Structural Reliability Assessment by a Modified Spectral Stochastic Meshless Local Petrov-Galerkin Method

Guang Yih Sheu

1Department of Accounting and Information System, Chang-Jung Christian University, Tainan, Chinese Taipei
2Department of Civil Engineering, Feng-Chia University, Taichung, Chinese Taipei
Email: xsheu@hotmail.com

Received January 29, 2013; revised March 9, 2013; accepted March 16, 2013

ABSTRACT

This study presents a new tool for solving stochastic boundary-value problems. This tool is created by modifying the previous spectral stochastic meshless local Petrov-Galerkin method using the MLPG5 scheme. This modified spectral stochastic meshless local Petrov-Galerkin method is selectively applied to predict the structural failure probability with the uncertainty in the spatial variability of mechanical properties. Except for the MLPG5 scheme, deriving the proposed spectral stochastic meshless local Petrov-Galerkin formulation adopts generalized polynomial chaos expansions of random mechanical properties. Predicting the structural failure probability is based on the first-order reliability method. Further comparing the spectral stochastic finite element-based and meshless local Petrov-Galerkin-based predicted structural failure probabilities indicates that the proposed spectral stochastic meshless local Petrov-Galerkin method predicts the more accurate structural failure probability than the spectral stochastic finite element method does. In addition, generating spectral stochastic meshless local Petrov-Galerkin results are considerably time-saving than generating Monte-Carlo simulation results does. In conclusion, the spectral stochastic meshless local Petrov-Galerkin method serves as a time-saving tool for solving stochastic boundary-value problems sufficiently accurately.

Keywords: Spectral Stochastic Meshless Local Petrov-Galerkin Method; Generalized Polynomial Chaos Expansion; First-Order Reliability Method; Structural Failure Probability; Reliability Index

1. Introduction

Available stochastic numerical methods for solving stochastic boundary-value problems include the Monte Carlo simulation, spectral stochastic finite element [1] and stochastic element-free Galerkin methods [2]. The Monte Carlo simulation may be simplest, since implementing it requires sampling the existing random fields and substituting the resulting samples into deterministic solutions. However, a perquisite of obtaining accurate Monte Carlo simulation results is sufficiently sampling the existing random fields; therefore, completing a Monte Carlo simulation is usually time-consuming. This perquisite brings about a motive of developing a time-saving tool for solving stochastic boundary-value problems.

Meanwhile, the spectral stochastic finite element or stochastic element-free Galerkin methods are developed by extending the finite element or element-free Galerkin methods. For example, deducing a spectral stochastic finite element couples a finite element formulation with such as polynomial chaos and Karhunen-Loève expansions of stochastic processes. These stochastic processes are assumed to represent the existing uncertainty.

A number of spectral stochastic finite element formulations are available for some branches in engineering science and mechanics. References [3,4] are two recent examples. Nevertheless, applying these two stochastic numerical methods needs a finite element discretization or background cells for the numerical integration. To provide more freedom in solving stochastic boundary-value problems, a truly-meshless stochastic numerical method may be a promising alternative. In a published study [5], a spectral stochastic meshless local Petrov-Galerkin method has been developed by coupling a meshless local Petrov-Galerkin formulation and radial basis function-based meshfree shape functions with polynomial chaos expansions [6] of stochastic processes. Since the meshless local Petrov-Galerkin method is truly meshless [7], the spectral stochastic meshless local Petrov-Galerkin method is also truly meshless. Nonetheless, the spectral stochastic meshless local Petrov-Galerkin results of two elastostatic problems are more accurate than spectral
stochastic finite element results of the same problems. In addition, generating the spectral stochastic meshless local Petrov-Galerkin results is considerably time-saving.

Based on the published conclusion [8] that the MLPG5 scheme may substitute for the finite element method to solve boundary-value problems, the current study further derives a two-dimensional spectral stochastic meshless local Petrov-Galerkin formulation in elastostatics using the MLPG5 scheme. The resulting spectral stochastic meshless local Petrov-Galerkin formulation is selectively applied to predict the structural failure probability with the uncertainty in the spatial variability of mechanical properties. In addition to the MLPG5 scheme, deriving the proposed spectral stochastic meshless local Petrov-Galerkin formulation adopts the generalized polynomial chaos expansions [6] of random mechanical properties to deduce a 2D spectral stochastic meshless local Petrov-Galerkin results is considerably time-saving.

In Section 2, deriving a meshless local Petrov-Galerkin formulation in elastostatics using the MLPG5 scheme is presented. In Section 3, coupling the resulting expressions in Section 2 with generalized polynomial chaos expansions of random mechanical properties to deduce a spectral stochastic meshless local Petrov-Galerkin formulation is presented. In Section 4, the algorithm for implementing the first-order reliability method is reviewed. Section 5 inspects the accuracy of spectral stochastic meshless local Petrov-Galerkin-based results. Based on this inspection, Section 6 presents the conclusion.

2. Meshless Local Petrov-Galerkin Formulation

Suppose the linearly elastic and isotropic material. In addition, the infinitesimal strain assumption holds. Describe any physical parameter as functions of \( x \) and \( \theta \) within a problem domain \( \Omega \) in which \( x = (x_1, x_2) \) is a vector of spatial coordinates and \( \theta \) is an event in the probability space. The succeeding study introduces the stress equations of equilibrium to derive a meshless local Petrov-Galerkin formulation. These stress equations have the following tensor form [10]:

\[
\sigma_{i,j} + b_j = 0 \tag{1}
\]

where \( \sigma \) is the stress field corresponding to the displacement field \( u \) and \( b \) is the body force. The boundary conditions are

\[
T_i = \sigma_{i,j} n_j = T_{oi} \quad \text{on } \Gamma_T \nonumber
\]

\[
u_i = U_{oi} \quad \text{on } \Gamma_U
\tag{2}
\]

where \( \Gamma_T \) is the natural boundary, \( \Gamma_U \) is the essential boundary, \( T \) are the tractions, \( U_{oi} \) and \( T_{oi} \) are known functions, \( n \) are the components of a unit vector \( n \) outward normal to \( \Gamma \), and \( \Gamma = \Gamma_T \cup \Gamma_U \).

