{(0)}(t) \right)^2 + \varepsilon w^2 \int_{-\infty}^{\infty} \left( x^{(1)}(t; t_1) \right)^2 dt_1 + 2\varepsilon w^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left( x^{(2)}(t; t_1, t_2) \right)^2 dt_1 dt_2 + \cdots
\]

\[
L x^{(1)}(t, t_1) + 2\varepsilon w^2 x^{(0)}(t) x^{(1)}(t, t_1) + 4\varepsilon w^2 \int_{-\infty}^{\infty} x^{(1)}(t; t_2) x^{(2)}(t; t_1, t_2) dt_2 + 4\varepsilon w^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^{(1)}(t; t_1, t_2) x^{(2)}(t; t_2, t_3) dt_2 dt_3 + \cdots
\]
\[\begin{align*}
Lx^{(3)}(t, t_1, t_2) + & \varepsilon w^2 \left[ 2x^{(0)}(t)x^{(2)}(t, t_1, t_2) + x^{(1)}(t, t_1)x^{(1)}(t, t_2) + 4 \int_{-\infty}^{\infty} x^{(2)}(t, t_1, t_2) x^{(3)}(t; t_2, t_3) dt_3 \right] \\
+ 2 \int_{-\infty}^{\infty} x^{(1)}(t; t_3)x^{(3)}(t; t_1, t_2, t_3) dt_3 + 2 \int_{-\infty}^{\infty} x^{(0)}(t; t_3)x^{(3)}(t; t_2, t_3, t_4) dt_3 + 2 \int_{-\infty}^{\infty} x^{(1)}(t; t_3)x^{(3)}(t; t_1, t_2, t_3) dt_3 \\
+ 2 \int_{-\infty}^{\infty} \int x^{(3)}(t; t_1, t_2, t_3) dt_3 dt_4 + 4 \int_{-\infty}^{\infty} \int x^{(3)}(t; t_1, t_2, t_3) x^{(3)}(t; t_2, t_3, t_4) dt_3 dt_4 \\
+ 3 \int_{-\infty}^{\infty} \int x^{(3)}(t; t_1, t_2, t_3) dt_3 dt_4 + 3 \int_{-\infty}^{\infty} \int x^{(3)}(t; t_1, t_2, t_3) x^{(3)}(t; t_2, t_3, t_4) dt_3 dt_4 \\
+ 2 \int_{-\infty}^{\infty} \int \int x^{(3)}(t; t_1, t_2, t_3) x^{(3)}(t; t_2, t_3, t_4) dt_3 dt_4 + 3 \int_{-\infty}^{\infty} \int \int x^{(3)}(t; t_1, t_2, t_3) x^{(3)}(t; t_2, t_3, t_4) dt_3 dt_4 \\
= & \mathcal{F}^{(2)}(t, t_1, t_2)
\end{align*}\]

\[\begin{align*}
Lx^{(3)}(t, t_1, t_2, t_3) + & \varepsilon w^2 \left[ 2x^{(0)}(t)x^{(2)}(t, t_1, t_2, t_3) + x^{(1)}(t, t_1)x^{(1)}(t, t_2, t_3) + x^{(1)}(t, t_3)x^{(1)}(t, t_2, t_1) + x^{(0)}(t; t_3)x^{(3)}(t; t_2, t_3, t_4) dt_3 \\
+ 4 \int_{-\infty}^{\infty} x^{(2)}(t; t_4, t_1)x^{(3)}(t; t_2, t_3, t_4) dt_4 + 4 \int_{-\infty}^{\infty} x^{(0)}(t; t_4, t_1)x^{(3)}(t; t_2, t_3, t_4) dt_4 + 4 \int_{-\infty}^{\infty} x^{(1)}(t; t_4, t_1)x^{(3)}(t; t_2, t_3, t_4) dt_4 \\
+ 4 \int_{-\infty}^{\infty} x^{(0)}(t; t_4, t_1)x^{(3)}(t; t_2, t_3, t_4) dt_4 + 4 \int_{-\infty}^{\infty} x^{(1)}(t; t_4, t_1)x^{(3)}(t; t_2, t_3, t_4) dt_4 \\
+ 4 \int_{-\infty}^{\infty} \int x^{(3)}(t; t_1, t_2, t_3) x^{(3)}(t; t_2, t_3, t_4) dt_3 dt_4 + 4 \int_{-\infty}^{\infty} \int x^{(3)}(t; t_1, t_2, t_3) x^{(3)}(t; t_2, t_3, t_4) dt_3 dt_4 \\
= & \mathcal{G}^{(3)}(t, t_1, t_2, t_3) + \mathcal{G}^{(3)}(t, t_1, t_2, t_3) + \mathcal{G}^{(3)}(t, t_2, t_3, t_4)
\end{align*}\]

