The Expected Discounted Tax Payments on Dual Risk Model under a Dividend Threshold*

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ABSTRACT

In this paper, we consider the dual risk model in which periodic taxation are paid according to a loss-carry-forward system and dividends are paid under a threshold strategy. We give an analytical approach to derive the expression of $g_\delta(u)$ (i.e., the Laplace transform of the first upper exit time). We discuss the expected discounted tax payments for this model and obtain its corresponding integro-differential equations. Finally, for Erlang (2) inter-innovation distribution, closed-form expressions for the expected discounted tax payments are given.

Keywords: Dual Risk Model; Expected Discounted Tax Payments; Dividend; Threshold Strategy

1. Introduction

Consider the surplus process of an insurance portfolio

$$R(t) = u - ct + S(t) \quad (1.1)$$

which is dual to the classical Cramér-Lundberg model in risk theory that describes the surplus at time $t$, where $u \geq 0$ is the initial capital, the constant $c > 0$ is the rate of expenses, and $S(t) = \sum_{i=1}^{N(t)} Y_i$ is aggregate profits process with the innovation number process $N(t)$ being a renewal process whose inter-innovation times $T_i (i = 1, 2, \ldots)$ have common distribution $F$. We also assume that the innovation sizes $\{Y_i, i \geq 1\}$, independent of $\{T_i, i \geq 1\}$, forms a sequence of i.i.d. exponentially distributed random variables with exponential parameter $\beta (> 0)$. There are many possible interpretations for this model. For example, we can treat the surplus as the amount of capital of a business engaged in research and development. The company pays expenses for research, and occasional profit of random amounts arises according to a Poisson process.

Due to its practical importance, the issue of dividend strategies has received remarkable attention in the literature. De Finetti [1] considered the surplus of the company that is a discrete process and showed that the optimal strategy to maximize the expectation of the discounted dividends must be a barrier strategy. Since then, researches on dividend strategies have been carried out extensively. For some related results, the reader may consult the following publications therein: Bühlmann [2], Gerber [3], Gerber and Shiu [4,5], Lin et al. [6], Lin and pavlova [7], Dickson and Waters [8], Albrecher et al. [9], Dong et al. [10] and Ng [11]. Recently, quite a few interesting papers have been discussing risk models with tax payments of loss carry forward type. Albrecher et al. [12] investigated how the loss-carry-forward tax payments affect the behavior of the dual process (1.1) with general inter-innovation times and exponential innovation sizes. More results can be seen in Albrecher and Hipp [13], Albrecher et al. [14], Ming et al. [15], Wang and Hu [16] and Liu et al. [17,18].

Now, we consider the model (1.1) under the additional assumption that tax payments are deducted according to a loss-carry-forward system and dividends are paid under a threshold strategy. We rewrite the objective process as $\{R_{\gamma,\delta}(t), t \geq 0\}$, that is, the insurance company pays tax at rate $\gamma \in [0,1]$ on the excess of each new record high of the surplus over the previous one; at the same time, dividends are paid at a constant rate $\alpha$ whenever the surplus of an insurance portfolio is more than $b$ and otherwise no dividends are paid. Then the surplus process of our model $\{R_{\gamma,\delta}(t), t \geq 0\}$ can be expressed as

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for \( t \geq 0 \), with \( R_{r,b}(0) = u \), where \( 1_{\{A\}} \) is the indicator function of event \( A \) and \( R_{r,b}(t) \) is the surplus immediately before time \( t \).

For practical consideration, we assume that the positive safety loading condition

\[
c < E(Y_t)/E(T_t),
\]

holds all through this paper. The time of ruin is defined as \( T_{r,b} = \inf \{ t \geq 0 : R_{r,b}(t) \leq 0 \} \) with \( T_{r,b} = \infty \) if \( R_{r,b}(t) > 0 \) for all \( t \geq 0 \).