If \( N_T \) nodes locate within \( \Omega \) and \( \Omega_{cs} \) represents a local quadrature domain for a node \( x_I \) \( (I = 1 \text{ to } N_T) \), a local weak form of Equation (1) is

\[
\int_{\Omega_{cs}} (\sigma_{i,j} + b_j) w_i \mathrm{d}\Omega = 0 \tag{3}
\]

where \( w_i \) is the test function associated with \( x_I \). Subsequently, this study similarly manipulates a published radial basis function-based interpolation formula [5] to construct the meshfree shape function \( N \). Since the resulting \( N \) satisfies the Kronecker delta function property \(( \delta_{ij} = 0 \text{ for } I \neq J, \delta_{ij} = 1 \text{ for } I = J, \text{ and } I, J \text{ denote the } I\text{-th and } J\text{-th nodes})

Equation (3) contains neither Lagrange multipliers nor penalty parameters for imposing the essential boundary condition. Further simplifying Equation (3) by the divergence theorem results in

\[
\int_{\Omega_{cs}} \sigma_{i,j} w_{ij} \mathrm{d}\Omega - \int_{\Gamma_{ST}} \sigma_{ij} T_i w_j \mathrm{d}S - \int_{\Omega_{cs}} b_i w_i \mathrm{d}\Omega = 0 \tag{4}
\]

where \( \Gamma_{ST} = \Omega_{cs} \cap \Gamma_T, \Gamma_{SU} = \Omega_{cs} \cap \Gamma_U, L_S = \Gamma_S - \Gamma_{ST} - \Gamma_{SU}, \text{ and } \Gamma_S \) is the boundary of \( \Omega_{cs} \). Theoretically speaking, the shape of \( \Omega_{cs} \) can be arbitrary in computing Equation (4). However, choosing each \( \Omega_{cs} \) as a rectangular centered at \( x_I \) \( (I = 1 \text{ to } N_T) \) can simplify the numerical integration of Equation (4). In addition, \( \Omega_{cs} \) for \( x_I \) \( (I = 1 \text{ to } N_T) \) may be different from an interpolation domain \( \Omega_Q \) for approximating an unknown or a random field in the neighborhood of the same node. The difference between \( \Omega_{cs} \) and \( \Omega_Q \) is further illustrated in Figure 1. Also different in interpolation domains or points may be chosen for approximating an unknown or representing a random field.

Now substituting \( w_{ij}(x) = H(x) = c \) \( (x \in \Omega_{cs}) \) and \( w(x) = 0 \) \( (x \notin \Omega_{cs}) \) \( (I = 1 \text{ to } N_T) \) [7] into Equation (4) results in

\[
-\int_{\Gamma_{ST}} T_i \mathrm{d}S - \int_{\Gamma_{SU}} T_i \mathrm{d}S + \int_{\Omega_{cs}} b_i \mathrm{d}\Omega
\tag{5}
\]

where \( H \) denotes the Heaviside step function, and \( c \) is an arbitrary constant \(( c = 1 \text{ is used in the succeeding study})

Equation (5) outlines a distinguishing characteristic of the MLPG5 scheme. If the last term of this equation van-
ishes, this equation contains no domain integrals. Therefore, if Equation (5) is adopted to derive a spectral stochastic meshless local Petrov-Galerkin formulation, computing the resulting spectral stochastic meshless local Petrov-Galerkin formulation is more time-saving than computing the published spectral stochastic meshless local Petrov-Galerkin formulation [5].

Moreover, substituting
\[
T' = C_n \cdot n_j = (\lambda \delta_{ij} \epsilon_{ik} + 2G \epsilon_{ij}) n_j
\]
into Equation (5) yields
\[
-\int_{\Omega} nDBud\Gamma - \int_{\Gamma_{SL}} nDBd\Gamma = \int_{\Omega} Fd\Gamma + \int_{\Gamma_{SL}} bd\Omega
\]
where \( \lambda \) is the Lamé constant, \( G \) is the shear modulus, \( u = [u_1, u_2]^T \), \( T_0 = [T_{01}, T_{02}]^T \), \( b = [b_1, b_2]^T \), and
\[
\begin{bmatrix}
\frac{\partial}{\partial x_1} \\
\frac{\partial}{\partial x_2} \\
\frac{\partial}{\partial x_3}
\end{bmatrix}
\]

Next, similarly manipulating a published radial basis function-based interpolation formula [5], \( u \) over \( \Omega_Q \) for a node is approximated by
\[
u = N^T U = \begin{bmatrix} R^T & P^T \end{bmatrix} \begin{bmatrix} U \\ 0 \end{bmatrix}
\]
\[
= \begin{bmatrix} R^T & P^T \end{bmatrix} \begin{bmatrix} R_0 & P_0 \\ 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} U \\ 0 \end{bmatrix}
\]
\[
N^T = \begin{bmatrix} N_1, N_2, \ldots, N_M \end{bmatrix}, \quad U = [u_1, u_2, \ldots, u_M]^T, \quad M \text{ is the total number of nodes within } \Omega_Q, \quad \text{the subscript } i \text{ is the } i\text{-th node, } \quad R^T = [R_1, R_2, \ldots, R_M], \quad P^T = [P_1, P_2, \ldots, P_m] \text{ is a complete monomial basis of order } m, \quad R_i, \quad i = 1 \text{ to } M \text{ represent the radial basis function, and}
\]
in which \( x_1 \) to \( x_M \) represent those \( M \) nodes within \( \Omega_Q \) for \( x_i \), and \( r \) to \( r_M \) represents the Euclidean distance between \( x_i \) and each node within \( \Omega_Q \) for \( x_i \). Constructing \( N \) for further details can be seen in the published study [5].

