Advances in Computed Tomography, 2019, 8, 1-9
http://www.scirp.org/journal/act
ISSN Online: 2169-2483
ISSN Print: 2169-2475

A Reweighted Total Variation Algorithm with the Alternating Direction Method for Computed Tomography

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How to cite this paper: Li, X.Z. and Zhu, J.H. (2019) A Reweighted Total Variation Algorithm with the Alternating Direction Method for Computed Tomography. *Advances in Computed Tomography*, **8**, 1-9.

https://dx.doi.org/10.4236/act.2019.81001

Received: November 23, 2018
Accepted: January 20, 2019
Published: January 23, 2019

Abstract

A variety of alternating direction methods have been proposed for solving a class of optimization problems. The applications in computed tomography (CT) perform well in image reconstruction. The reweighted schemes were applied in l_1-norm and total variation minimization for signal and image recovery to improve the convergence of algorithms. In this paper, we present a reweighted total variation algorithm using the alternating direction method (ADM) for image reconstruction in CT. The numerical experiments for ADM demonstrate that adding reweighted strategy reduces the computation time effectively and improves the quality of reconstructed images as well.

Keywords

Computed Tomography, Nonmonotone Alternating Direction Algorithm, Reweighted Algorithm

1. Introduction

In many applications, the problem of recovering a sparse unknown vector x from a set of measurements b is presented. The l_1 minimization,

$$\min_x \|x\|_1 \quad \text{subject to} \quad Ax = b, \tag{1}$$

where A represents a measurement matrix, has been widely used in solving the problem [1–5]. This method has a unique sparsest solution x under certain conditions [6]. However, the l_1-norm regularization provides a less sparsity representation than the l_0-norm regularization and may lose some detailed features and contrast. One major improved l_1 minimization algorithm for finding the sparsest solution efficiently is the reweighted l_1 minimization [7]. Some modifications of the reweighted l_1-minimization algorithms have been studied in recent years. A more general form of the reweighted l_1-minimization, in which the weights are raised to a certain power between 0 and 1, was considered in [8,9]. Numerical experiments in the areas of sparse signal recovery, the error correction in sample transition and the total minimization for sparse image gradients have shown that the performance of recovery using reweighted l_1 minimization is better than the regular l_1 minimization [7].

In computed tomography (CT), algebraic approaches for image reconstruction involves solving a consistent system of linear equations

$$\Phi f + e = u, \tag{2}$$

where Φ is an $m \times n^2$ projection matrix, $f \in R^{n^2}$ represents a 2D $n \times n$ image to be reconstructed, e an additive noise with $||e||_2 \leq \varepsilon$ for some known $\varepsilon > 0$, and $u \in R^m$ the noisy projection data. For limited-view reconstruction, the system is underdetermined with $m \ll n^2$ and has infinitely many solutions. Optimization_based Iterative methods are usually used to find an optimal solution representing the original image.

The image f in System (2) can be reconstructed by solving the following total variation (TV) minimization with an l_2-norm constraints,

$$\min_f ||f||_{TV} \quad \text{subject to} \quad ||\Phi f - u||^2 \leq \varepsilon, \tag{3}$$

where $|| * ||$ stands for $|| * ||_2$ for simplicity. Tomography images can be approximately modeled to be essentially piecewise constant so the gradients are sparse. As $||f||_{TV}$ is the l_1-norm of the gradient of f, the TV minimization is also known as the l_1-minimization method. A generalized l_1 greedy algorithm in the compressed sensing framework [10] was introduced to incorporate the threshold feature of the l_1 greedy algorithm [11] and the inversely proportional weights used in the reweighted l_1-minimization algorithm for CT. Error analysis of reweighted l_1 greedy algorithm for noisy reconstruction was studied in [12]. The reweighted l_1-norm was incorporated into non-local TV minimization to streghten the structural details and the tissue contrast and thus to enhance the CT reconstructing performance [13].

The alternating direction method (ADM) is a variant of the classic augmented Lagrangian method for structured optimization. For a convex optimization problem where variables appearing separately are coupled in the constraint, the minimization with respect to all variables are time-consuming. The ADM minimizes the augmented Lagrangian function with respect to variables separately via iteration to reduce computation time, widely used in l_1-norm minimization [14,15]. An ADM with fast convergence under certain strong conditions was proposed in [16]. A unified alternating direction method by majorization minimization was proposed and its convergence was analyzed [17]. The nonmonotone alternating direction method (NADA) was proposed for solving a class of equality-constrained nonsmooth optimization problems and applied to the total variation minimization effectively [18, 19]. The advantage of the NADA lies in that the objective function is not required to be differentiable while reducing the computation complexity and improving the efficiency.

