Decomposition of Point-Symmetry Using Ordinal Quasi Point-Symmetry for Ordinal Multi-Way Tables

Yusuke Saigusa, Kouji Tahata, Sadao Tomizawa

Department of Information Sciences, Tokyo University of Science, Chiba, Japan
Email: saigusaysk@gmail.com, kouji_tahata@is.noda.tus.ac.jp, tomizawa@is.noda.tus.ac.jp

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Abstract

For multi-way tables with ordered categories, the present paper gives a decomposition of the point-symmetry model into the ordinal quasi point-symmetry and equality of point-symmetric marginal moments. The ordinal quasi point-symmetry model indicates asymmetry for cell probabilities with respect to the center point in the table.

Keywords
Decomposition, Multi-Way Table, Ordinal Quasi Point-Symmetry, Point-Symmetry

1. Introduction

Consider an \( R_1 \times R_2 \times \cdots \times R_T \) table with ordered categories. Let \( i = (i_1, \cdots, i_T) \) for \( i_k = 1, \cdots, R_k \) and \( k = 1, \cdots, T \), and let \( p_{i_k} \) denote the probability that an observation will fall in \( i_k \)th cell of the table. Let \( X_{i_k} \) denote the \( k \)th variable of the table for \( k = 1, \cdots, T \). Denote the \( h \)th-order \((h=1, \cdots, T-1)\) marginal probability \( P(X_{i_{k_1}} = i_{k_1}, \cdots, X_{i_{k_h}} = i_{k_h}) \) by \( p_{i_{k_1}, \cdots, i_{k_h}} \) with \( 1 \leq k_1 < \cdots < k_h \leq T \).

In the case of \( R_1 = \cdots = R_T \) \((= R)\), the symmetry (\( S^2 \)) model is defined by

\[ p_i = \psi_i \]

for any \( i \), where \( \psi_i = \psi_j \) for any permutation \( j = (j_1, \cdots, j_T) \) of \( i \) (Bhapkar and Darroch, [1]; Agresti, [2], p. 439). We may also refer to this model as the permutation-symmetry model.

The \( h \)th-order marginal symmetry \((MS_h)\) model is defined by, for a fixed \( h \) \((h=1, \cdots, T-1)\),

\[ p_{i_{k_1}, \cdots, i_{k_h}} = \psi_{i_{k_1}, \cdots, i_{k_h}} \]

where \( \psi_{i_{k_1}, \cdots, i_{k_h}} = \psi_{j_{k_1}, \cdots, j_{k_h}} \) for any permutation \( j = (j_1, \cdots, j_T) \) of \( i \) (Bhapkar and Darroch, [1]; Agresti, [2], p. 439).
is any permutation of \((i_1, \ldots, i_b)\), and for any \((s_1, \ldots, s_j)\) and \((t_1, \ldots, t_b)\) (Bhapkar and Darroch, [1]). The \(h\)th-order quasi symmetry (\(QS^h\)) model is defined by, for a fixed \(h\) \((h = 1, \cdots, T - 1)\),

\[
p_{i} = \mu \left( \prod_{k=1}^{T} \alpha_k(i_k) \right) \left( \prod_{l=1}^{h} \prod_{k_h \leq \cdots \leq k_1 < T} \alpha_{k_h \cdots k_1}(i_{k_h} \cdots i_{k_1}) \right) \left( \prod_{l=1}^{h} \prod_{k_h \leq \cdots \leq k_1 < T} \alpha_{k_h \cdots k_1}(i_{k_h} \cdots i_{k_1}) \right) \psi_i \quad \text{for any } i,
\]

where \(\psi_i = \psi_j\) for any permutation \(j\) of \(i\) (Bhapkar and Darroch, [1]). Bhapkar and Darroch [1] gave the theorem that:

1) For the \(R^T\) table and a fixed \(h\) \((h = 1, \cdots, T - 1)\), the \(S^T\) model holds if and only if both the \(QS^h\) and \(MS^h\) models hold.