Substituting Equation (8) into Equation (6) and writing the resulting expressions more succinctly in matrix algebra yield
\[
K_i u_i = F_i
\]
where \( K \) and \( F \) are, respectively, the stiffness and force matrices, the subscript \( I \) represents the contribution to \( K \) or \( F \) at \( x_i \) (\( I = 1 \) to \( N_I \)), \( u_i = [u_i, u_2, \ldots, u_M]^T \), \( K_i \) and \( F_i \) are derived by
\[
K_i = \int_{\Omega} nDB(N) d\Gamma - \int_{\Gamma_{SL}} nDB(N) d\Gamma
\]
\[
F_i = \int_{\Omega} F d\Gamma + \int_{\Gamma_{SL}} bd\Omega
\]
where the subscripts \( i \) and \( j \) denote \( i\)-th and \( j\)-th node within \( \Omega_Q \) for \( x_i \) (\( I = 1 \) to \( N_I \)); respectively and
\[
B(N) = \begin{bmatrix} \frac{\partial N}{\partial x_1} \\ \frac{\partial N}{\partial x_2} \\ \frac{\partial N}{\partial x_3} \\
\end{bmatrix}
\]

Repeatedly deriving Equation (10) for all \( N_I \) nodes and assembling all the resulting expressions based on a global numbering system yield
\[
K_{(2N_I+2N_J)} u_{(2N_I+1)} = F_{(2N_I+1)}
\]
Since this study accounts for the uncertainty in the spatial variability of mechanical properties in predicting the structural failure probability \( p_f \), the generalized polynomial chaos expansion is introduced to represent random mechanical properties. The next section presents the relevant derivation.

3. Spectral Stochastic Meshless Local Petrov-Galerkin Formulation

Observing the derivation of Equation (13) needs mechanical properties \( G \) and \( \lambda \). Thus, the generalized polynomial chaos expansions of \( G \) and \( \lambda \) are [5]
\[
G = \sum_{j=0}^{N_G} G_j \Psi_j(\xi), \quad \lambda = \sum_{j=0}^{N_\lambda} \lambda_j \Psi_j(\xi)
\]
where \( \xi = (\xi_1, \xi_2, \ldots, \xi_n) \), \( \Psi_j \) represent the multivariate orthogonal polynomial of \( \xi \), \( \xi_1, \xi_2, \ldots, \xi_n \) denote multi-dimensional uncorrelated random variables having zero mean and unit variance (for facilitating the computation of mean values and standard deviations of \( G \) and \( \lambda \), \( N_{PC} \) is equal to \( (n + P)!/n!(P-1)! \), \( P \) is the highest order of \( \Psi \), and \( n \) is the total number of uncorrelated random
variables.

For facilitating the construction of Equation (14), \( \Psi_i \), \( \hat{G}_i \), and \( \hat{\lambda}_i \) are, respectively, set to 1, \( \mu_G \), and \( \mu_{\lambda} \) in which \( \mu_G \) and \( \mu_{\lambda} \) are mean values of \( G \) and \( \lambda \), respectively. Furthermore, computing \( \hat{G}_i \) (i = 1 to \( N_{PC} \)) needs the orthogonal relationship, \( \{ \hat{\Psi}_i, \hat{\Psi}_j \} = \{ \hat{\Psi}_i \} \hat{\delta}_{ij} \) where \( \{ \hat{\cdot} \} \) is the ensemble average. For example, \( \hat{G}_i \) (i = 0 to \( N_{PC} \)) are computed by

\[
\hat{G}_i = \left\{ \frac{\langle G \Psi_i \rangle}{\langle \Psi_i^2 \rangle} \right\} 
\]

where \( \{ \cdot \} \) is computed as follows: If \( f \) and \( g \) are two functions, \( \{ \cdot \} \) is computed by

1. Continuous case:

\[
\langle f(\xi), g(\xi) \rangle = \int \cdots \int f(\xi)g(\xi)W(\xi_1) \cdots W(\xi_n) d\xi_1 \cdots d\xi_n \quad (16a)
\]

2. Discrete case:

\[
\langle f(\xi), g(\xi) \rangle = \sum \sum \cdots \sum f(\xi)g(\xi)W(\xi_1) \cdots W(\xi_n) \quad (16b)
\]

where \( W(\xi_1) \cdots W(\xi_n) \) are the weighting functions. Since the succeeding study focuses on the continuous random fields, Table 1 [6] lists examples of orthogonal polynomials, statistical distributions and weighting functions to generate \( \Psi_i \) (i = 0 to \( \infty \)), \( \xi_1 \cdots \xi_n \), and \( W(\xi_1) \cdots W(\xi_n) \) respectively. Substituting Equation (14) into Equation (11) yields

\[
K_i = \sum_{k=0}^{N_{PC}} \left[ -\hat{\Psi}_k nD_k B(N) d\Gamma - \hat{\psi}_k nD_k B(N) d\Gamma \right] 
\]

\[
= \sum_{k=0}^{N_{PC}} \left( \hat{K}_i \right)_k \Psi_k 
\]

in which \( D_L \) represents the computation of \( D \) using \( \hat{G}_L \) and \( \hat{\lambda}_L \) (L = 0 to M) and

\[
\left( \hat{K}_i \right)_L = \sum_{k=0}^{N_{PC}} nD_k B(N) d\Gamma - \hat{\psi}_k nD_k B(N) d\Gamma 
\]