For initial surplus \( u > 0 \), let \( D = \int_0^{T_{r,b}} e^{-\delta t} dD(t) \) be the present value of all dividends until ruin, and \( \delta > 0 \) is the discount factor. Denote by \( V_r(u,b) \) the expectation of \( D \), that is,

\[
V_r(u,b) = E[D | R_{r,b}(0) = u].
\]

It needs to be mentioned that we shall drop the subscript \( \gamma \) whenever \( \gamma \) is zero.

The rest of this paper is organized as follows. In Section 2.1, we derive the expression of \( g_r(u) \) \( (i.e. \) the Laplace transform of the first upper exit time). We also discuss the expected discounted tax payments for this model and obtain its satisfied integro-differential equations. Finally, for Erlang (2) inter-innovation distribution, closed-form expressions for the the expected discounted tax payments are given.

2. Main Results and Proofs

Let \( g(u,u_0) = E[e^{-\delta T_{r,b}(u,u_0)}] \) denote the Laplace transform of the upper exit time \( T_{r,b}(u,u_0) \), which is the time until the ruin process \( \{R(t), t \geq 0\} \) starting with initial capital \( u \leq u_0 \) up-crosses the level \( u_0 \geq b \) for the first time without leading to ruin before that event. In particular, \( g_0(u,u_0) = \lim_{\delta \rightarrow 0} g(u,u_0) \) is the probability that the process \( \{R(t), t > 0\} \) up-crosses the level \( u_0 \geq b \) before ruin.

For general innovation waiting times distribution, one can derive the integral equations for \( g(u,u_0) \). When \( u < b \),

\[
g(u,u_0) = \int_0^{u/(c-\alpha)} e^{-\delta t} f_{T_1}(t) dt \left\{ \int_0^{u-ct+y,u_0} g(u-ct+y,u_0) e^{-\beta y} dy + \int_{u-ct+y,u_0}^{\infty} \beta e^{-\beta y} dy \right\}. \tag{2.1}
\]

When \( b \leq u \leq u_0 \),

\[
g(u,u_0) = \int_0^{(u-b)\xi} e^{-\delta t} f_{T_1}(t) dt \left\{ \int_0^{u-ct+y,u_0} g(u-ct+y,u_0) e^{-\beta y} dy + e^{-\beta(u_0-u)+ct} \right\} + \int_{(u-b)\xi}^{(u-b)\xi+(c-\alpha)\xi} e^{-\delta t} f_{T_1}(t) dt \left\{ \int_0^{(u-b)\xi+(c-\alpha)\xi-(b-\alpha)\xi} g(u-ct+y,u_0) e^{-\beta y} dy + e^{-\beta(u_0-u)+(c-\alpha)\xi-(b-\alpha)\xi} \right\}. \tag{2.2}
\]

It follows from Equation (2.1) and from Equation (2.2) that \( g(u,u_0) \) is continuous on \((0,u_0)\) as a function of \( u \) and that

\[
g(0^+,u_0) = 0, g(b^-,u_0) = g(b^-,u_0). \tag{2.3}
\]

For certain distributions \( F_{T_1} \), one can derive integro-differential equations for \( g(u,u_0) \) and \( V(u,b) \). Let us assume that the i.i.d innovation waiting times have a common generalized Erlang \((n)\) distribution, i.e. the \( T_i \)'s are distributed as the sum of \( n \) independent and exponentially distributed r.v.'s \( S_i := \eta_1 + \eta_2 + \ldots + \eta_n \) with \( \eta_i \) having exponential parameters \( \lambda_i > 0 \).

The following theorem 2.1 gives the integro-differential equations for \( g(u,u_0) \) when \( T_i \)'s have a generalized Erlang \((n)\) distribution.