Tahata, Yamamoto and Tomizawa [3] considered the \(h\)th-linear ordinal quasi symmetry (\(LQS^h\)) model, which was defined by, for a fixed \(h\) \((h = 1, \cdots, T - 1)\),

\[
p_{i} = \mu \left( \prod_{k=1}^{T} \alpha_k(i_k) \right) \left( \prod_{l=1}^{h} \prod_{k_h \leq \cdots \leq k_1 < T} \alpha_{k_h \cdots k_1}(i_{k_h} \cdots i_{k_1}) \right) \psi_i \quad \text{for any } i,
\]

where \(\psi_i = \psi_j\) for any permutation \(j\) of \(i\). This model is a special case of the \(QS^h\) model. The \(LQS^h\) model is the ordinal quasi symmetry model when \(h = 1\) (Agresti, [4], p. 244). Tahata et al. [3] also considered the \(h\)th-order marginal moment equality (\(MME^h\)) model, which was expressed as, for a fixed \(h\) \((h = 1, \cdots, T - 1)\),

\[
\mu_{k_1, \cdots, k_h} = \mu_{l_1, \cdots, l_h} \quad (l = 1, \cdots, h),
\]

where \(\mu_{k_1, \cdots, k_h} = E(X_{k_1} \cdots X_{k_h})\) for \(1 \leq k_1 < \cdots < k_h \leq T\). Tahata et al. [3] obtained the theorem that:

2) For the \(R^T\) table and a fixed \(h\) \((h = 1, \cdots, T - 1)\), the \(S^T\) model holds if and only if both the \(LQS^h\) and \(MME^h\) models hold.

Various decompositions of the symmetry model are given by several statisticians, e.g. Caussinus [5], Bishop, Fienberg and Holland ([6], Ch.8), Read [7], Kateri and Papaioannou [8], and Tahata and Tomizawa [9].

For the \(R_i \times R_j \times \cdots \times R_T\) table, the point-symmetry (\(P^T\)) model is defined by

\[
p_{i} = \gamma_i \quad \text{for any } i,
\]

where \(\gamma_i = \gamma_j\) and \(i^T = (i_1^*, \cdots, i_T^*)\) with \(i_k^* = R_k + 1 - i_k\) for \(k = 1, \cdots, T\) (Wall and Lienert, [10]; Tomizawa, [11]). This model indicates the point-symmetry of cell probabilities with respect to the center point of multi-way table.

For the \(R^T\) table, Tahata and Tomizawa [12] considered the \(h\)th-order marginal point-symmetry (\(MP^h\)) model defined by, for a fixed \(h\) \((h = 1, \cdots, T - 1)\),

\[
p_{i}^{(k_1, \cdots, k_h)} = p_{i}^{(k_1, \cdots, k_h)} \quad (1 \leq k_1 < \cdots < k_h \leq T; \ i = 1, \cdots, R; \ l = k_1, \cdots, k_h).
\]

Tahata and Tomizawa [12] also considered the \(h\)th-order quasi point-symmetry (\(QP^h\)) model defined by, for a fixed \(h\) \((h = 1, \cdots, T - 1)\),

\[
p_{i} = \mu \left( \prod_{k=1}^{T} \alpha_k(i_k) \right) \left( \prod_{l=1}^{h} \prod_{k_h \leq \cdots \leq k_1 < T} \alpha_{k_h \cdots k_1}(i_{k_h} \cdots i_{k_1}) \right) \gamma_i \quad \text{for any } i,
\]

where \(\gamma_i = \gamma_j\). Tahata and Tomizawa [12] gave the theorem that:

3) For the \(R^T\) table and a fixed \(h\) \((h = 1, \cdots, T - 1)\), the \(P^T\) model holds if and only if both the \(QP^h\) and \(MP^h\) models hold.

Theorem 3) is Theorem 1) with structures in terms of permutation-symmetry, i.e. the \(S^T\), \(QS^h\) and \(MS^h\) models, replaced by structures in terms of point-symmetry, i.e. the \(P^T\), \(QP^h\) and \(MP^h\) models. However, a theorem in terms of point-symmetry corresponding to Theorem 2) is not obtained yet. So we are now interested in the decomposition of the \(P^T\) model.

In the present paper, Section 2 proposes three models. Section 3 gives a new decomposition of the \(P^T\) model. Section 4 provides the concluding remarks.
2. Models

Let \( S = \left\{ h \mid h = 2m - 1, m = 1, \ldots, \left\lfloor \frac{T}{2} \right\rfloor \right\} \), where \( \lfloor x \rfloor \) denotes the largest integer less than or equal to \( x \).