Since the expressions of \( F_i \) doesn’t contain \( G \) and \( \lambda \), substituting the generalized polynomial chaos expansions of \( G \) and \( \lambda \) into \( F_i \) is unnecessary. Meanwhile, the generalized polynomial chaos expansion of \( u \) is

\[
u = \sum_{j=0}^{N_{PC}} \hat{u}_j \Psi_j
\]

Substituting Equations (17) and (19) into Equation (11) results in

\[
\sum_{L=0}^{N_{PC}} \sum_{j=0}^{N_{PC}} \hat{K}_L \hat{u}_j \Psi_L \Psi_j = \sum_{L=0}^{N_{PC}} \hat{F}_L \langle \Psi_L \Psi_k \rangle 
\]

in which \( k = 0 \) to \( N_{PC} \). Solving Equation (21) can obtain \( \hat{u}_j \) (J = 0 to \( N_{PC} \)). Collecting the resulting \( \hat{u}_j \) constructed the generalized polynomial chaos expansion of \( u \).

4. First Order Reliability Method

This study introduces structural reliability assessment problems to evaluate the performance of Equation (21). Estimating this structural reliability follows the first-order reliability method [9]; therefore, this section summarizes the first-order reliability method.

Given a traction \( T_0 \) bearing on a structure subjected to the uncertainty in the spatial variability of \( G \) and \( \lambda \), a vector \( X \) having components \( X = (X_1, X_2) \) is created. In addition, a performance state function \( g(X) \) is defined to identify the failure (\( g(X) < 0 \)) and safe states (\( g(X) > 0 \)) of the structure. For example, a book [11] emphasizes \( T_0 \) and its resistance \( R \); thus, \( g(X) \) is

\[
g(X) = T_0 - R
\]

Moreover, a space of different \( X \) values is plotted and the location of \( g(X) \) is marked. Figure 2 [12] illustrates the special case of \( X = (X_1, X_2) \). In this figure, suppose the point \( A \) denotes the point \( (\mu_{X_1}, \mu_{X_2}) \) and the structure is safe at this point. Observing Figure 2 can know that the shortest distance between point \( A \) and \( g(X) \) is the range of \( X \) values within which a safe structural design is expected. Extending this observation, the shortest distance between \( (\mu_{X_1}, \mu_{X_2}, \cdots, \mu_{X_N}) \) and \( g(X) \) is the range of \( X \) values within which a safe structural design is expected. Hasofer and Lind (1974) [9] defined the shortest distance between \( (\mu_{X_1}, \mu_{X_2}, \cdots, \mu_{X_N}) \) and \( g(X) \), in units of directional standard deviations as the reliability index \( \beta \). They concluded that searching \( \beta \) is a constrained optimization problem in the form as

\[
\beta = \min_{X \in \mathbb{R}^N} \sqrt{(X - \mu)^	op C^{-1} (X - \mu)} 
\]

subjected to \( g(X) = 0 \)

Table 1. Examples of polynomials and corresponding weight functions and statistical distributions for generating the generalized polynomial chaos [6].

<table>
<thead>
<tr>
<th>Distribution</th>
<th>Polynomial</th>
<th>( W(\xi) )</th>
<th>Interval</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gaussian</td>
<td>Hermite ( H_n(\xi) )</td>
<td>( \exp(-\hat{\xi}^2) )</td>
<td>( (-\infty, \infty) )</td>
</tr>
<tr>
<td>gamma</td>
<td>Laguerre ( L_n(\xi) )</td>
<td>( \exp(-\xi) )</td>
<td>( [0, \infty] )</td>
</tr>
<tr>
<td>beta</td>
<td>Jacobi ( G_{\rho,q}(\xi) )</td>
<td>( (1-\xi)^{\rho}(1+\xi)^{q} )</td>
<td>( [a, b] )</td>
</tr>
<tr>
<td>uniform</td>
<td>Legendre ( P_n(\xi) )</td>
<td>1</td>
<td>( [a, b] )</td>
</tr>
</tbody>
</table>

Note that \([a, b]\) denotes a specific interval.
where $F$ denotes the failure region on the space of $X$, $\boldsymbol{\mu} = \{\mu_{X_1}, \mu_{X_2}, \cdots, \mu_{X_N}\}$, and $C$ is the covariance matrix.

A number of algorithms have been developed to solve Equation (23) or similar equations. This study chooses a popular algorithm suggested by Lowe and Tang [12]. Briefly, this algorithm is based on the Rackwitz-Fiessler equivalent normal transformation [13] but the concepts of coordinate transformation and frame-of-reference rotation are not applied. Correlation is accounted for by setting up the quadratic form directly. Similarly manipulating the previous study [12], three steps are performed to solve Equation (23):

1) Modify Equation (23) with standard normal random variables. Transform the vector $X$ into a new vector of $Y$ having standard normal random variables $Y_i$ ($i = 1$ to $2N_f$) in the form as

$$Y_i = \frac{X_i - \mu_{X_i}}{S_{X_i}} = \Phi^{-1} \left[ CDF \left( X_i \right) \right]$$

in which $\Phi(X) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{X} \exp(-\zeta^2/2) d\zeta$ is the cumulative probability function of a standard normal distribution, $CDF$ is the cumulative probability function computed at $X_i$. Using the new vector $Y$, Equation (23) is modified to

$$\beta = \min_{x \in F} \sqrt{Y^T \rho^{-1} Y} \quad \text{subjected to} \ g \left( X \right) = 0$$

in which $\rho$ is the correlation matrix evaluated at $Y$.