**Theorem 2.1** Let \( I \) and \( D \) denote the identity operator and differentiation operator respectively. Then \( g(u,u_0) \) satisfies the following equation for \( 0 < u < b \)
\[
\prod_{k=1}^{n} \left[ 1 + \frac{\delta}{\lambda_k} \right] \left( I + \frac{c - \alpha}{\lambda_k} D \right) g(u, u_0) \\
= \int_{y_0-y}^{y_0} g(u+y, u_0) \beta e^{-\beta y} dy + e^{-\beta(y_0-y)},
\]
(2.4)

for \( u \geq b \).

**Proof** First, we rewrite \( g(u,u_0) \) as \( g_k(u,u_0) \) when \( T_1 = S_n - S_{k-1} \) with \( S_0 = 0 \) in the surplus process (1.2) with \( \gamma = 0 \). Thus, we have \( g_k(u,u_0) = g(u,u_0) \). When \( 0 < u < b \),
\[
g_k(u,u_0) = \int_{y_0-y}^{y_0} \lambda_y e^{-(\lambda_y+\delta)y} g_{k+1}(u+c(u-y),u_0)\,dy
\]
(2.5)

for \( k = 1, 2, ..., n-1 \), and
\[
g_n(u,u_0) = \int_{y_0-y}^{y_0} \lambda_y e^{-(\lambda_y+\delta)y} \left( \int_{y_0-y}^{y_0} g(u+y, u_0) \beta e^{-\beta y} dy + \int_{y_0-y}^{\infty} \beta e^{-\beta y} dy \right) \,dy.
\]
(2.7)

By changing variables in from Equation (2.6) and from Equation (2.7), we have for \( 0 < u < b \),
\[
g_k(u,u_0) = \int_{y_0-y}^{y_0} \frac{\lambda_y}{c-\alpha} e^{-(\lambda_y+\delta)y} g_{k+1}(x,u_0)\,dx,
\]
(2.8)

for \( k = 1,2,\ldots,n-1 \), and
\[
g_n(u,u_0) = \int_{y_0-y}^{y_0} \frac{\lambda_y}{c-\alpha} e^{-(\lambda_y+\delta)y} dx \cdot \left[ \int_{y_0-y}^{y_0} g(x+y, u_0) \beta e^{-\beta y} dy + \int_{y_0-y}^{\infty} \beta e^{-\beta y} dy \right].
\]
(2.9)

Then, differentiating both sides of from Equation (2.8) and from Equation (2.9) with respect to \( u \), one gets
\[
g_k(u,u_0) = \int_{y_0-y}^{(u-b)/c} \lambda_y e^{-(\lambda_y+\delta)y} g_{k+1}(u-ct,u_0)\,dt + \int_{(u-b)/c}^{(u-b)(c+b)(c)} \lambda_y e^{-(\lambda_y+\delta)y} g_{k+1}(b-\alpha(t-(u-b)/c),u_0)\,dt,
\]
(2.12)

for \( k = 1,2,\ldots,n-1 \), and
\[
g_n(u,u_0) = \int_{y_0-y}^{(u-b)/c} \lambda_y e^{-(\lambda_y+\delta)y} \left\{ \int_{y_0-y+ct}^{(u-b)/c} g(u-ct+y,u_0) \beta e^{-\beta y} dy + e^{-\beta(y_0-y-ct)} \right\} + \int_{(u-b)/c}^{(u-b)(c+b)(c)} \lambda_y e^{-(\lambda_y+\delta)y} \left\{ \int_{y_0-y}^{(u-b)(c+b)(c)} g(u+y,u_0) \beta e^{-\beta y} dy + e^{-\beta(y_0-y)(u-ct-y)} \right\}
\]
(2.13)