Consider the model defined by, for a fixed odd number \( h \) \((h \in S)\),

\[
\mu_{i,h,k_1,\ldots,k_i} = \mu_{i,h,k_1,\ldots,k_i}^* \quad (1 \leq k_1 < k_2 < \cdots < k_i \leq T; l = 1, 3, \cdots, h),
\]

where

\[
\mu_{i,h,k_1,\ldots,k_i} = E \left( X_{k_1} X_{k_2} \cdots X_{k_i} \right), \quad \mu_{i,h,k_1,\ldots,k_i}^* = E \left( X_{k_1}^* X_{k_2}^* \cdots X_{k_i}^* \right),
\]

and \( X_k^* = R_k + 1 - X_k \) for \( k = 1, \cdots, T \). We shall refer to this model as the \( h \)th-order marginal moment point-symmetry (\( \text{MMP}_h \)) model. Note that if the \( \text{MP}_h \) model holds then the \( \text{MMP}_h \) model holds. Under the \( \text{MMP}_h \) model, we see, for any \( k \) \((k = 1, \cdots, T)\),

\[
\mu_k = \frac{R_k + 1}{2}.
\]

Then we obtain, for any \( k_1 \) and \( k_2 \) \((1 \leq k_1 < k_2 \leq T)\),

\[
\mu_{i,h,k_1,k_2} - \mu_{i,h,k_1,k_2}^* = \sum_{i_1 \neq i_2} \sum_{2 \leq k_1 \leq k_2} \left( i_1 \mu_{i_1,k_1} - i_2 \mu_{i_2,k_2} \right) = 0.
\]

Under the \( \text{MMP}_3 \) model, we see, for any \( k_1, k_2 \) and \( k_3 \) \((1 \leq k_1 < k_2 < k_3 \leq T)\),

\[
\mu_{i,h,k_1,k_2,k_3} = -\left( \frac{1}{2} \left( R_{k_1} + 1 \right) \left( R_{k_2} + 1 \right) \left( R_{k_3} + 1 \right) - \left( R_{k_1} + 1 \right) \left( R_{k_2} + 1 \right) \mu_{i,k_3} - \left( R_{k_2} + 1 \right) \left( R_{k_3} + 1 \right) \mu_{i,k_1} - \left( R_{k_1} + 1 \right) \left( R_{k_3} + 1 \right) \mu_{i,k_2} \right).
\]

Then we obtain, for any \( k_1, k_2, k_3 \) and \( k_4 \) \((1 \leq k_1 < k_2 < k_3 < k_4 \leq T)\),

\[
\mu_{i,h,k_1,k_2,k_3,k_4} - \mu_{i,h,k_1,k_2,k_3,k_4}^* = \sum_{i_1 \neq i_2} \sum_{i_3 \neq i_4} \sum_{2 \leq k_1 \leq k_2} \sum_{2 \leq k_3 \leq k_4} \left( i_1 \mu_{i_1,k_1} - i_2 \mu_{i_2,k_2} - i_3 \mu_{i_3,k_3} + i_4 \mu_{i_4,k_4} \right) = 0.
\]

Thus we are not interested in the \( \text{MMP}_h \) model with \( h \) being even. Therefore we shall consider the \( \text{MMP}_h \) model with \( h \) being odd.

Consider the model defined by

\[
p_i = \mu \left( \prod_{x=1}^{T} \alpha_x^{\gamma_i} \right)^{\beta_i} \quad \text{for any } i,
\]

where \( \gamma_i = \gamma_j \). We shall refer to this model as the ordinal quasi point-symmetry (\( \text{OQP}_T \)) model. In the case of \( T = 2 \), this model is identical to the model proposed by Tahata and Tomizawa [13]. The special case of the \( \text{OQP}_T \) model obtained by putting \( \alpha_1 = \cdots = \alpha_T = 1 \) is the \( \mathbf{P} \) model. Also the \( \text{OQP}_T \) model is the special case of the \( \text{QP}_T \) model obtained by putting \( \alpha_1 = \alpha_2 = 1 \). The \( \text{OQP}_T \) model may be expressed as

\[
\log \frac{P_i}{P_i'} = \beta_0 + \sum_{k=1}^{T} \beta_k \quad \text{for any } i,
\]

with \( \beta_0 = -\sum_k (R_k + 1) \log \alpha_k \) and \( \beta_k = \log \alpha_k^2 \). From this equation, we can see the log-odds that an ob-
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...where \( \gamma_i = \gamma_i' \). We shall refer to this model as the \( h \)-linear ordinal quasi point-symmetry (LQP\(^h\)) model. Especially, when \( h = 1 \), the LQP\(^1\) model is identical to the OQP\(^T\) model. Also the LQP\(^1\) model is the special case of the QT\(^h\) model obtained by putting
\[
\alpha_k(i_k) = \alpha_k(i_k^1), \quad \alpha_k(i_k, i_k^1) = \alpha_k(i_k^1), \quad \ldots, \quad \alpha_k(i_k^1, \ldots, i_k^1) = 1, \quad \text{and} \quad \{ \alpha_k(i_k^1, i_k^2) = 1 \}.
\]

Figure 1 shows the relationships among models.