2) Start from $Y_i = 0$ ($i = 1$ to $2N_f$) (or $X_i = \mu_{X_i}$) to search the $X$ value causing $g(X) = 0$. In searching such an $X$ value, increase $Y$ values and calculate the corresponding $X$ value by Table 2 [12]. Note that this table is edited by choosing those probability distributions, which may be applied in the current study.

3) To incorporate with a spectral stochastic meshless local Petrov-Galerkin FORTRAN code, a VA10AD subroutine [14] is introduced to automate the above two steps.

After finding the $X$ value causing $g(X) = 0$ and computing the corresponding $\beta$ value from Equation (25), the structural failure probability $p_f$ is estimated by

$$p_f = CDF \left( X \right) = \int_{g \left( X \right) < 0} PDF \left( X \right) dX$$

$$\approx \Phi \left( -\beta \right) = 1 - \Phi \left( \beta \right)$$

where $PDF$ is the probability density function.

### Table 2. Obtaining $X_i$ from $Y_i$ based on $CDF(X_i) = \Phi(Y_i)$ [12].

<table>
<thead>
<tr>
<th>Distribution</th>
<th>$X_i = CDF^{-1} (\Phi(Y_i))$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Normal</td>
<td>$X_i = \mu_i + Y S_i$</td>
</tr>
<tr>
<td>Lognormal</td>
<td>$X_i = \exp(\lambda + \zeta Y_i)$, $\zeta = \frac{\ln \left[ \frac{1}{1 + \frac{S_i}{\mu_i}} \right]}{\ln \frac{\ln 0.5}{\ln 1}}$</td>
</tr>
<tr>
<td>Extreme value</td>
<td>$X_i = \eta - \frac{\ln \left[ -\ln(\Phi(Y_i)) \right]}{\alpha}$, $\zeta = \frac{\pi}{\sqrt{6S_{ii}}}$</td>
</tr>
<tr>
<td>Exponential</td>
<td>$X_i = -\mu_i \ln[1 - \Phi(Y_i)]$</td>
</tr>
<tr>
<td>Uniform</td>
<td>$f_{X_i}(X) = \begin{cases} 1/(b-a) - \frac{X-a}{b-a} &amp; X \in [a,b] \ 0 &amp; \text{otherwise} \end{cases}$</td>
</tr>
</tbody>
</table>

$f_{X_i}$ is probability density function.
5.1. Bending of a Cantilever Beam by a Parabolically Distributed Traction

Suppose the cantilever beam has the length $L$, width $h$, and unit thickness. Figure 3 illustrates the layout of this cantilever beam and boundary conditions in which $Q$ denotes the integration of parabolically distributed traction along the $x_2$ direction and the point B is subsequently used to define the performance state function $g(X)$. If any uncertainty is neglected, the analytical solution of $u_2$ is [10]

$$u_2 = -\frac{Q(\lambda + G)}{6G(\lambda + 2G)I} \left[ 3\lambda x_i^2(L-x_i) + \frac{5S_i}{2(\lambda + 2G)} \left( h^3 x_i + \frac{3}{4} (3L-x_i)x_i^2 \right) \right]$$

where $I = h^3/12$ is the moment of inertia and $Q$ is integration of the parabolically distributed traction along the $x_2$ direction. Since only the analytical solution of $u_2$ is adopted subsequently to implement the Monte Carlo simulation, analytical solutions of $u_1$ and $\sigma_i$ ($i, j = 1$ to 2) are not listed here. Interested readers can find these analytical solutions in the book [10].

Nevertheless, this study accounts for the uncertainty in random $G$ and $\lambda$ in predicting $p_f$. Assume $G$ and $\lambda$ vary according to

$$G = \mu_G \left[ 1 + \alpha_G(x) \right] \quad \text{and} \quad \lambda = \mu_\lambda \left[ 1 + \alpha_\lambda(x) \right]$$

where $\alpha_G$ and $\alpha_\lambda$ are two homogeneous Gaussian random fields with zero mean and having the following covariances $C_G$ and $C_\lambda$:

$$C_G = \text{cov} \left[ G(x_1, x_2), G(x_1 + \xi_1, x_2 + \xi_2) \right] = S_G^2 \exp \left( -\frac{\xi_1^2}{d_L^2} - \frac{\xi_2^2}{d_h^2} \right)$$

$$C_\lambda = \text{cov} \left[ \lambda(x_1, x_2), \lambda(x_1 + \xi_1, x_2 + \xi_2) \right] = S_\lambda^2 \exp \left( -\frac{\xi_1^2}{d_L^2} - \frac{\xi_2^2}{d_h^2} \right)$$

in which cov represents the covariance, $(x_1, x_2)$ and $(x_1 + \xi_1, x_2 + \xi_2)$ are two points on the cantilever beam, $S_G$ and $S_\lambda$ are respectively, standard deviations of $G$ and $\lambda$, and $d_i$ ($i = 1$ to 4) are four correlation parameters.

To predict $p_f$ of the cantilever beam with the uncertainty in random $G$ and $\lambda$, essential data are listed below

1) Define the problem domain $\Omega$ as $0 \leq x_1 \leq L$ and $-h/2 \leq x_2 \leq h/2$.

