The First Order Autoregressive Model with Coefficient Contains Non-Negative Random Elements: Simulation and Estimation

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ABSTRACT
This paper considered an autoregressive time series where the slope contains random components with non-negative values. The authors determine the stationary condition of the series to estimate its parameters by the quasi-maximum likelihood method. The authors also simulate and estimate the coefficients of the simulation chain. In this paper, we consider modeling and forecasting gold chain on the free market in Hanoi, Vietnam.

Keywords: Random Coefficient Autoregressive Model; Quasi-Maximum Likelihood; Consistency

1. Introduction
It is well-known that many time series in finance such as stock returns exhibit leptokurtosis, time-varying volatility and volatility clusters. The generalized autoregressive conditional heteroscedasticity (GARCH) and the random coefficient autoregressive (RCA) model have been capturing three characteristics of financial returns.

The RCA models have been studied by several authors [1-3]. Most of their theoretic properties are well-known, including conditions for the existence and the uniqueness of a stationary solution, or for the existence of moments for the stationary distribution. In this paper, we address the stationary conditions for the RCA model, the existence and the uniqueness of a stationary solution and parameter estimation problem for the RCA model with the coefficient have a non-negative random elements.

2. Stationary Conditions of the Series
Consider time series \( \{Y_t\} \) satisfying
\[
Y_t = (\phi + b_t) Y_{t-1} + e_t
\]
\[
e_t \sim N(0, \sigma_e^2), b_t \sim N(0, \sigma_b^2)
\]
where \( \{(b_t, e_t)\}_{t=0}^{\infty} \) are random vectors with independent identical distribution defined in a certain \( (\Omega, \mathcal{F}, P) \) probability space.

Firstly, we consider the property of the stochastic variable
\[
Y = \sum_{i=0}^{\infty} e_{-i} \prod_{j=0}^{i-1} (\phi + b_{-j})
\]
Let \( \log^+ x = \max(\log x, 0) \).

Lemma 1. Suppose that condition (2) satisfied,
\[
E \log^+ |e_0| < \infty \text{ and } E \log^+ |\phi + b_0| < \infty
\]
\[
(4)
\]
If
\[
-\infty \leq E \log |\phi + b_0| < 0
\]
\[
(5)
\]
\( Y \) determined by (3) will be absolute convergence with probability 1.

Proof. Assume \(-\infty \leq E \log |\phi + b_0| < 0\), according to the law of great numbers, existing stochastic variable \( i_0 \) such that:
\[
\log |\phi + b_{i_0}| + \log |\phi + b_{i_1}| + \cdots + \log |\phi + b_{i_r}| \leq \frac{1}{2} r^\gamma
\]
with every \( i \geq i_0 \)

where \(-\infty < \gamma = E \log |\phi + b_0| < 0\). Then
\[
|Y| \leq \sum_{i=0}^{i_0} \left( |e_{-i}| \prod_{j=0}^{i-1} |\phi + b_{-j}| \right) + \sum_{i=i_0}^{\infty} \left( |e_{-i}| \prod_{j=0}^{i-1} |\phi + b_{-j}| \right)
\]
\[
\leq \sum_{i=0}^{i_0} \left( |e_{-i}| \prod_{j=0}^{i-1} |\phi + b_{-j}| \right) + \sum_{i=i_0}^{\infty} \left( |e_{-i}| e^{r/2} \right)
\]
\[
(7)
\]
We will prove \( P \left( \sum_{i=0}^{\infty} |e_{-i}| e^{r/2} < \infty \right) = 1. \) Indeed, due to \( 0 < e^{r/2} < 1 \) and in accordance with lemma Borel-Cantelli, sufficient condition here means proving
\[
\sum_{i=0}^{\infty} P \left( |e_{-i}| > \zeta^i \right) < \infty \text{ with } \zeta > 1.
\]
We have:
with the almost sure convergence of $Y_{k}$. Therefore, (7) is always true.

**Lemma 2.** Suppose that (2) and (5) meet $E|b_{0}|<\infty$ and $E|e_{0}|<\infty$ with some $\epsilon>0$. Then, existing $\delta>0$ such that $E|Y|<\infty$.