3. Decomposition of Point-Symmetry

We obtain the following theorem:

**Theorem 1.** For the \( R \times R \times \cdots \times R \) table and a fixed odd number \( h \ (h \in S) \), the PT model holds if and only if both the LQP\(^h\) and MMP\(^h\) models hold.

**Proof.** If the PT model holds, then both the LQP\(^h\) and MMP\(^h\) models hold. Assuming that both the LQP\(^h\) and MMP\(^h\) models hold, then we shall show the PT model holds. Let \( q = \{ q_i \} \) denote cell probabilities which satisfy both the LQP\(^h\) and MMP\(^h\) models. The LQP\(^h\) model is expressed as
\[
\log q_i = \log \mu r_i + \sum_{k=1}^t \log \alpha_i + \sum_{k=1}^t \sum_{i_k \in \mathbb{R} \times \cdots \times \mathbb{R}} i_1 \cdot i_2 \cdot i_3 \log \alpha_{i_1 i_2 i_3} + \cdots + \sum_{k=1}^t \sum_{i_k \in \mathbb{R} \times \cdots \times \mathbb{R}} i_k \log \alpha_{i_k},
\]
where \( \gamma_i = \gamma_i' \). Let
\[
c = \sum_{i_1 = 1}^{q_1} \cdots \sum_{i_t = 1}^{q_t} r_i, \quad \pi_i = \frac{\gamma_i}{c}.
\]
Note that \( \pi_i = \{ \pi_i \} \) satisfy \( 0 < \pi_i < 1, \sum \pi_i \pi_i = 1 \) and \( \pi_i = \pi_i' \). Then the LQP\(^h\) model is also expressed as
\[
\log \left( \frac{q}{\pi} \right) = \log \mu c + \sum_{k=1}^t \log \alpha_i + \sum_{k=1}^t \sum_{i_k \in \mathbb{R} \times \cdots \times \mathbb{R}} i_1 \cdot i_2 \cdot i_3 \log \alpha_{i_1 i_2 i_3} + \cdots + \sum_{k=1}^t \sum_{i_k \in \mathbb{R} \times \cdots \times \mathbb{R}} i_k \log \alpha_{i_k},
\]

Figure 1. Relationships among various models. Note: “\( M_i \rightarrow M_i' \)” indicates that model \( M_i \) implies model \( M_i' \).
The \( M_{k}^{T} \) model is expressed as
\[
\mu_{i_{k_{1}}, \ldots, i_{k_{T}}}^{0} = \mu_{i_{k_{1}}, \ldots, i_{k_{T}}}^{*} \quad (1 \leq k_{1} < k_{2} < \cdots < k_{l} \leq T; l = 1, 3, \ldots, h),
\]
where
\[
\mu_{i_{k_{1}}, \ldots, i_{k_{T}}}^{0} = \sum_{q_{i_{k_{1}}} = 1}^{q_{i_{k_{1}}}} \cdots \sum_{q_{i_{k_{T}}} = 1}^{q_{i_{k_{T}}}} i_{k_{1}}^{(i_{k_{1}})} \cdots i_{k_{T}}^{(i_{k_{T}})}, \quad \mu_{i_{k_{1}}, \ldots, i_{k_{T}}}^{*} = \sum_{q_{i_{k_{1}}} = 1}^{q_{i_{k_{1}}}} \cdots \sum_{q_{i_{k_{T}}} = 1}^{q_{i_{k_{T}}}} i_{k_{1}}^{*} \cdots i_{k_{T}}^{*} q_{i_{k_{1}}}^{(i_{k_{1}})} \cdots q_{i_{k_{T}}}^{(i_{k_{T}})}.
\]

Then we denote \( \mu_{i_{k_{1}}, \ldots, i_{k_{T}}}^{0} \) by \( \mu_{i_{k_{1}}, \ldots, i_{k_{T}}}^{p} \).