Let us take the simple case of evaluating the only Gaussian part (first order approximation) of the solution process of the previous case study, mainly

\[x(t; \omega) = x^{(0)}(t) + \int_{-\infty}^{\infty} x^{(1)}(t_1) H^{(1)}(t_1) dt_1 .\]

In this case, the governing equations are

\[\begin{align*}
Lx^{(0)}(t) + & \varepsilon w^2 \left[ x^{(0)}(t) + \int_{-\infty}^{\infty} \left[ x^{(0)}(t_1) \right]^2 dt_1 \right] = \mathcal{G}^{(0)}(t) \\
Lx^{(1)}(t, t_1) + & \varepsilon w^2 2x^{(0)}(t)x^{(1)}(t, t_1) = \mathcal{G}^{(1)}(t, t_1)
\end{align*}\]

The ensemble average is

\[\mu_4(t) = x^{(0)}(t)\]

and the variance is

\[\sigma^2(t) = \int_{-\infty}^{\infty} \left[ x^{(1)}(t, t_1) \right]^2 dt_1\]

It has to be noticed that all the previous equations are deterministic linear ones in the general form

\[\dot{x} + 2w_0x + w_0^2x = F(t)\]

with deterministic initial conditions \(x(0) = x_0, \dot{x}(0) = \dot{x}_0\). It has the general solution

\[x(t; \omega) = x^{(0)}(t) + \int_{-\infty}^{\infty} x^{(1)}(t_1) H^{(1)}(t_1) dt_1 + \int_{-\infty}^{\infty} \int x^{(2)}(t_1, t_2) H^{(2)}(t_1, t_2) dt_1 dt_2 .\]
the governing equations become

\[ L x^{(0)}(t) + 2\varepsilon w^2 \mathbb{E} \left[ x^{(0)}(t)^2 \right] + \int_{-\infty}^{\infty} x^{(1)}(t; t_1) \, dt_1 = (23) \]

\[ 1 \to \varepsilon \to 2 \to \varepsilon \to \infty \]

\[ L x^{(1)}(t; t_1) + 2\varepsilon w^2 x^{(0)}(t) x^{(1)}(t; t_1) \]

\[ + 4\varepsilon w^2 \int_{-\infty}^{\infty} x^{(1)}(t; t_1) x^{(2)}(t; t_1; t_2) \, dt_1 \, dt_2 = G^{(0)}(t) \]

\[ L x^{(2)}(t; t_1; t_2) \]

\[ + \varepsilon w^2 \left[ 2 x^{(0)}(t) x^{(2)}(t; t_1; t_2) + x^{(1)}(t; t_1) x^{(3)}(t; t_2) \right] \]

\[ + 4 \varepsilon \int_{-\infty}^{\infty} x^{(2)}(t; t_1; t_2) x^{(2)}(t; t_2; t_3) \, dt_1 \, dt_2 \]

\[ = G^{(2)}(t; t_1; t_2) \]

The ensemble average is still got by Equation (19) while the variance is got as

\[ \sigma^2(t) = \int_{-\infty}^{\infty} \mathbb{E} \left[ x^{(i)}(t; t_1)^2 \right] \, dt_1 + 2 \int_{-\infty}^{\infty} \mathbb{E} \left[ x^{(2)}(t; t_1; t_2) \right] \, dt_1 \, dt_2 \]

(26)

The WHEP technique uses the following expansion for its deterministic kernels as corrections made under each approximation order.

\[ x^{(i)}(t) = x_0^{(i)} + \varepsilon x_1^{(i)} + \varepsilon^2 x_2^{(i)} + \varepsilon^3 x_3^{(i)} + \cdots, \quad i = 0, 1, 2, 3, \ldots (27) \]

Example:

Let us take \( F(t; q) = e^{-t} + \varepsilon \delta(t; \omega), \quad \varepsilon = 0.3 \) (28) in the previous case-study and then solving using the WHEP technique. The following results are obtained, see Figures 1-3.

