Strong Law of Large Numbers for a 2-Dimensional Array of Pairwise Negatively Dependent Random Variables

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ABSTRACT

In this paper, we obtain the strong law of large numbers for a 2-dimensional array of pairwise negatively dependent random variables which are not required to be identically distributed. We found the sufficient conditions of strong law of large numbers for the difference of random variables which independent and identically distributed conditions are regarded. In this study, we consider the limit as \( mn \to \infty \) which is stronger than the limit as \( m, n \to \infty \) when \( m, n \) are natural numbers.

Keywords: Strong Law of Large Numbers; Negatively Dependent; 2-Dimensional Array of Random Variables

1. Introduction and Main Results

Let \( \{X_i\}_{i \in \mathbb{N}} \) be a sequence of random variables. We say \( \{X_i\}_{i \in \mathbb{N}} \) satisfies the strong law of large numbers (SLLN) if there exist sequences of real numbers \( \{a_n\}_{n \in \mathbb{N}} \) and \( \{b_n\}_{n \in \mathbb{N}} \) such that

\[
\frac{S_n - a_n}{b_n} \xrightarrow{a.s.} 0 \quad \text{as} \quad n \to \infty,
\]

where \( S_n = \sum_{i=1}^{n} X_i \) and the abbreviation a.s. stands for almost surely.

To study the strong law of large numbers, there is a simple question come in mind. When does the sequence \( \{X_i\}_{i \in \mathbb{N}} \) satisfy the SLLN? Many conditions of the sequence \( \{X_i\}_{i \in \mathbb{N}} \) have been found for this question. The SLLN are investigated extensively in the literature especially to the case of a sequence of independent random variables (see for examples in [1-3]). After concepts of dependence was introduced, it is interesting to study the SLLN with condition of dependence.

A sequence \( \{X_{ij}\}_{i,j \in \mathbb{N}} \) of random variables is said to be pairwise positively dependent (pairwise PD) if for any \( a, b \in \mathbb{R} \) and \( i \neq j \),

\[
P(X_i > a, X_j > b) \geq P(X_i > a)P(X_j > b)
\]

and it is said to be pairwise negatively dependent (pairwise ND) if for any \( a, b \in \mathbb{R} \) and \( i \neq j \),

\[
P(X_i > a, X_j > b) \leq P(X_i > a)P(X_j > b).
\]

Theorem 1.1-1.5 are examples of SLLN for a sequence of pairwise PD and pairwise ND random variables.

Theorem 1.1. (Birkel, [4]) Let \( \{X_i\}_{i \in \mathbb{N}} \) be a sequence of pairwise PD random variables with finite variances. Assume

1) \( \sup_{i \in \mathbb{N}} E(|X_i - E(X_i)|) < \infty \),

2) \( \sum_{i=1}^{\infty} \text{Var}(X_i) < \infty \).

Then \( S_n - E(S_n) \xrightarrow{a.s.} 0 \) as \( n \to \infty \).

Theorem 1.2. (Azarnoosh, [5]) Let \( \{X_{ij}\}_{i,j \in \mathbb{N}} \) be a sequence of pairwise ND random variables with finite variances. Assume

1) \( \sup_{i \in \mathbb{N}} E(|X_i|) < \infty \),

2) \( \sum_{i=1}^{\infty} \frac{\text{Var}(X_i)}{i^2} < \infty \).

Then \( \frac{S_n - E(S_n)}{n} \xrightarrow{a.s.} 0 \) as \( n \to \infty \).

Theorem 1.3. (Nili Sani, Azarnoosh and Bozorgnia, [6]) Let \( \{a_n\}_{n \in \mathbb{N}} \) be a positive and increasing sequence such that \( a_n \to \infty \) as \( n \to \infty \).