Consider arbitrary cell probabilities \( p = \{ p_{i} \} \) which satisfy the \( M_{k}^{T} \) model and
\[
\mu_{i_{k_{1}}, \ldots, i_{k_{T}}}^{p} = \mu_{i_{k_{1}}, \ldots, i_{k_{T}}}^{0} \quad (1 \leq k_{1} < k_{2} < \cdots < k_{l} \leq T; l = 1, 3, \ldots, h),
\]
where
\[
\mu_{i_{k_{1}}, \ldots, i_{k_{T}}}^{p} = \sum_{q_{i_{k_{1}}} = 1}^{q_{i_{k_{1}}}} \cdots \sum_{q_{i_{k_{T}}} = 1}^{q_{i_{k_{T}}}} i_{k_{1}}^{(i_{k_{1}})} \cdots i_{k_{T}}^{(i_{k_{T}})}, \quad \mu_{i_{k_{1}}, \ldots, i_{k_{T}}}^{*} = \sum_{q_{i_{k_{1}}} = 1}^{q_{i_{k_{1}}}} \cdots \sum_{q_{i_{k_{T}}} = 1}^{q_{i_{k_{T}}}} i_{k_{1}}^{*} \cdots i_{k_{T}}^{*} q_{i_{k_{1}}}^{(i_{k_{1}})} \cdots q_{i_{k_{T}}}^{(i_{k_{T}})}.
\]

From (1), (2) and (3),
\[
\sum_{q_{i} = 1}^{q_{i_{k_{1}}}} \cdots \sum_{q_{i_{k_{T}}} = 1}^{q_{i_{k_{T}}}} (p_{i} - q_{i}) \log \left( \frac{q_{i}}{\pi_{i}} \right) = 0.
\]

Let \( K(\cdot) \) denote the Kullback-Leibler information, e.g., it between \( q \) and \( \pi \) is
\[
K(q; \pi) = \sum_{q_{i} = 1}^{q_{i_{k_{1}}}} \cdots \sum_{q_{i_{k_{T}}} = 1}^{q_{i_{k_{T}}}} q_{i} \log \left( \frac{q_{i}}{\pi_{i}} \right).
\]

From (4),
\[
K(p; \pi) = K(p; q) + K(q; \pi).
\]

Thus, for fixed \( \pi \),
\[
K(q; \pi) = \min_{p} K(p; \pi),
\]
and then \( q \) uniquely minimize \( K(p; \pi) \) (see Darroch and Ratcliff, [14]).

Let \( q^{*} = \{ q_{i}^{*} \} \). Then, in a similar way as described above, we obtain
\[
K(q^{*}; \pi) = \min_{p} K(p; \pi),
\]
and then \( q^{*} \) uniquely minimize \( K(p; \pi) \), hence \( q = q^{*} \). Namely \( q \) satisfy the \( P^{T} \) model. The proof is completed.

For the analysis of data, the test of goodness-of-fit of the \( L_{k}^{T} \) model is achieved based on, e.g., the likelihood ratio chi-square statistic which has a chi-square distribution with the number of degrees of freedom
\[
\frac{1}{2} \left( \prod_{i=1}^{T} R_{i} - 1 \right) - \sum_{i=1}^{T} R_{i} \quad (R_{i} : \text{odd for } k = 1, \ldots, T),
\]
\[
\frac{1}{2} \sum_{i=1}^{T} R_{i} \quad (\text{otherwise}).
\]

Also the number of degrees of freedom for the \( M_{k}^{T} \) model is
\[
\sum_{i=1}^{T} \left( \frac{T}{2} - 1 \right).
\]
We point out that, for a fixed $h$, the number of degrees of freedom for the $P^T$ model is equal to sum of those for the $LQP_h^T$ and $MMP_h^T$ models.

4. Concluding Remarks

For multi-way contingency tables, we have proposed the $MMP_h^T$, $OQP_h^T$ and $LQP_h^T$ models. Under the $OQP_h^T$ model, the log-odds that an observation falls in a cell instead of in its point-symmetric cell is described as a linear combination of category indicators. For a fixed odd number $h (h \in S)$, the $LQP_h^T$ model implies the $QP_h^T$ model.

We have gave the theorem that the $P^T$ model holds if and only if both the $LQP_h^T$ and $MMP_h^T$ models. For the analysis of data, the decomposition given in the present paper may be useful for determining the reason when the $P^T$ model fits data poorly.

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