2) Generate two cases of meshless discretizations and one case of finite element discretization. Figure 4 illustrates these meshless (top and middle sub-figures) and finite element discretizations (bottom sub-figure) in which the meshless discretization of randomly located nodes (middle sub-figure) is obtained by randomly distributing the meshless discretization of equally spaced nodes (bottom sub-figure).

3) Experiment to represent $G$, $\lambda$, and $u$ by the Laguerre polynomial chaos.

4) Set a complete monomial basis $p^T = [1, x_1, x_2]$ ($m = 3$). Setting such a low-order of $p$ is intentional. Observing the accuracy of corresponding numerical results is desired.

5) Construct $N$ by the Gaussian radial basis function; that is, $R_i = \exp \left( -\alpha r_i^2 \right)$ ($i = 1$ to $M$) where $\alpha_i$ ($\geq 0$) is a shape parameter.

6) Choose each $\Omega_\lambda$ as a circle centered at a point and each $\Omega_G$ as a rectangular centered at a node. The length and width of each $\Omega_\lambda$ and radius of each $\Omega_G$ are set subsequently.

7) Define the performance state function $g(X)$ by

$$g(X) = u_1 - u_{2,B}$$

where $u_1$ is a threshold of displacement and the subscript $B$ denotes the point B in Figure 3.

8) Generate Monte Carlo simulation results to serve as the accuracy standard in comparing spectral stochastic finite element-based and spectral stochastic meshless local Petrov-Galerkin-based predicted $p_f$. Following a book [11] the first step of implementing a Monte Carlo simulation is sampling of $G$ and $\lambda$ according to Equation (28). Each sample of $G$ and $\lambda$ are then substituted into

![Figure 3. Bending of a cantilever beam by a parabolically distributed at its ends (not to scale, $i = 1$ to 2).](image)

![Figure 4. Meshless and finite element discretizations for analyzing the first benchmark problem.](image)
Equation (27) to compute a sample of \( u_{2,B} \). If \( N_{\text{sample}} \) is the total number of samples of \( u_{2,B} \) and \( N(g(X) \leq 0) \) is the total number of samples of \( u_{2,B} \) causing the structural failure, \( p_f \) is computed by

\[
p_f = \frac{N(g(X) \leq 0)}{N_{\text{sample}}} \tag{31}
\]

Moreover, the resulting \( p_f \) can be inverted to compute the reliability index \( \beta \). If \( G \) and \( \lambda \) are sufficiently sampled, the Monte Carlo simulation-based \( p_f \) and \( \beta \) approach their exact values.

9) Unless otherwise stated, the following parameters are adopted: \( L = 48 \text{ m}, h = 12 \text{ m}, N_{\text{PC}} = 10, \mu_G = 11.5 \text{ MPa}, \mu_s = 17.3 \text{ MPa}, \alpha_s = 0.03, H_s = 9.6 \text{ m}, B_s = 6 \text{ m}, r_Q = 6 \text{ m}, N_{\text{sample}} = 10^6, N_q = 16, d_i = 1 \) (\( i = 1 \) to \( 4 \)), \( Q = 10^3 \) kN where \( H_s \) and \( B_s \) are; respectively, the height and width of each \( \Omega_s \), \( r_Q \) is the radius of each \( \Omega_s \), and \( N_q \) is the total number of quadrature points in each \( \Omega_s \) or finite element.

Moreover, in order to state quantitatively the accuracy of spectral stochastic meshless local Petrov-Galerkin or spectral stochastic finite element results, two error estimators \( \Delta \) and \( \delta \) are defined below

\[
\Delta(\%) = \left( \frac{(p_f)_{\text{MCS}} - (p_f)_{\text{SSMLPG}}}{(p_f)_{\text{MCS}}} \right) \tag{32}
\]

\[
\delta(\%) = \left( \frac{(p_f)_{\text{MCS}} - (p_f)_{\text{SSFEM}}}{(p_f)_{\text{MCS}}} \right)
\]

in which the subscripts \( \text{MCS}, \text{SSMLPG}, \text{SSFEM} \) denote the Monte Carlo simulation, spectral stochastic meshless local Petrov-Galerkin and spectral stochastic finite element methods, respectively.

Figure 5 compares variation of the predicted PDF of \( u_2 \) at the point B versus different prediction methods

![FIGURE 5](image)

Figure 5. Variation of the predicted probability density function of \( u_2 \) at the point B versus different prediction methods (First benchmark problem, meshless discretization: the top sub-figure of Figure 4, \( d_1 = d_2 = d_3 = d_4 = 1 \)).

Figure 6 (in the next page) presents variation of the predicted \( p_f \) at the point B versus different \( u_2 \) values, prediction methods, and \( S_{\ell}/\mu_{\ell} = 0.12, 0.24, 0.32 \). Furthermore, Table 3 compares the time spent to produce the spectral stochastic meshless local Petrov-Galerkin-based and Monte Carlo simulation-based predicted \( p_f \) with \( S_{\ell}/\mu_{\ell} = 0.12 \).

Benefiting from adopting the MLPG5 scheme to derive a spectral stochastic meshless local Petrov-Galerkin formulation, Table 3 indicates that generating spectral stochastic meshless local Petrov-Galerkin results is considerably time-saving, even if the Monte Carlo simulation is implemented using analytical solutions. Meanwhile, Figure 5 illustrates the necessity of predicting \( u_2 \) with the uncertainty in the spatial variability of \( G \) and \( \lambda \). When the \( S_{\ell}/\mu_{\ell} \) values increase, the standard deviation of \( u_2 \) increase; thus, obtaining the predicted \( u_2 \), which is different from its mean value, becomes more and more possible. In addition, Figure 6 presents that the spectral stochastic meshless local Petrov-Galerkin method predicts more accurate \( p_f \) than the spectral stochastic finite element method does. For example, if computing \( \Delta \) and \( \delta \) values with \( S_{\ell}/\mu_{\ell} = S_{\ell}/\mu_{\ell} = 0.12 \), the resulting \( \Delta \) value approximately peaks at 0.36 %; whereas, the resulting \( \delta \) value peaks about at 13.97 %.