Let \( \{X_{ij}\}_{i,j \in \mathbb{N}} \) be a sequence of pairwise ND random variables with finite variances such that

1) \( \sup_{n \in \mathbb{N}} \left\{ \sum_{i=1}^{\infty} E(|X_i - E(X_i)|) \right\} < \infty \),

2) \( \sum_{i=1}^{\infty} \frac{\text{Var}(X_i)}{a_n^2} < \infty \).
Then \( \frac{S_n - E(S_n)}{a_n} \xrightarrow{a.s.} 0 \) as \( n \to \infty \).

In this work, we study the SLLN for a 2-dimensional array of pairwise ND random variables. We say that \( \{X_{i,j}\}_{i,j \in N} \) satisfies the SLLN if there exist double sequences of real numbers \( \{a_{m,n}\}_{m,n \in N} \) and \( \{b_{m,n}\}_{m,n \in N} \) such that \( \frac{S_{m,n} - a_{m,n}}{b_{m,n}} \xrightarrow{a.s.} 0 \) as \( m,n \to \infty \) where

\[
S_{m,n} = \sum_{i=1}^{m} \sum_{j=1}^{n} X_{i,j}.
\]

In 1998, Kim, Beak and Seo investigated SLLN for a 2-dimensional array of pairwise PD random variables and it was generalized to a case of weighted sum of 2-dimensional array of pairwise PD random variables by Kim, Beak and Han in one year later. The followings are their results.

A double sequence \( \{X_{i,j}\}_{i,j \in N} \) is said to be pairwise positively dependent (pairwise PD) if for any \( a,b \in R \) and \( (i,j) \neq (k,l) \),

\[
P\left(X_{i,j} > a, X_{k,l} > b \right) \geq P\left(X_{i,j} > a\right)P\left(X_{k,l} > b\right).
\]

**Theorem 1.4.** (Kim, Beak and Seo, [7]) Let \( \{X_{i,j}\}_{i,j \in N} \) be a 2-dimensional array of pairwise PD random variables with finite variances. Assume

1) \( \sup_{i,j \in N} E\left[X_{i,j} - E\left(X_{i,j}\right)\right] < \infty \),

2) \( \sum_{i,j \in N} \left( \sum_{k,l \in N \backslash \{i,j\}} \operatorname{Cov}\left(X_{i,j}, X_{k,l}\right) (i \times j)^2 \right) < \infty \).

Then \( \frac{S_{m,n} - E(S_{m,n})}{m \times n} \xrightarrow{a.s.} 0 \) as \( m \times n \to \infty \).

**Theorem 1.5.** (Kim, Beak and Han, [8]) Let \( \{a_{i,j}\}_{i,j \in N} \) be a 2-dimensional array of positive numbers and

\[
b_{m,n} = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{i,j} \quad \text{such that} \quad \frac{a_{m,n}}{b_{m,n}} \to 0 \quad \text{and} \quad b_{m,n} \to \infty \quad \text{as} \quad m,n \to \infty.
\]

Let \( \{X_{i,j}\}_{i,j \in N} \) be a 2-dimensional array of pairwise PD random variables with finite variances such that

1) \( \sup_{i,j \in N} E\left[X_{i,j} - E\left(X_{i,j}\right)\right] < \infty \),

2) \( \sum_{i,j \in N} \left( \sum_{k,l \in N \backslash \{i,j\}} \frac{a_{i,j}a_{k,l} \operatorname{Cov}\left(X_{i,j}, X_{k,l}\right)}{b_{k,l}^2} \right) < \infty \).

Then \( \frac{W_{m,n} - E(W_{m,n})}{b_{m,n}} \xrightarrow{a.s.} 0 \) as \( m \times n \to \infty \) where

\[
W_{m,n} = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{i,j} X_{i,j}.
\]

Observe that, for a double indexed sequence of real number \( \{a_{m,n}\}_{m,n \in N} \), the convergence as \( m \times n \to \infty \) implies the convergence as \( m,n \to \infty \). However, a double sequence \( \{a_{m,n}\}_{m,n \in N} \) where \( a_{m,n} = \frac{(-1)^m (m+n)}{m \times n} \) shows us that the converse is not true in general.