4. The Homotopy Perturbation Method (HPM)

In this technique, a parameter \( P \in [0, 1] \) is embedded in a homotopy function \( \nu(r, p) : \phi \times [0, 1] \to \mathbb{R} \) which satisfies

Figure 1. (a) The first order approximation of the mean at \( \varepsilon \) correction for different correction levels; (b) The first order approximation of the mean at \( \varepsilon^3 \) correction for different correction levels; (c) The first order approximation of the mean at \( \varepsilon^3 \) correction; (d) The first order approximation of the mean at \( \varepsilon^2, \varepsilon^3 \) correction; (e) The first order approximation of the mean at \( \varepsilon, \varepsilon^2, \varepsilon^3 \) correction; (f) The first order approximation of the mean at \( \varepsilon, \varepsilon^2, \varepsilon^3 \) correction.
Figure 2. (a) The first order approximation of the variance at $\varepsilon$ correction for different correction levels; (b) The first order approximation of the variance at $\varepsilon^2$ correction for different correction levels; (c) The first order approximation of the variance at $\varepsilon^3$ correction for different correction levels; (d) The first order approximation of the variance at $\varepsilon, \varepsilon^2, \varepsilon^3$ correction.

Figure 3. (a) The first order approximation of the variance at $\varepsilon, \varepsilon^2, \varepsilon^3$ correction; (b) The first order approximation of the variance at $\varepsilon, \varepsilon^2, \varepsilon^3$ correction.

\[
H(v, p) = (1 - p)[L(v) - L(u_0)] + p[A(v) - f(r)] = 0
\]

where $u_0$ is an initial approximation to the solution of the equation

\[
A(u) - f(r) = 0, \quad r \in \phi
\]

with boundary conditions

\[
B\left(u, \frac{\partial u}{\partial n}\right) = 0, \quad r \in \Gamma
\]

in which $A$ is a nonlinear differential operator which can be decompose into a linear operator $L$ and a nonlinear operator $N$, $B$ is a boundary operator, $f(r)$ is a known analytic function and $\Gamma$ is the boundary of $\phi$. The homotopy introduces a continuously deformed solution for the case of $p = 0$, $L(v) - L(u_0) = 0$, to the case of $p = 1$, $A(v) - f(r) = 0$, which is the original Equation (30). This is the basic idea of the homotopy method which is to deform continuously a simple problem (and easy to solve) into the difficult problem under study [35].

The basic assumption of the HPM method is that the solution of the original Equation (29) can be expanded as a power series in $p$ as:

\[
v = v_0 + pv_1 + p^2v_2 + p^3v_3 + \cdots
\]

Now, setting $p = 1$, the approximate solution of Equation (23) is obtained as:

\[
u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + v_3 + \cdots
\]

The rate of convergence of the method depends greatly on the initial approximation $u_0$.

The idea of the imbedded parameter can be utilized to solve nonlinear problems by imbedding this parameter to the problem and then forcing it to be unity in the obtained approximate solution if converge can be assured. A simple technique enables the extension of the applicability of the perturbation methods from small valued ap-
lications to general ones.

**Example**

Considering the same previous example of Sub-Section 3.1.1, one can get the following results w.r.t. homotopy perturbation:

\[
A(x) = L(x) + \varepsilon w^2 x^2 ,
\]

\[
L(x) = \ddot{x} + 2w\varepsilon \dot{x} + w^2 x ,
\]

\[
N(x) = \varepsilon x^2 ,
\]

\[
f(r) = F(t; \omega) .
\]

The homotopy function takes the following form:

\[
H(v, p) = (1 - p) \left[ L(v) - L(u_0) \right] + p \left[ A(v) - f(r) \right] = 0
\]
or equivalently,

\[
L(v) - L(u_0) + p \left[ L(u_0) + \varepsilon w^2 v^2 - F(t; \omega) \right] = 0 .
\] (34)