In this paper, we present a reweighted TV algorithm using ADM in CT. The rest of this paper is organized as follows. Notations and preliminary are introduced in Section 2. A reweight scheme corporation into the total variation algorithm using NADA is developed in Section 3. Numerical experiments in Section 4 show the adopt of weights improves the performance of the TV algorithm using NADA in computational time and error measurements. Finally a discussion in Section 5 concludes the paper.

2. Preliminary

In this section we introduce notations and review approaches for solving a TV minimization problem (3) using the reweight scheme and ADM in the literature, respectively.

In the case of sparse signal recovery, it is extremely important to identify the locations of nonzero entries of a sparse vector x. After a few iterations, the larger entries in magnitude are naturally considered to be nonzero entries. The process should concentrate on other entries smaller in magnitude. The idea of the reweighted l_1 minimization is to make the weights small for the entries of x larger in magnitude and the weights large for those smaller in magnitude. In other words, the weights are roughly inversely proportional to the magnitude of the previous iterate solution. So reweighted l_1 minimization speeds up the convergence of recovery. The minimization problem (1) is modified as

$$\min_x \|Wx\|_1 \quad \text{subject to} \quad Ax = b,$$

where at the k-th iteration $W = diag\{w_1^{(k)}, ..., w_{n^2}^{(k)}\}$ is the diagonal weight matrix with each $w_i^{(k)} = \dfrac{1}{\varepsilon + ||x^{(k-1)}||}$ and $\varepsilon > 0$.

In the case of image reconstruction in CT, the total variation of an image f is defined as

$$||f||_{TV} = \|\nabla f\|_1 = \sum_{i=1}^{n^2} ||D_i f||,$$

where ∇f is the gradient of f, and D_i is a forward difference operator at a pixel i in both horizontal and vertical directions. The gradient ∇f of an essentially piecewise constant image f is sparse so a reweighted scheme could be adopted to speed up the convergence and improve the efficiency of image reconstruction. Thus, the term to be minimized in (3) is revised as $||W\nabla f||_1$, where at the k-th iteration $W = diag\{w_1^{(k)}, ..., w_{n^2}^{(k)}\}$ with all $w_i^{(k)} = \dfrac{1}{\varepsilon + ||D_i f^{k-1}||}$.

Minimization (3) with constraints can be expressed as the minimization of an augmented Lagrangian function as, with Lagrangian multipliers λ_i, auxiliary vectors v_i, and regularization parameters β, $\mu > 0$,

$$L(f, v, \lambda) = \frac{\beta}{2}||\Phi f - u||_2^2 + \sum_{i=1}^{n^2} \{\|v_i\| + \lambda_i^T (D_i f - v_i) + \mu||D_i f - v_i||^2\}, \tag{4}$$

where λ_i, $v_i \in R^2$. Given f^k, v^k, and λ^k in the k-th iteration, ADM minimizes $L(f, v, \lambda)$ with respect to f and v separately, then updates λ. In other words, ADM iterates as

$$f^{k+1} = \arg\min_f L(f, v^k, \lambda^k);$$
$$v^{k+1} = \arg\min_v L(f^{k+1}, v, \lambda^k);$$
$$\lambda^{k+1} = \lambda^k + \beta(Df^{k+1} - v^{k+1}).$$

3. Reweighted TV Algorithm Using NADA

In this section we develop a new reweighted TV algorithm using NADA for solving the following minimization

$$\min_f ||W\nabla f||_1 \quad \text{subject to} \quad ||\Phi f - u||^2 \leq \varepsilon, \tag{5}$$

where the l_1-norm $||W\nabla f||_1$ is a weighted total variation of f, and W is a weight matrix with all positive diagonals.

Dealing with the constraint as usual ADM, we first rewrite the minimization above with a penalty parameter $\mu > 0$ as

$$\min_f ||W\nabla f||_1 + \frac{\mu}{2}||\Phi f - u||^2,$$

or

$$\min_f \sum_{i=1}^{n^2} ||v_i|| + \frac{\mu}{2}||\Phi f - u||^2 \quad \text{subject to} \quad w_i D_i f = v_i, \quad 1 \leq i \leq n^2. \tag{6}$$

Applying Lagrangian vectors $\lambda_i \in R^2$, for $1 \leq i \leq n^2$, we define

$$L_w(f, v, \lambda) = \frac{\mu}{2}||\Phi f - u||^2 + \sum_{i=1}^{n^2} (||v_i||_2 - \lambda_i^T(w_i D_i f - v_i) + \frac{\beta}{2}||w_i D_i f - v_i||^2). \tag{7}$$