Our goal is to obtain the SLLN for a 2-dimensional array of random variables in case of pairwise ND.

A double sequence \( \{X_{i,j}\}_{i,j \in N} \) is said to be pairwise negatively dependent (pairwise PD) if for any \( a,b \in R \) and \( (i,j) \neq (k,l) \),

\[
P\left(X_{i,j} > a, X_{k,l} > b \right) \leq P\left(X_{i,j} > a\right)P\left(X_{k,l} > b\right).
\]

The followings are SLLNs for a 2-dimensional array of pairwise ND random variables which are all our results.

**Theorem 1.6.** Let \( \{a_{m,n}\}_{m,n \in N} \) and \( \{b_{m,n}\}_{m,n \in N} \) be increasing sequences of positive numbers such that \( a_m, b_n \geq e \) which \( a_m \to \infty \) as \( m \to \infty \) and \( b_n \to \infty \) as \( n \to \infty \). Let \( \{X_{i,j}\}_{i,j \in N} \) be a 2-dimensional array of pairwise ND random variables with finite variances. If there exist real numbers \( p,q \) such that

\[
\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{\text{Var}(X_{i,j})}{a_i^p b_j^q} < \infty,
\]

then for any double sequence \( \{c_{m,n}\}_{m,n \in N} \) such that

\[
c_{m,n} \geq \frac{p}{a_m^p} \times \frac{q}{b_n^q}
\]

for every \( m,n \in N \),

\[
\frac{S_{m,n} - E(S_{m,n})}{c_{m,n}} \xrightarrow{a.s.} 0 \quad \text{as} \quad m \times n \to \infty.
\]

The next theorem is the SLLN for the difference of random variables which independent and identically distributed conditions are regarded.

**Theorem 1.7.** Let \( \{X_{i,j}\}_{i,j \in N} \) and \( \{Y_{i,j}\}_{i,j \in N} \) be 2-dimensional arrays of random variables on a probability space \( (\Omega, F, P) \). If

\[
\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} P\left(X_{i,j} \neq Y_{i,j}\right) < \infty,
\]

then

\[
\frac{1}{m \times n} \sum_{i=1}^{m} \sum_{j=1}^{n} \left( X_{i,j} - Y_{i,j} \right) \xrightarrow{a.s.} 0 \quad \text{as} \quad m \times n \to \infty.
\]
Corollary 1.8 and Corollary 1.9 follow directly from Theorem 1.6 by choosing \( c_{m,n} = (a_m + b_n) \) and \( c_{m,n} = a_m \times b_n \) where \( a_m \to \infty \) and \( b_n \to \infty \) as \( m \to \infty \) and \( n \to \infty \), respectively.

**Corollary 1.8.** Let \( (a_m)_{m \in \mathbb{N}} \) and \( (b_n)_{n \in \mathbb{N}} \) be increasing sequences of positive numbers such that \( a_m, b_n \geq \varepsilon \) which \( a_m \to \infty \) as \( m \to \infty \) and \( b_n \to \infty \) as \( n \to \infty \).

Let \( (X_{i,j})_{i,j \in \mathbb{N}} \) be a 2-dimensional array of pairwise ND random variables with finite variances. If there exist \( p,q \in \mathbb{N} \) such that

\[
\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \text{Var}(X_{i,j}) < \infty,
\]

then for any \( k \geq p+q \),

\[
\frac{S_n - \mathbb{E}(S_n)}{(a_m + b_n)^{k}} \xrightarrow{a.s.} 0 \quad \text{as} \quad m \times n \to \infty.
\]

**Corollary 1.9.** Let \( (X_{i,j})_{i,j \in \mathbb{N}} \) be a 2-dimensional array of pairwise ND random variables with finite variances. If

\[
\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \text{Var}(X_{i,j}) < \infty,
\]

then

\[
\frac{S_n - \mathbb{E}(S_n)}{81(m \times n)^2} \xrightarrow{a.s.} 0 \quad \text{as} \quad m \times n \to \infty.
\]

2. **Auxiliary Results**

In this section, we present some materials which will be used in obtaining the SLLN’s in the next section.