Nevertheless, the performance of both Equation (21) and \( S_{\ell}/\mu_{\ell} = S_{\ell}/\mu_{\ell} = 0.12 \).
and spectral stochastic finite element method becomes gradually unsatisfactory when $S_G/\mu_G$ and $S_d/\mu_d$ increases. If $S_G/\mu_G$ and $S_d/\mu_d$ values measure the degree of uncertainty. Figure 6 outlines that the degree of uncertainty can apparently reduce the accuracy of predicted $u$ or $p_f$. Furthermore, observing Equations (29a) to (29b) can find that decreasing $d_i$ ($i = 1$ to 4) values and increasing $S_G/\mu_G$ and $S_d/\mu_d$ values have similar effects on the accuracy of predicted $u$ or $p_f$.

Next, replacing Lauguerre polynomial chaos with Hermite polynomial chaos to represent a random field. Calculating $\Delta$ values using data in Figure 7 finds that the performance of Lauguerre polynomial chaos is more satisfactory. If $G$ and $\lambda$ are represented using the Hermite polynomial chaos, the corresponding $\Delta$ value peaks at about 2.758%.

Next, replacing meshless discretization of equally spaced nodes (the top sub-figure of Figure 4) with meshless discretization of randomly located nodes (the middle sub-figure of Figure 4). Figure 7 re-compare variation of Monte Carlo simulation-based and spectral stochastic meshless local Petrov-Galerkin-based predicted $p_f$ values versus different $u_\delta$ values, and $S_G/\mu_G = S_d/\mu_d = 0.12$.

Computing the $\Delta$ value using data in Figure 8 finds that the resulting $\Delta$ value peaks at about 2.367%. Consequently, Equation (21) still predicts $p_f$ sufficiently accurately, even if a meshless distribution of discrete nodes is adopted. Moreover, in an attempt of more understanding the effects of different nodal spacings on the accuracy of predicted $p_f$, the problem domain $\Omega$ is re-discretized using equally spaced nodes and $N_T = 27 \times 9 = 52 \times 13 = 85 \times 17$. Figure 9 depicts the corresponding variation of Monte Carlo simulation-based and spectral stochastic meshless local Petrov-Galerkin-based predicted $p_f$ versus different $h$ values, $u_\delta = 0.95$ cm, and $S_G/\mu_G = S_d/\mu_d = 0.12$ where $h$ denotes the spacing of any two connecting nodes.

Figure 9 reports that the effects of different $h$ values on the accuracy of predicted $p_f$ are not noticeable. For example, if the $N_T$ value increases from 52 (3 $\times$ 9) to 85 (5 $\times$ 17), the corresponding $\Delta$ value only changes slightly; consequently, adopting more nodes for improving the accuracy of Monte Carlo simulation-based or spectral stochastic meshless local Petrov-Galerkin-based predicted $p_f$ is laborious.

5.2. Bending of a Dam Caused by Fluid Pressure

Suppose the dam has the length $L$, width $h$, and unit thick-

![Figure 7](image7.png)
Figure 7. Variation of the predicted structural failure probability $p_f$ at the point B versus different types of the polynomial chaos (first benchmark problem, meshless discretization: the upper sub-figure of Figure 4, $d_1 = d_2 = d_3 = d_4 = 1$).

![Figure 8](image8.png)
Figure 8. Variation of the predicted structural failure probability $p_f$ at the point B versus randomly nodal distribution (first benchmark problem, meshless discretization: the middle sub-figure of Figure 4, $d_1 = d_2 = d_3 = d_4 = 1$).

![Figure 9](image9.png)
Figure 9. Variation of the predicted structural failure probability $p_f$ at the point B versus different nodal spacing $h$ (m) (First benchmark problem, meshless discretization: the upper sub-figure of Figure 4, $d_1 = d_2 = d_3 = d_4 = 1$).
ness. This dam is fixed at one end and subjected to fluid pressure. Figure 10 illustrates the layout of problem domain $\Omega$ and boundary conditions in which C and D are subsequently used to define the performance state function and $\gamma_f$ is the unit weight of fluid.

If any uncertainty is neglected, the analytical solutions of $u_1$ and $u_2$ are [10]

$$u_1 = \frac{(\lambda + G)(x_1^2 - L^2)}{G(3\lambda + 2G)h^2} \left( x_2^2 + \frac{3h^2}{20} \right)$$

$$+ \frac{(x_1^2 - L^2)^2}{3} \left[ \frac{\lambda (x_1^2 - L^2) y_f}{4G(3\lambda + 2G)} \right] \left( -\frac{1}{2} + \frac{x_2}{h} \left( \frac{2x_2}{h^2} - \frac{3}{2} \right) \right)$$

(33a)

$$u_2 = \frac{(\lambda + G)(x_1 - L)y_f x_2}{2G(3\lambda + 2G)} \left( -1 + \frac{x_2}{h} \left( \frac{x_2^2}{h^2} - \frac{3}{2} \right) \right)$$

$$+ \frac{\lambda y_f x_2^2}{2G(3\lambda + 2G)h} \left[ \left( \frac{x_2^2}{h^2} - \frac{1}{2} \right) + \frac{(x_1 - L)}{2} \left( \frac{x_2^2}{h^2} - \frac{3h^2}{10} \right) \right]$$

(33b)

However, suppose $G$ and $\lambda$ vary according to two uniform distributions:

$$G = \mu_G \left[ 1 + S_0^2 (\zeta_1 + \zeta_2) \right]$$

$$\lambda = \mu_\lambda \left[ 1 + S_0^2 (\zeta_1 + \zeta_2) \right]$$

(34)

where $-1 < \zeta_i (i = 1 \text{ to } 4) < 1$ represent four random variables.