Thus, Minimization (6) is converted to minimizing the objective function in (7),

$$\min_f L_w(f, v, \lambda). \tag{8}$$

For given f^k, v^k, and λ^k, the proposed algorithm iterates as

$$f^{k+1} = \arg\min_f L_w(f, v^k, \lambda^k), \tag{9}$$

$$v^{k+1} = \arg\min_v L_w(f^{k+1}, v, \lambda^k), \tag{10}$$

$$\lambda^{k+1} = \arg\min_\lambda L_w(f^{k+1}, v^{k+1}, \lambda). \tag{11}$$

In this new algorithm, a modified reweighted scheme is incorporated after a few non-weighted iterations in order to speed up the convergence of the algorithm and improve the efficiency. The standard reweighted scheme adopts a matrix $W = diag\{w_1, ..., w_{n^2}\}$, where w_i is essentially inversely proportional to the norm $||D_i f^k||$. In this project, the values of w_i's need to be rescaled so that the weights of four terms in the objective function $L_w(f, v, \lambda)$ remain close to the case where W is the identity matrix. The standard reweighted scheme is modified to calculating a vector w with $w_i = \dfrac{1}{\varepsilon + ||D_i f^k||}$ $(1 \leq i \leq n^2)$ and rescaling w by a factor $r = \frac{2}{\text{mean}(w)}$. The reweighted matrix W in this paper is obtained this way.

Minimizations (9) and (10) are solved by adopting the nonmonotone line search scheme in the framework of NADA though there is an extra reweighted matrix W in the objective function $L_w(f, v, \lambda)$. The details are as follows. After choosing a direction $d^k = -\frac{\partial}{\partial f} L_w(f^k, v^k, \lambda^k)$, search a step size s_k uniformly bounded above such that $L_w(f^k + s_k d^k, v^k, \lambda^k) \leq C_k - s_k \delta ||d^k||_2^2$, for a scalar C_k and $0 < \delta < 1$. Then update $f^{k+1} = f^k + s_k d^k$ and compute a scalar C_{k+1} such that

$L_w(f^{k+1}, v^k, \lambda^k) \le C_{k+1} \le C_k$. Combined with solution of (10), the sequence $\{L_w(f^k, v^k, \lambda^k)\}$ is bounded above by a monotonically non-increasing sequence $\{C_k\}$ though $\{L_w(f^k, v^k, \lambda^k)\}$ itself is not decreasing. The advantages of the NADA remain so that $L_w(f, v, \lambda)$ is not required to be differentiable and that the usage of the nonmonotone line search scheme reduces the computation complexity and improves the efficiency.

To find $v^{k+1} = \arg\min_v L_w(f^{k+1}, v, \lambda^k)$ in (10), we solve

$$\frac{\partial L_w}{\partial v_i} = \frac{v_i}{||v_i||} + \lambda_i - \beta w_i D_i f + \beta v_i = 0.$$

So $v_i(1 + \frac{1}{\beta||v_i||}) = w_i D_i f - \frac{\lambda_i}{\beta}$. It follows that $||v_i|| = ||w_i D_i f - \frac{\lambda_i}{\beta}|| - \frac{1}{\beta}$. Thus, we have

$$v_i = \max\{||w_i D_i f - \frac{\lambda_i}{\beta}|| - \frac{1}{\beta},\ 0\} \frac{w_i D_i f - \frac{\lambda_i}{\beta}}{||w_i D_i f - \frac{\lambda_i}{\beta}||}, \qquad 1 \le i \le n^2. \quad (12)$$

Similarly, the solution of **(11)** is given by

$$\lambda_i^{k+1} = \lambda_i^k + \beta(w_i D_i f^{k+1} - v^{k+1}), \qquad 1 \le i \le n^2. \quad (13)$$

Now we present a new algorithm to solve $\min_f L_w(f, v, \lambda)$ using NADA.

Algorithm (Reweighted Total Variation Algorithm Using NADA)

1. input Φ, u, ε
2. initialize $\beta, \mu, f^0, v^0, \lambda^0, \varepsilon, tol, k, maxit$
3. perform a few iterations of regular TV algorithm without weight using NADA
4. while $k < maxit$
   - 4.1 set a reweighted matrix W
   - 4.2 update $f^{k+1} = \arg\min_f L_w(f, v^k, \lambda^k)$ by NADA
   - 4.3 update v_i^{k+1} by (12)
   - 4.4 update λ^{k+1} by (13)
   - 4.5 if error $< tol$ then output f^{k+1}, stop
   - 4.6 increase $k = k + 1$

   end

4. Numerical Experiments

In this section, the reweighted TV algorithm using NADA for CT reconstruction is implemented in MATLAB. The reweighted TV minimization (5) (or (8)) is compared with the regular TV minimization (3) (or minimization of (4)) for their performances. The numerical experiments are conducted with the 2D Shepp-Logan phantom [20] of size 128×128 on an Intel Core i7 3.40 GHz PC. The MATLAB code is developed based on the software package TVAL3 [19].