**Proof.**

Suppose $M(t)=E|\phi|+|b_{0}|, 0\leq t \leq \epsilon$.

We have $M(0)=1$, and owing to (5): $M'(0+)<0$. $M(t)$ is a decreasing function in the neighborhood of 0. Hence, existing $\delta>0$ such that $M(\delta)<1$. Generally less, suppose that $0<\delta \leq 1$. Due to the convex, we have $(a+b)^{\delta} \leq a^{\delta}+b^{\delta}$ for all $a, b \geq 0$.

$$|Y|^{\delta} \leq \left( \sum_{i=0}^{\infty} |e_{i}| \prod_{j=0}^{i-1} |\phi|+|b_{j}| \right)^{\delta} \leq \sum_{i=0}^{\infty} |e_{i}|^{\delta} \prod_{j=0}^{i-1} |\phi|+|b_{j}|$$.

Using condition (2) and $M(\delta)<1$, we obtain:

$$E|Y|^{\delta} \leq E|e_{0}|^{\delta} \sum_{i=0}^{\infty} E\left( \prod_{j=0}^{i-1} |\phi|+|b_{j}| \right)^{\delta} = E|e_{0}|^{\delta} \sum_{i=0}^{\infty} M'(\delta)<\infty.$$

**Lemma 3.** Assume (2) and (5) are satisfied with $\alpha \geq 1: E|e_{0}|<\infty, E|b_{0}|<\infty$ and $E|\phi+b_{0}|^{\alpha}<\infty$. Then $E|Y|^{\alpha}<\infty$.

**Proof.**

Due to condition (2) and inequality Minkowski

$$E|Y|^{\alpha} \leq \left( E|e_{0}|^{\alpha} \right)^{1/\alpha} \cdot \sum_{i=0}^{\infty} \left( E\left| \prod_{j=0}^{i-1} |\phi|+|b_{j}| \right|^{\alpha} \right)^{1/\alpha} <\infty.$$ Hence, $E|Y|^{\alpha}<\infty$.

**Theorem 1:** Suppose that (1), (4) and (5) satisfied with the almost sure convergence of

$$Y_{k}=\sum_{i=0}^{\infty} e_{k-i} \prod_{j=0}^{i-1} (\phi+|b_{j}|)$$

and process $\{Y_{k}: k \in Z\}$ is the stationary solution of (1).

**Proof.**

$Y_{k}$ is convergent absolutely, according to Lemma 1

We have: $Y_{k}=\sum_{i=0}^{\infty} e_{k-i} \prod_{j=0}^{i-1} (\phi+|b_{j}|)$. Therefore:

$$Y_{k}=e_{k}+\sum_{i=0}^{\infty} e_{k-i} \prod_{j=0}^{i-1} (\phi+|b_{j-i}|)$$

$$=e_{k}+e_{k-1}(\phi+|b_{0}|)+e_{k-2}(\phi+|b_{0}|)(\phi+|b_{1}|)+\cdots+e_{k-m}(\phi+|b_{0}|)(\phi+|b_{1}||\cdots(\phi+|b_{m-1}|)$$

$$=e_{k}+(\phi+|b_{0}|) \prod_{i=0}^{\infty} e_{k-i} \prod_{j=0}^{i-1} (\phi+|b_{j-i}|)$$

$$=e_{k}+(\phi+|b_{0}|) Y_{k-1}$$

$\Rightarrow Y_{k}$ is the single solution of (1).

Obviously, $\{Y_{k}\}$ is a stationary series and $\{Y_{k}\}$ is independent of $e_{i}, b_{i}, t > k$.

### 3. Estimation of Model Parameters

Suppose that

$$E(b_{0}, e_{0})=(0,0);$$

$$\text{cov}(b_{0}, e_{0})=\begin{bmatrix} \sigma_{e}^{2} & 0 \\ 0 & \sigma_{b}^{2} \end{bmatrix};$$

$$\sigma_{e}^{2}>0, \sigma_{b}^{2}>0.$$ 

In this section, we care about estimating vectors of $\theta=(\phi, \sigma_{e}^{2}, \sigma_{b}^{2})$ based on Quasi-Maximum Likelihood method.