**Proposition 2.1.** (Móricz, [9]) Let \( \lambda_{i,j} \) \( (i,j \in \mathbb{N}) \) be a double sequence of positive numbers such that for all \( i,j \in \mathbb{N} \),

\[
\lambda_{i+1,j} - \lambda_{i,j} \geq 0, \lambda_{i,j+1} - \lambda_{i,j} \geq 0,
\]

and \( \lambda_{i,j} \to \infty \) as \( \max\{i,j\} \to \infty \).

Let \( (a_{i,j})_{i,j \in \mathbb{N}} \) be a double sequence of real numbers. Assume that

1. \( \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{i,j} < \infty \),
2. \( \sum_{i=1}^{\infty} \frac{a_{i,j}}{\lambda_{i,j}} < \infty \) for every \( i \in \mathbb{N} \) and \( \sum_{j=1}^{\infty} \frac{a_{i,j}}{\lambda_{i,j}} < \infty \) for every \( j \in \mathbb{N} \). Then \( \frac{1}{\lambda_{m,n}} \sum_{i=1}^{m} \sum_{j=1}^{n} a_{i,j} \to 0 \) as \( m,n \to \infty \).

The following proposition is a Borel-Cantelli lemma for a sequence of double indexed events

**Proposition 2.2.** Let \( (E_{i,j})_{i,j \in \mathbb{N}} \) be a double sequence of events on a probability space \( (\Omega, F, P) \). Then

\[
\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} P(E_{i,j}) < \infty \implies \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} P(E_{i,j} | \{i, j\}) = 0
\]

where \( \mathcal{E}_{i,j} = \bigcap_{k=1}^{\infty} \bigcup_{i,j \in \mathbb{N}} E_{i,j} \).

**Proof.** Let \( L \in \mathbb{R} \) be such that \( L = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} P(E_{i,j}) \).

First note that

\[
\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} P(E_{i,j}) \leq \sum_{i,j \in \mathbb{N}} P(E_{i,j} | \{i, j\}) \leq \sum_{k=1}^{\infty} \sum_{i,j \in \mathbb{N}} P(E_{i,j} | \{i, j\}) = L.
\]

Therefore

\[
\lim_{k \to \infty} \sum_{i,j \in \mathbb{N}} P(E_{i,j} | \{i, j\}) = L.
\]

3. **Proof of Main Results**

**Proof of Theorem 1.6**

Let \( m,n \in \mathbb{N} \) and define \( f(m) = \lfloor \ln a_m \rfloor \) and \( g(n) = \lfloor \ln b_n \rfloor \).

Clearly, \( f \) and \( g \) are increasing whose facts

\[
f(m) \leq \ln a_m < f(m) + 1 \quad \text{and} \quad g(n) \leq \ln b_n < g(n) + 1
\]

which imply that \( e^{f(m)} \leq a_m < e^{f(m)+1} \) and \( e^{g(n)} \leq b_n < e^{g(n)+1} \).

Let \( \varepsilon > 0 \) be given. By using the fact that \( \text{Cov}(X_{i,j}, X_{k,l}) \leq 0 \) for \( (i, j) \neq (k, l) \) ([10], p. 313), we have

\[
\text{Var}\left( \sum_{i=1}^{m} \sum_{j=1}^{n} X_{i,j} \right) \leq \sum_{i=1}^{m} \sum_{j=1}^{n} \text{Var}(X_{i,j}).
\]

From this fact and Chebyshev’s inequality, we have
\[ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} P \left\{ \frac{S_{m,n} - E(S_{m,n})}{c_{m,n}} \geq \varepsilon \right\} \leq C, \]
\[ \leq C \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\text{Var}(S_{m,n})}{c_{m,n}^2} \leq C \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{a_m^2} \sum_{j=1}^{\infty} \text{Var}(X_{i,j}) \]
\[ \leq C \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{a_m^2} \sum_{j=1}^{\infty} \frac{1}{e^{(\alpha m + \beta n)}}. \] 