In each test, a random matrix $\Phi \in R^{m \times n^2}$ ($m \approx 0.3n^2$) is generated and $u = \Phi f + e$ is set, where the noise $e = 0.05$*mean(Φf)*randn(m). The parameters are taken as $\beta = 2^4$ and $\mu = 2^6$. The values of ε in the weight matrix is 0.01. If the relative error $||f - f_{recon}||_F / ||f||_F$ of the reconstructed image f_{recon} in Frobenius norm is less than $tol = 0.05$ then the iteration is terminated. For the reweighted TV algorithm, 15 iterations of the standard total variation minimization are set as the initial solution for the reweighted TV-minimization. The average results from 100 tests of the proposed algorithm and non-weighted TV regularization are used in the following table and figure for evaluations.

Let f_{ave} denote the average of the pixel values of a 2D image f. Experimental results are also evaluated using the root-mean-square error (RMSE), the normalized root mean square deviation (NRMSD), and the normalized mean absolute deviation (NMAD) which are defined as follows:

$$RMSE = \frac{||f - f_{recon}||_F}{\sqrt{m*n}},$$
$$NRMSD = \frac{||f - f_{recon}||_F}{||f_{ave} - f||_F},$$
$$NMAD = \frac{\sum_{m,n} |f(m,n) - f_{recon}(m,n)|}{\sum_{m,n} |f(m,n)|}.$$

These measurements reflect different aspects of the quality of the recovered images. RMSE evaluates the reconstruction quality on a pixel-by-pixel basis. NRMSD emphasizes large errors in a few pixels of the recovered image. NMAD focuses on small errors in the recovered image.

The original and reconstructed images are shown in Figure 1. The experimental results are summarized in Table 1. The CPU time is measured by MATLB built-in functions. The values of RMSE, N-RMSD, and NMAD for the reweighted TV minimization are improved from the corresponding values for the TV minimization. The CPU time is saved 34.9%. The numerical experiments demonstrate that the reweighted TV minimization is superior to the regular TV minimization.

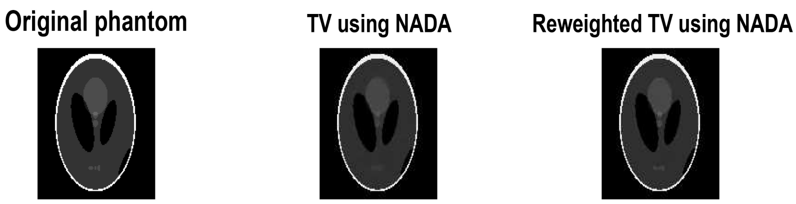

Figure 1. Shepp-Logan phantom and reconstructed images.

Table 1. Experimental data with shepp-logan phantom.

Algorithm	CPU Time	RMSE	NRMSD	NMAD
TV Using NADA	14.4 sec	0.0085	0.0088	0.0353
Reweighted TV using NADA	9.4 sec	0.0066	0.0069	0.0294

There are three important parameters in this proposed algorithm: β, μ, and rescaling factor r for w. Similar to a regular NADA without reweight, the values of β and μ should be between 2^4 and 2^{13}. The rescaling factor r is adopted between $\frac{1}{\text{mean}(w)}$ and $\frac{4}{\text{mean}(w)}$ in the experiments. It is a challenging question how to optimize the reconstructed images by reweighed TV using NADA. More extensive numerical experiments will be performed in future investigation.

5. Conclusion

The reweighted TV minimization has shown its capability of speeding up the convergence of sparse image reconstruction in CT since it was proposed more than ten years ago. In recent years the ADM was widely applied to a class of equality-constrained nonsmooth optimization problems and particularly image reconstruction in CT. As far as the authors are aware, the ADM hasn't been used to solve the reweighted TV minimization in the literature. This paper develops a reweighted total variation algorithm using the ADM method for image reconstruction in CT, as the major novelty of our work. The usage of the nonmonotone line search scheme reduces complexity and improves efficiency. Numerical simulation indicate that the proposed algorithm can effectively incorporate the reweighted scheme and improve the efficiency for image reconstruction in CT.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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