Again, by changing variables in Equation (2.12) and Equation (2.13) and then differentiating them with respect to \( u \), we obtain for \( u \geq b \)
\[
g_k(u,u_0) = g_{k+1}(u,u_0),
\]
(2.14)

for \( k = 1,2,\ldots,n-1 \), and

\[
\left[ 1 + \frac{\delta}{\lambda_k} \right] \left( I + \frac{c - \alpha}{\lambda_k} D \right) g_k(u,u_0) = g_{k+1}(u,u_0),
\]
(2.15)

for \( k = 1,2,\ldots,n-1 \), and

Using Equation (2.14) and Equation (2.15), we obtain Equation (2.5) for \( g(u,u_0) \) on \([b,\infty)\).
It needs to be mentioned that from the proof of Lemma 2.1, we know that
\[ g_k(0^+, u_0) = 0, g_k(b^+, u_0) = g_k(b^-, u_0), k = 2, 3, \ldots, n. \]
Therefore, Equations (2.10), (2.11), (2.14) and (2.15) yield
\[
\prod_{i=1}^{k} \left[ 1 + \frac{\delta}{\lambda_i} I + \frac{c - \alpha}{\lambda_i} D \right] g(b^+, u_0)
= \prod_{i=1}^{k} \left[ 1 + \frac{\delta}{\lambda_i} I + \frac{c - \alpha}{\lambda_i} D \right] g(b^-, u_0),
\]
with Equation (2.19).

With the preparations made above, we can now derive analytic expressions of the expected n -th moment of the accumulated discounted tax payments for the surplus process \( \{ R_{s,b}(t), t \geq 0 \} \). We claim that the process \( \{ R_{s,b}(t), t \geq 0 \} \) shall up-cross the initial surplus level \( u \) at least once to avoid ruin.

Now, let
\[
g_s(u) = E_s\left[ e^{-\delta u} \right]
\]
de note the Laplace transform of the first upper exit time \( \tau_{u,s} \), which is the time until the risk process \( \{ R_{s,b}(t), t \geq 0 \} \) starting with initial capital \( u \) reaches a new record high for the first time without leading to ruin before that event. In particular, \( g_s(u) = \lim_{\delta \to 0} g_s(u) \) is the probability that the process \( \{ R_{s,b}(t), t \geq 0 \} \) reaches a new record high before ruin. Then the closed-form expression of the quantity \( g_s(u) \) can be calculated as follows.

When \( u \geq b, g_s(u) = g(u, u) \). When \( 0 < u < b \), using a simple sample path argument, we immediately have,
\[
V(u, u) = g_s(u) \int_0^u \beta e^{-\beta y} V(u + y, u) dy,
\]
or, equivalently
\[
g_s(u) = \frac{V(u, u)}{\int_0^u \beta e^{-\beta y} V(u + y, u) dy}.
\]

Let \( \sigma_0 = 0 \) and define
\[
\sigma_n = \inf \left\{ t > \sigma_{n-1} : R_{s,b}(t) \geq \max_{0 < s \leq t} R_{s,b}(s) \right\},
\]
to be the n -th taxation time point. Thus,
\[
M_s(u, u) := E_s\left[ D_{\sigma,b} \right]^n
:= E_s\left[ \left( \frac{\gamma}{1 - \gamma} \sum_{n=1}^{\infty} e^{-\alpha s} \left( R_{s,b}(\sigma_n) - R_{s,b}(\sigma_{n-1}) \right) 1_{[\sigma_n < \tau_{u,s}]} \right)^n \right]
\]
denotes the n -th moment of the accumulated discounted tax payments over the life time of the surplus process \( \{ R_{s,b}(t), t \geq 0 \} \).

We will consider a recursive formula of \( M_s(u, b) \) in the following theorem 2.2.

