A New Method of Estimating the Asset Rate of Return

Moawia Alghalith, Tracy Polius
Economics, University of the West Indies, St Augustine, USA
E-mail: malghalith@gmail.com, tracy.polius@sta.uwi.edu
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Abstract

We present a new consumption-based method of estimating the asset rate of return.

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In this note, we present a new model that links the stock/portfolio rate of return to consumption. Our approach is more general than the existing models such as the consumption-CAPM models, that are based on very restrictive assumptions [1]. In so doing, we utilize a more advanced and appropriate theoretical and empirical framework than the ones used by previous literature. It is worth noting that previous literature mainly used simple linear regressions without a rigorous theoretical basis.

We use a stochastic factor model, which includes a risky asset (portfolio, a risk-free asset and a stochastic external economic factor [2,3]. Thus, we have a two-dimensional standard Brownian motion

\[ \{W_t, W_s, \mathcal{F}_t\}_{t \leq s \leq T} \]

on the probability space \((\Omega, \mathcal{F}, P)\), where \(\{\mathcal{F}_t\}_{t \leq T}\) is the augmentation of filtration. The risk-free asset price process is

\[ S_0 = e^{\pi t} \]

where \(r(Z_t) \in C^1(R)\), is the rate of return and \(Z_t\) is the stochastic economic factor.

The dynamics of the risky asset price is given by

\[ dS_t = S_t \{\mu(Z_t) dt + \sigma(Z_t) dW_t^1\}, \]

where \(\mu(Z_t)\) and \(\sigma(Z_t)\) are the rate of return and the volatility, respectively. The stochastic economic factor process is defined as

\[ dZ_t = a(Z_t) dt + \rho dW_t^1 + \sqrt{1 - \rho^2} dW_t^2, Z_t = z, \]

where \(|\rho| < 1\) is the correlation factor between the two Brownian motions, \(\gamma\) is a parameter, and \(a(Z_t) \in C^1(R)\) has a bounded derivative.

The wealth process is given by

\[ X^{x,c} = x + \int_0^T \left\{ r(Y_t) X^{x,c} + \left( \mu(Y_t) - r(Y_t) c_t \right) \right\} ds + \int_0^T \pi_t \sigma(Y_t) dW^1_t, \]

where \(x\) is the initial wealth, \(\{\pi_t, \mathcal{F}_t\}_{t \leq T}\) is the portfolio process and \(\{c_t, \mathcal{F}_t\}_{t \leq T}\) is the consumption process, with \(\int_0^T \pi^2_t ds < \infty, \int_0^T c_t ds < \infty\) and \(c \geq 0\). The trading strategy \((\pi_t, c_t) \in A(x, y)\) is admissible.

The investor’s objective is to maximize the expected utility of terminal wealth and consumption

\[ v(T, x, y) = \sup_{\pi_t \in A} E \left[ U_1 \left( X^{x,c}_T \right) + \int_0^T U_2 (c_t) ds \right], \]

where \(v(.)\) is the (smooth) value function, \(U(.)\), bounded and strictly concave utility function.

The value function satisfies the Hamiltonian-Jacobi-Bellman PDE

\[ v_t + r(y) x v_x + a(z) v_y + \frac{1}{2} v_{yy} + \sup_{\pi_t \in A} \left\{ \frac{1}{2} \pi_t^2 \sigma^2(z) v_{xx} + \left[ \pi_t \left( \mu(z) - r(z) \right) - c_t \right] v_x \right\} = 0, \]

\[ v(T, x, z) = U(x), \]

Hence, the optimal solutions are

\[ \pi^{*}_t = \frac{\left( \mu(z) - r(z) \right) v_x}{\sigma^2(z) v_{xx}} - \frac{\gamma \rho v_{xy}}{\sigma(z) v_{xx}}, \]
Using the result of Alghalith [3], the optimal portfolio can be expressed as
\[
\pi^*_t = -\frac{(\mu(z) - r(z))(\alpha_i + \alpha_z c^*_t)}{\sigma^2(z)} \frac{\alpha_i \rho}{\sigma(z)},
\]
and thus
\[
\frac{\partial \mu(z)}{\partial c^*_t} = \alpha_i (\mu(z) - r(z))(\alpha_i + \alpha_z c^*_t) + \alpha_z (\pi^*_t \sigma^2(z) + \alpha_i \rho \sigma(z)) \left(\frac{\alpha_i + \alpha_z c^*_t}{\sigma(z)}\right)^2.
\]

1. Empirical Example

We used quarterly data for Jamaica for the period March 1998 to June 2010 for real private aggregate consumption (in millions of dollars), stock index (JSI) and the Treasury bill rate (r), and GDP (as the stochastic factor Z). We also computed the volatility of the index and the correlation factor between GDP and the JSI.

Using (7)–(8), we estimated each of the following non-linear equations
\[
\pi^*_t = -\frac{(\mu(z) - r(z))(\beta_i + \beta_t c^*_t)}{\sigma^2(z)} \frac{\beta_i \rho}{\sigma(z)} + \epsilon_t,
\]
\[
\mu(z) = r - \frac{\pi^*_t \sigma^2(z) + \beta_t \rho \sigma(z)}{\beta_i + \beta_t c^*_t} + \epsilon_t,
\]
where \(\beta_i\) are the parameter to be estimated, while the other variables are observed data, and \(\epsilon\) is the estimation error. Using the estimated values \(\hat{\beta}_i\), we obtain the following comparative statics
\[
\frac{\partial \mu(z)}{\partial c^*_t} = \hat{\beta}_i (\mu(z) - r(z))(\hat{\beta}_i + \hat{\beta}_t c^*_t) + \hat{\beta}_t (\pi^*_t \sigma^2(z) + \hat{\beta}_i \rho \sigma(z)) \left(\frac{\hat{\beta}_i + \hat{\beta}_t c^*_t}{\sigma(z)}\right)^2.
\]

In contrast, to previous studies that used simple linear regressions, the results support the existence of a very weak relationship between private consumption in Jamaica and the rate of return of the stock index (see Table 1).

Table 1. Empirical results.

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2. References