For each \( i, j \in N \), let
\[ A_i = \{ s \in N : e^{f(i)} \geq a_i \} \]
\[ B_j = \{ t \in N : e^{f(j)} \geq b_j \} \]
and \( \tilde{i} = \min A_i \) and \( \tilde{j} = \min B_j \). Since \( i \in A_i \) and \( j \in B_j \), we have \( \tilde{i} \leq i \) and \( \tilde{j} \leq j \). From this facts and (3.1), we have
\[ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} P \left\{ \frac{S_{m,n} - E(S_{m,n})}{c_{m,n}} \geq \varepsilon \right\} \leq C \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \text{Var}(X_{i,j}) \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{e^{(\alpha m + \beta n)}} \sum_{j=1}^{\infty} \frac{1}{e^{(\alpha m + \beta n)}}. \] 

Thus, for each \( i, j \in N \), we have
\[ \frac{1}{e^{f(i)}} < e^{-1} \frac{1}{a_i} \]
and
\[ \frac{1}{e^{f(j)}} < e^{-1} \frac{1}{b_j}. \]

From these facts and (3.2) together with our assumption 2, we have
\[ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} P \left\{ \frac{S_{m,n} - E(S_{m,n})}{c_{m,n}} \geq \varepsilon \right\} \leq C \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{1}{a_m^2} \frac{1}{2} \sum_{j=1}^{\infty} \frac{1}{e^{(\alpha m + \beta n)}}. \] 

By Proposition 2.2 with
\[ E_{m,n} = \left\{ \frac{S_{m,n} - E(S_{m,n})}{c_{m,n}} \geq \varepsilon \right\}, \]
we have \( P \{ E_{m,n} \text{i.o.} \} = 0 \) and this hold for every \( \varepsilon > 0 \). By using the same idea with Theorem 4.2.2 ([11], p. 77), we can prove that
\[ \frac{S_{m,n} - E(S_{m,n})}{c_{m,n}} \xrightarrow{a.s.} 0 \text{ as } m \times n \to \infty. \]

**Proof of Theorem 1.7**

Let \( \Omega_0 = \bigcap_{k=1}^{\infty} \Omega \times \{ X_{i,j} \neq Y_{i,j} \} \). By Proposition 2.2, we have
\[ P(\Omega_0) = 1 - P(\Omega_0) = 1 - P(\{ X_{i,j} \neq Y_{i,j} \} \text{i.o.}) = 1. \]

For every \( \omega \in \Omega_0 \), we will show that
\[ \sum_{i,j} \frac{1}{i \times j} (X_{i,j}(\omega) - Y_{i,j}(\omega)) < \infty, \]
for every \( i \in N \),
\[ \sum_{i,j} \frac{1}{i \times j} (X_{i,j}(\omega) - Y_{i,j}(\omega)) < \infty, \]
and for every \( j \in N \),
\[ \sum_{i,j} \frac{1}{i \times j} (X_{i,j}(\omega) - Y_{i,j}(\omega)) < \infty. \]

From (3.3), (3.4) and (3.5), we can apply Proposition 2.1 with \( \lambda_{i,j} = i \times j \) that
\[ \frac{1}{m \times n} \sum_{i=1}^{m} \sum_{j=1}^{n} (X_{i,j} - Y_{i,j}) \xrightarrow{a.s.} 0 \]
as \( m \times n \to \infty \). We here note that \( a_{m,n} \to a \) as \( m \times n \to \infty \) implies \( a_{m,n} \to a \) as \( m \times n \to \infty \). Hence
\[ \frac{1}{m \times n} \sum_{i=1}^{m} \sum_{j=1}^{n} (X_{i,j} - Y_{i,j}) \xrightarrow{a.s.} 0 \]
as \( m \times n \to \infty \).