**Theorem 2.2** When \( 0 < u < b \), we have
\[
M_s(u, b) = \frac{n \gamma}{1 - \gamma} g_{s,b}(u) \int_{0}^{\gamma} \int_{0}^{b} \tau_{u,s}^{(1 - e^{-\alpha s}) \alpha} d\tau \int_{s}^{b} M_{s+1}(s) e^{-\beta \tau_{u,s}^{(1 - e^{-\alpha s}) \alpha}} ds + \int_{0}^{b} M_{s+1}(s) e^{-\beta \tau_{u,s}^{(1 - e^{-\alpha s}) \alpha}} ds \int_{s}^{b} M_{s+1}(s) e^{-\beta \tau_{u,s}^{(1 - e^{-\alpha s}) \alpha}} ds,
\]
and when \( u \geq b \), we have
\[
M_n(u, b) = \frac{ny}{1-\gamma} g_{ub}(u) \int_{\gamma}^{\infty} M_{n-1}(s) \frac{\beta}{1-\gamma s} e^{-\beta \gamma (1-\gamma s) s} ds.
\] (2.27)

**Proof** Given that the after-tax excess of the surplus level over \( u \) at time \( \tau_u \) is exponentially distributed with mean \((1-\gamma) / \beta\) due to the memoryless property of the exponential distribution. By a probabilistic argument, one easily shows that for \( u > 0 \)
\[
M_n(u, b) = n g_{ub}(u) \int_{\gamma}^{\infty} D_{\gamma, b}(u + x) \frac{1}{1-\gamma} e^{-\beta \gamma (1-\gamma x) x} dx
\]
and 2
\[
= n g_{ub}(u) \int_{\gamma}^{\infty} \left[ D_{\gamma, b}(x) + \frac{\gamma}{1-\gamma} (x-u) \right] e^{-\beta \gamma (1-\gamma x) x} dx
\]
(2.28)

Differentiating with respect to \( u \) yields
\[
M'_n(u, b) = \left( g'_{ub}(u) + \frac{\beta}{1-\gamma} (1 - g_{ub}(u)) \right) M_n(u, b)
\]
\[
- \frac{ny}{1-\gamma} g_{ub}(u) \left[ \int_{\gamma}^{\infty} \frac{\beta}{1-\gamma} e^{-\beta \gamma (1-\gamma x) x} \left[ \frac{\gamma}{1-\gamma} (x-u) \right] e^{-(s-1) (x-u)} dx \right]
\]
\[
= \left( g'_{ub}(u) + \frac{\beta}{1-\gamma} (1 - g_{ub}(u)) \right) M_n(u, b)
\]
\[- \frac{ny}{1-\gamma} g_{ub}(u) M_{n-1}(u, b).
\]
(2.29)

When \( 0 < u < b \), we have
\[
M_n(u, b) = g_{ub}(u) e^{-\frac{\beta}{1-\gamma} \gamma (1-\gamma s) b} \left( C + \frac{ny}{1-\gamma} \int_{\gamma}^{\infty} M_{n-1}(s) \frac{\beta}{1-\gamma s} e^{-\beta \gamma (1-\gamma s) s} ds \right).
\] (3.20)

When \( u \geq b \), the general solution of Equation (3.20) can be expressed as
\[
M_n(u, b) = g_{ub}(u) \left( M_{n}(\infty) e^{-\frac{\beta}{1-\gamma} \gamma (1-\gamma s) b} \right.
\]
\[
+ \frac{ny}{1-\gamma} \int_{\gamma}^{\infty} M_{n-1}(s) \frac{\beta}{1-\gamma s} e^{-\beta \gamma (1-\gamma s) s} ds \left. \right)
\]
(2.31)

Due to the facts that \( M_n(\infty) < \infty \) and \( 0 < g_{ub}(\infty) < \infty \), we have for \( u \geq b \)
\[
M_n(u, b) = \frac{ny}{1-\gamma} g_{ub}(u) \int_{\gamma}^{\infty} M_{n-1}(s) \frac{\beta}{1-\gamma s} e^{-\beta \gamma (1-\gamma s) s} ds.
\] (2.32)

Now, it remains to determine the unknown constant \( C \) in Equation (3.20). The continuity of \( M(u, b) \) on \( b \) and Equation (3.22) lead to
\[
C = \frac{ny}{1-\gamma} g_{ub}(u) \int_{\gamma}^{\infty} M_{n-1}(s) \frac{\beta}{1-\gamma s} e^{-\beta \gamma (1-\gamma s) s} ds.
\] (2.33)

Plugging Equation (2.33) into Equation (3.20), we arrive at Equation (2.26).