Letting \( v = v_0 + p v_1 + p^2 v_2 + p^3 v_3 + \cdots \), substituting in Equation (34) and equating the equal powers of \( p \) in both sides of the equation, one can get the following results:
1) \( L(v_0) = L(y_0) \), in which one may consider the following simple solution:

\[
v_0 = y_0 , \quad v_0(0) = x_0 , \quad \dot{y}_0(0) = \ddot{x}_0 .
\]

2) \( L(v_1) = F(t; \omega) - L(v_0) - \varepsilon w^2 v_0^2 , \quad v_1(0) = 0 , \quad \dot{v}_1(0) = 0 .
\)

3) \( L(v_2) = -2\varepsilon w^2 v_0 v_1 , \quad v_2(0) = 0 , \quad \dot{v}_2(0) = 0 .
\)

4) \( L(v_3) = -\varepsilon w^2 \left( v_1^2 + 2v_0 v_2 \right) , \quad v_3(0) = 0 , \quad \dot{v}_3(0) = 0 .
\)

5) \( L(v_4) = -2\varepsilon \left( v_0 v_1 + v_1 v_2 \right) , \quad v_4(0) = 0 , \quad \dot{v}_4(0) = 0 .
\)

The approximate solution is

\[
x(t; \omega) = \lim_{p \to 1} v = v_0 + v_1 + v_2 + v_3 + \cdots
\]

which can be considered to any approximation order. One can notice that the algorithm of the solution is straightforward and that a lot of flexibilities can be made. For example, we have many choices in guessing the initial approximation together with its initial conditions. For zero initial conditions, we can choose \( v_0 = 0 \) which leads to:

\[
x(t; \omega) \equiv x_3 = v_0 + v_1 + v_2 + v_3 + v_4 + \cdots
\]

\[
= \frac{1}{0} \int h(t-s) F(s; \omega) ds - \varepsilon w^2 \frac{1}{0} \int h(t-s) v_2^2 (s; \omega) ds
\]

\[
- 2\varepsilon w^2 \frac{1}{0} \int h(t-s) v_1 (s; \omega) v_3 (s; \omega) ds
\]

(35)

**Figures 4-7** are obtained for \( \varepsilon = 0.5 \) : [42].

5. **Comparisons between WHEP and HPM Methods**

Figure [8] shows comparisons between the WHEP and HPM methods for different values of the nonlinearity strength, \( \varepsilon \). As the nonlinearity strength increases, the deviation between the two methods is also increasing.

![Figure 4](image1.png)

**Figure 4.** (a) The first and second order approximation of the mean for different correction levels; (b) The first and second order approximation of the variance at for different correction levels.

![Figure 5](image2.png)

**Figure 5.** (a) The third order approximation of the mean for different correction levels; (b) The third order approximation of the variance for different correction levels.
Figure 6. (a) A comparison between first, second order and the third order of the mean at $\varepsilon = 0.1$; (b) Comparison between first, second order and the third order of the variance at $\varepsilon = 0.1$.

Figure 7. (a) A comparison between first, second order and the third order of the mean at $\varepsilon = 0.3$; (b) A comparison between first, second order and the third order of the variance at $\varepsilon = 0.3$; (c) A comparison between first, second order and the third order of the mean at $\varepsilon = 0.7$; (d) A comparison between first, second order and the third order of the variance at $\varepsilon = 0.7$. 
This is due to the convergence condition of the WHEP technique which depends on \( \varepsilon \). For small values of \( \varepsilon \), the WHEP technique converges but after a certain value of \( \varepsilon \) it will diverge. The HPM is more accurate for higher values of \( \varepsilon \). The HPM has advantages when used in solving differential equations with large nonlinearities.

6. Conclusion

The quadratic nonlinear oscillator with stochastic excitation is considered. The solution was obtained using the WHEP technique with different orders and different number of corrections. The HPM is used also with different approximations. The WHEP technique is more efficient but it converges only for certain limit of the nonlinearity strength. The HPM is more difficult in the stochastic differential equations but it is more preferable for higher values of the nonlinearity strength. The two methods are shown to be efficient in estimating the stochastic response of the quadratic nonlinear oscillators.

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