To prove (3.3), (3.4) and (3.5), let \( \omega \in \Omega_i \). Then there exists \( k_\omega \in N \) such that for \( i, j \in N \),
\[ i \times j \geq k_\omega \Rightarrow X_{i,j}(\omega) = Y_{i,j}(\omega). \] 

Thus, for each \( \omega \in \Omega_0, \{ X_{i,j}(\omega) \}_{i,j \in \mathbb{N}} \) and \( \{ Y_{i,j}(\omega) \}_{i,j \in \mathbb{N}} \) are different only finitely many terms. This implies that (3.3) holds.

For fixed \( i \in N \), we can find a large \( j_0 \in N \) such that (3.6) holds for all \( j \geq j_0 \) which means that there are only finitely many different terms of \( \{ X_{i,j}(\omega) \}_{i,j \in \mathbb{N}} \) and \( \{ Y_{i,j}(\omega) \}_{i,j \in \mathbb{N}} \). So for fixed \( i \in N \),
\[ \sum_{j=1}^{\infty} \frac{1}{i \times j} (X_{i,j}(\omega) - Y_{i,j}(\omega)) < \infty. \]
Similarly, for fixed \( j \in N \),
\[
\sum_{i=1}^{\infty} \frac{1}{i \times j} \left( X_{i,j}(\omega) - Y_{i,j}(\omega) \right) < \infty.
\]

Now (3.4) and (3.5) are now proved and this ends the proof. \( \square \)

**Remark 3.1.** In case of \( m \) fixed and \( n \to \infty \), by considering the limit as \( m \times n \to \infty \), we also obtain the corresponding results for a case of 1-dimensional pairwise ND random variables.

4. Example

**Example 4.1** A box contains \( pq \) balls of \( p \) different colors and \( q \) different sizes in each color. Pick 2 balls randomly.

Let \( X_{i,j}, i = 1,2,\ldots,p \) and \( j = 1,2,\ldots,q \) be a random variable indicating the presence of a ball of the \( i^{th} \) color and the \( j^{th} \) size such that

\[
\hat{X}_{i,j} = \begin{cases} 
1, & \text{if the } i^{th} \text{ color and the } j^{th} \text{ size of ball is picked}, \\
0, & \text{otherwise}.
\end{cases}
\]

For \( i, j \in N \), let \( X_{i,j} \) be a random variable defined by

\[
X_{i,j} = \begin{cases} 
\hat{X}_{i,j}, & \text{if } 1 \leq i \leq p \text{ and } 1 \leq j \leq q, \\
0, & \text{otherwise}.
\end{cases}
\]

**Proof.** By a direct calculation, we have \( X_{i,j} \)'s are pairwise ND random variables, i.e. for \( i,j,k,l \in R \) that \( (i,j) \neq (k,l) \) and \( a,b \in R \),

\[
P\left( X_{i,j} > a, X_{k,l} > b \right) \leq P\left( X_{i,j} > a \right) P\left( X_{k,l} > b \right).
\]

Note that

\[
E\left( X_{i,j} \right) = \frac{pq - 1}{pq} \left( \frac{2}{2} \right) = \frac{1}{pq}
\]

and

\[
\text{Var}\left( X_{i,j} \right) = \frac{2}{pq} - \frac{4}{pq^2}.
\]

Hence,

\[
\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \text{Var}\left( X_{i,j} \right) = \lim_{m,n \to \infty} \left( \sum_{i=1}^{m} \sum_{j=1}^{n} \text{Var}\left( X_{i,j} \right) \right) = \frac{2}{pq} - \frac{4}{pq^2} \sum_{i=1}^{\infty} \frac{1}{i^2} < \infty.
\]

By applying Theorem 1.6, for any double sequence \( (c_{m,n})_{m,n \in N} \) such that \( c_{m,n} \geq 81 (m \times n)^2 \) for every \( m \)

\[
n \in N, \text{ we have } S_{m,n} - E\left( S_{m,n} \right) \xrightarrow{a.s.} 0 \text{ as } m \times n \to \infty.
\]

\( \square \)

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