The special case \( n = 1 \) leads to an expression for the expected discounted total sum of tax payments over the life time of the risk process
\[
M_n(u, b) = g_{ub}(u) \int_{\gamma}^{\infty} \frac{\beta}{1-\gamma} g_{ub}(u) e^{-\beta \gamma (1-\gamma s) s} ds.
\] (2.34)

for all \( u > 0 \).

### 3. Explicit Results for Erlang(2) Innovation Waiting Times

In this section, we assume that \( W_t \)'s are Erlang(2) distributed with parameters \( \lambda_1 \) and \( \lambda_2 \). We also assume that \( \lambda_1 < \lambda_2 \) (without loss of generality).

**Example 3.1** Note that
\[
(\beta I - D) \left( \int_{0}^{u} g(u + y, u_0) \beta e^{-\gamma y} dy + e^{-\gamma(u+u_0)} \right) = \beta g(u, u_0).
\]
(3.1)

Applying the operator \( (\beta I - D) \) to Equations (2.4) and (2.5) gives
\[
(\beta I - D) \prod_{k=1}^{n} \left[ 1 + \frac{\delta}{\lambda_k} \right] I + \frac{c-\alpha}{\lambda_k} D \right] g(u, u_0) = \beta g(u, u_0),
\]
\( 0 \leq u < b, \)
(3.2)

and
\[
(\beta I - D) \prod_{k=1}^{n} \left[ 1 + \frac{\delta}{\lambda_k} \right] I + \frac{c-\alpha}{\lambda_k} D \right] g(u, u_0) = \beta g(u, u_0),
\]
\( u \geq b, \)
(3.3)

The characteristic equation for Equation (3.2) is
\[
(\beta - r) \prod_{k=1}^{n} \left[ 1 + \frac{\delta}{\lambda_k} \right] I + \frac{c-\alpha}{\lambda_k} r = \beta
\]
(3.4)

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without loss of generality, we assume that \( \lambda_1 < \lambda_2 \). We know that Equation (3.4) has three real roots, say \( r_1, r_2 \) and \( r_3 \) which satisfies

\[
\beta > r_1 > 0 > r_2 > -\frac{\lambda_1 + \delta}{c - \alpha} > -\frac{\lambda_2 + \delta}{c - \alpha} > r_3 > -\frac{\lambda_1 + \delta}{c - \alpha} - \frac{\lambda_2 + \delta}{c - \alpha}.
\]

With \( c \) replace \( c - \alpha \) in Equation (3.4), we get the characteristic equation of Equation (3.3), whose roots are denoted by \( r_4, r_5 \) and \( r_6 \) with

\[
\beta > r_2 > 0 > r_3 > -\frac{\lambda_1 + \delta}{c} > -\frac{\lambda_2 + \delta}{c} > r_6
\]

Thus, we have

\[
g(u, u_0) = c_1 e^{\beta u} + c_2 e^{\alpha u} + c_3 e^{\alpha u}, 0 \leq u < b, \quad (3.5)
\]

and

\[
g(u, u_0) = c_4 e^{\beta u} + c_5 e^{\alpha u} + c_6 e^{\alpha u}, u \geq b, \quad (3.6)
\]

where \( c_1, c_2, c_3, c_4, c_5, c_6 \) are arbitrary constants. To determine the arbitrary constants, we insert Equations (3.5) and (3.6) into Equation (2.3) and obtain

\[
c_1 + c_2 + c_3 = 0, \quad (3.7)
\]

and

\[
e^{\beta u} c_1 + e^{\alpha u} c_2 + e^{\alpha u} c_3 - e^{\beta u} c_4
\]

\[
e^{\alpha u} c_5 - e^{\beta u} c_6 = 0. \quad (3.8)
\]