To predict $p_f$ of the dam with the uncertainty in random $G$ and $\lambda$, the essential data are provided below:

1) Define the problem domain $\Omega$ as $-h/2 \leq x_2 \leq h/2$ and $0 \leq x_1 \leq L$.

2) Generate a meshless discretization and a finite element discretization. Figure 11 presents these meshless and finite element discretizations.

3) Represent $G$, $\lambda$, and $u_i$ ($i = 1 \text{ to } 2$) by the Legendre polynomial chaos.

4) Still set a complete monomial basis $p^T = [1, x_1, x_2]$ but adopt the multiquadric radial basis function to construct $\phi_i$ that is, $R_i = \left[ r_i^2 + (\alpha_i d_i)^2 \right]^{q/2} (i = 1 \text{ to } M)$ where $\alpha_i$ ($\geq 0$) and $q$ are two shape parameters and $d_i$ is the characteristic length related to the nodal spacing in an $\Omega$.

5) Define two performance state functions $g_1(X)$ and $g_2(X)$ as follows:

$$g_1(X) = u_{2,\delta} - u_{2,C}$$

(35a)

$$g_2(X) = u_{1,\delta} - u_{1,D}$$

(35b)

where $u_{i,\delta} (i = 1 \text{ to } 2)$ are two thresholds of displacements and the subscripts $C$ and $D$ denote the points $C$ and $D$ in Figure 10.

6) Similarly manipulate point (8) in Section 5.1 but replace Equation (27) with Equations (33a) to (33b) to generate the Monte Carlo simulation-based predicted $p_f$. The resulting Monte Carlo simulation-based predicted $p_f$ serves as the accuracy standard in comparing the spectral stochastic meshless local Petrov-Galerkin and spectral stochastic finite element results.

7) Unless otherwise stated, the following data are used: $L = 30 \text{ m}$, $h = 10 \text{ m}$, $\gamma_f = 9.81 \text{ kN/m}^3$, $N_{PC} = 10$, $\mu_G = 11.5$ MPa, $\mu_\lambda = 17.3$ MPa, $\alpha_c = 1.0$, $d_c = 3.0$, $q = 1.03$, $H_S = 5 \text{ m}$, $B_S = 5 \text{ m}$, $r_Q = 5 \text{ m}$, $N_{\text{sample}} = 10^4$, and $N_f = 16$.

Figure 12 (in the next page) compares variation of the $p_f$ at the point C with respect to $S_C/\mu_G = S_d/\mu_\lambda = 0.12, 0.24, 0.32$, different prediction methods and $u_{2,\delta}$ values. Figure 13 (in the next page) compares variation of the $p_f$ at the point D with respect to $S_C/\mu_G = S_d/\mu_\lambda = 0.12, 0.24, 0.32$, different prediction methods and $u_{1,\delta}$ values.

Observing Figure 12 confirms that the performance of
spectral stochastic meshless local Petrov-Galerkin method is more satisfactory than the performance of spectral stochastic finite element method. Even if different statistical distributions are encountered, Figures 6, 12, and 13 present the spectral stochastic meshless local Petrov-Galerkin results are more accurate than the spectral stochastic finite element results. In addition, careful inspection of spectral stochastic finite element results in Figures 12 to 13 finds that the errors between Monte Carlo simulation and spectral stochastic finite element results majorly source from inaccurate spectral stochastic finite element-based predicted mean values of $u$ at the points C and D. Resolving this problem may need high-order finite elements. But, to the author’s knowledge, similar experiences seem to be seldom seen.

6. Conclusions

Prior to the previous [5] and current studies, available stochastic numerical methods include the Monte Carlo simulation, spectral stochastic finite element, and stochastic element-free Galerkin methods. The Monte Carlo simulation is simplest. As demonstrated in Sections 5.1 and 5.2, implementing the Monte Carlo simulation only needs deterministic solutions. Even so, as outlined by Table 3, completing the Monte Carlo simulation is still more time-consuming than generating the spectral stochastic meshless local Petrov-Galerkin results. Producing such results attributes to that the total number of samples for implementing a Monte Carlo simulation is usually very large.

Meanwhile, applying the spectral stochastic finite element method is easy, since numerous resources (computer software and experiences) are available. Nevertheless, based on these resources, this study finds that the spectral stochastic finite element results of some problems are less accurate than spectral stochastic meshless local Petrov-Galerkin results of the same problems. Sections 5.1 and 5.2 provide two examples.

Together with the previous study [5], the succeeding study provides a new alternative for solving stochastic boundary-value problems. This new stochastic numerical method is truly-meshless. As demonstrated in Sections 5.1 and 5.2, no finite elements or background cells for the numerical integration are created in applying the spectral stochastic meshless local Petrov-Galerkin method. However, the spectral stochastic meshless local Petrov-Galerkin method successfully spend less time but still predict the accurate structural failure probability $p_f$ in Sections 5.1 and 5.2.

In conclusion, the spectral stochastic meshless local Petrov-Galerkin method is a time-saving tool for solving stochastic boundary-value problems.

REFERENCES