Apply Equation (2.10) together with Equations (2.3) and (3.5) when \( k = 1 \), we get

\[
r_1 c_1 + r_2 c_2 + r_3 c_3 = 0. \quad (3.9)
\]

Insert Equation (3.5) into Equation (2.4), we have

\[
-\frac{\beta e^{\alpha u}}{\beta - r_1} c_1 + \frac{\beta e^{\alpha u}}{\beta - r_2} c_2 + \frac{\beta e^{\alpha u}}{\beta - r_3} c_3 - \frac{\beta e^{\alpha u}}{\beta - r_4} c_4 = 1. \quad (3.10)
\]

In addition, plugging Equations (3.5) and (3.6) into Equation (2.16) yields

\[
(c - \alpha) r_2 e^{\beta u} c_1 + (c - \alpha) r_1 e^{\alpha u} c_2 + (c - \alpha) r_5 e^{\alpha u} c_3
\]

\[
-c r_6 e^{\beta u} c_4 - c r_2 e^{\alpha u} c_5 - c r_6 e^{\alpha u} c_6 = 0, \quad (3.11)
\]

and

\[
-\frac{\beta e^{\alpha u}}{\beta - r_2} c_2 - \frac{\beta e^{\alpha u}}{\beta - r_5} c_5 - \frac{\beta e^{\alpha u}}{\beta - r_6} c_6 = 0. \quad (3.12)
\]

Some calculations give
with
\[ \theta_i(r_i, r_j, r_k) = \frac{c(r_i - r_j)}{\beta - r_i} + \frac{(c-\alpha)r_i - cr_k}{\beta - r_j} + \frac{cr_j - (c-\alpha)r_2}{\beta - r_k}, \]

(3.14)

\[ g_\delta(u) = g(u, u), \]

By Equations (3.6) and (3.13), we have for \( u \geq b \)
\[ g_\delta(u) = l_4(b)e^{\alpha_\delta} + l_5(b)e^{\alpha_\gamma} + l_6(b)e^{\alpha_\delta}, \]

(3.15)

**Remark 3.1**

Now, we give the explicit results for
\[ l_4(b) = \frac{(r_i - r_2)e^{\alpha_\delta}(r_i, r_2, r_3) + (r_2 - r_i)e^{\alpha_\delta}(r_2, r_i, r_3) + (r_2 - r_3)e^{\alpha_\delta}(r_2, r_3, r_i)}{e^{\alpha_\delta} + e^{\alpha_\gamma} + e^{\alpha_\delta}}. \]

(3.16)

\[ l_5(b) = \frac{(r_2 - r_i)e^{\alpha_\delta}(r_2, r_i, r_3) + (r_i - r_2)e^{\alpha_\delta}(r_i, r_2, r_3) + (r_2 - r_3)e^{\alpha_\delta}(r_2, r_3, r_i)}{e^{\alpha_\delta} + e^{\alpha_\gamma} + e^{\alpha_\delta}}. \]

\[ l_6(b) = \frac{(r_i - r_2)e^{\alpha_\delta}(r_i, r_2, r_3) + (r_2 - r_i)e^{\alpha_\delta}(r_i, r_2, r_3) + (r_2 - r_3)e^{\alpha_\delta}(r_2, r_3, r_i)}{e^{\alpha_\delta} + e^{\alpha_\gamma} + e^{\alpha_\delta}}. \]

For \( 0 < u < b \), using the explicit expressions of \( V(b, b) \) in Liu et al. [17], we obtain
\[ g_\delta(u) = \frac{V(u, u)}{\int_0^b e^{-\beta y}V(u + y, u)dy} \]

(3.17)

\[ = \frac{(cr_2 + \delta)r_6 - (cr_0 + \delta)r_5}{\beta - r_6} + \frac{(r_i - r_2)e^{\alpha_\delta} + (r_2 - r_i)e^{\alpha_\gamma} + (r_2 - r_i)e^{\alpha_\delta}}{l_1(r_2 - r_3)e^{\alpha_\delta} + l_2(r_2 - r_3)e^{\alpha_\gamma} + l_3(r_2 - r_3)e^{\alpha_\delta}}. \]

with
\[ l_i = \frac{\beta \theta_i(i, 5)}{\beta - r_6} + \frac{\beta \theta_i(i, 6)}{\beta - r_5} + \theta_i(i, 5, 6), i = 1, 2, 3, \]

(3.18)

where
\[ \theta_k(r_i, r_j, r_k) = \frac{c(r_i - r_j)}{\beta - r_i} + \frac{(c-\alpha)r_i - cr_k}{\beta - r_j} + \frac{cr_j - (c-\alpha)r_2}{\beta - r_k}, \]

\[ 1 \leq i < j < k \leq 6, \]

and
\[ \theta_3(r_i, r_j) = \frac{(cr_j + \delta)r_i}{\beta - r_i} - \frac{(c-\alpha)r_i + \delta)r_j}{\beta - r_i}, 1 \leq i < j \leq 6. \]

We point out that when the innovation times are exponentially distributed, one can follow the same steps to get the explicit expressions of \( g_\delta(u) \), which coincide with the results in Albrecher et al. (2008).

**Example 3.2**

(The expected discounted tax payments.) Following from Equation (2.34) of Theorem 2.2 and Remark 3.1, we have for \( 0 < u < b \),
\[ M_i(u, b) = \frac{\gamma}{1 - \gamma} \left\{ \frac{(cr + \delta)r_u}{\beta - r_u} - \frac{(cr + \delta)r_i}{\beta - r_i} \right\} \left( (r_i - r_u)e^{\alpha_u} + (r_i - r_i)e^{\alpha_i} + (r_i - r_i)e^{\alpha_i} \right) \]

\[ \times \int_b^u \exp \left\{ -\beta \int_{-\gamma}^{\infty} \left( \frac{(cr + \delta)r_u}{\beta - r_u} - \frac{(cr + \delta)r_i}{\beta - r_i} \right) \left( (r_i - r_u)e^{\alpha_u} + (r_i - r_i)e^{\alpha_i} + (r_i - r_i)e^{\alpha_i} \right) \right\} \, ds \]

\[ \times \exp \left\{ -\beta \int_{-\gamma}^{\infty} \left( \frac{l_i(b)e^{\alpha_i} + l_i(b)e^{\alpha_i} + l_i(b)e^{\alpha_i}}{\beta - r_i} \right) \left( e^{\alpha_i} + e^{\alpha_i} + e^{\alpha_i} \right) \right\} \, ds \quad (3.19) \]

And, for \( u \geq b \), we have

\[ M_i(u, b) = \frac{\gamma}{1 - \gamma} \frac{l_i(b)e^{\alpha_i} + l_i(b)e^{\alpha_i} + l_i(b)e^{\alpha_i}}{\beta - r_i} \int_{-\gamma}^{\infty} \exp \left\{ -\beta \int_{-\gamma}^{\infty} \left( \frac{l_i(b)e^{\alpha_i} + l_i(b)e^{\alpha_i} + l_i(b)e^{\alpha_i}}{\beta - r_i} \right) \left( e^{\alpha_i} + e^{\alpha_i} + e^{\alpha_i} \right) \right\} \, ds \quad (3.20) \]

Then we can get that when \( W \)'s are Erlang (2) distributed with parameters \( \lambda_i \) and \( \lambda_i \), the expresses of \( g_{\alpha}(u) \) can be given by Equations (3.15) and (3.17) and the expected discounted tax payments can be given by Equation (3.20).

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REFERENCES