With $k \in Z$, we have:

$$E(Y_{k} | F_{k-1}) = E((\phi+|b_{0}|) Y_{k-1}+e_{k} | F_{k-1})$$

$$= (\phi+|b_{0}|) Y_{k-1}$$

but $E|b_{0}|=\frac{2}{\sqrt{\pi}} \sigma_{b}$, so

$$E(Y_{k} | F_{k-1}) = (\phi+\sigma_{b} \frac{2}{\sqrt{\pi}} Y_{k-1})$$

$$\text{Var}(Y_{k} | F_{k-1}) = E\left[ \left( Y_{k}-(\phi+\sigma_{b} \frac{2}{\sqrt{\pi}} Y_{k-1}) \right)^{2} | F_{k-1} \right]$$

$$E\left[ \left( |b_{0}|-\sigma_{b} \frac{2}{\sqrt{\pi}} Y_{k-1}+e_{k} \right)^{2} | F_{k-1} \right]$$

$$= \left[ 1+\frac{2}{\pi} \right] \sigma_{e}^{2} Y_{k-1}^{2}+\sigma_{b}^{2}$$

$$= \left( 1-\frac{2}{\pi} \right) \sigma_{b}^{2} Y_{k-1}^{2}+\sigma_{e}^{2}.$$
\[ L_n(u) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi(1-\frac{4}{\pi})}} \times \exp \left\{ -\frac{1}{2} \left[ Y_i - s \right] \times Y_i^2 + y \right\} \]

Maximum likelihood estimators determined by:
\[ \sup_{\text{acf}} L_n(u) = L_n(\hat{\theta}_n) \]  
(8)

where \( \Gamma \) is a certain optional appropriate area of \( \mathbb{R}^3 \).

Let \[ H_n(u) = \frac{1}{n} \sum_{i=1}^{n} g_i(u) \]

\[ g_i(u) = \frac{\left[ Y_i - \left( s + \frac{2\sqrt{x}}{\pi^2} \right) \times Y_i^{-1} \right]^2}{\left( 1-\frac{2}{\pi} \right) \times Y_i^{-1} + y} + \log \left[ \left( 1-\frac{2}{\pi} \right) \times Y_i^{-1} + y \right]. \]

Then (8) can be written as \[ \inf_{\text{acf}} H_n(u) = H_n(\hat{\theta}_n) \]

Assume \[ \Gamma = \left\{ (s, x, y): -s_0 \leq s \leq s_0, 1 \leq x \leq x_0, 1 \leq y \leq y_0 \right\} \]
with \( s_0 > 0, x_0 > 1, y_0 > 1 \).

Now, the consistence of maximum livelihood estimates \( \hat{\theta}_n \) is said.

**Theorem 2.** Suppose (2), (4), (5), (8), (9) satisfied and \( P\{\phi + b_o\}e_o = 0 \} < 1 \) and \( \theta \in \Gamma \). We have \( \hat{\theta}_n \to \theta \) a.s \((n \to \infty)\).

**Proof.**

Let \[ Y_{i,1}(u, \alpha, \beta) = \frac{Y_i^{\alpha}}{\left( 1-\frac{2}{\pi} \right) \times Y_i^{-1} + y}, \alpha = 0, 1, \cdots, 2\gamma, \gamma \in N \]
and \( \beta(\alpha, \gamma) = E\frac{Y_i^{\alpha}}{\left( 1-\frac{2}{\pi} \right) \times Y_i^{-1} + y}, \alpha = 0, 1, \cdots, 2\gamma. \)

We will prove \( E\inf_{\text{acf}} g_i(u) > -\infty \) and \( E\sup_{\text{acf}} g_i(u) \) be continuous on \( \Gamma \).

Indeed,
\[ E\sup_{\text{acf}} \log \left[ \left( 1-\frac{2}{\pi} \right) \times Y_i^{-1} + y \right] \]
\[ \leq E\log \left[ \left( 1-\frac{2}{\pi} \right) \times Y_i^{-1} + y \right] + E\log \left[ \left( 1-\frac{2}{\pi} \right) \times Y_i^{-1} + y \right] \]
\[ + \log (y_0) < \infty \]

\[ E\inf_{\text{acf}} g_i(u) = E\inf_{\text{acf}} \left\{ \left[ Y_i - \left( s + \frac{2\sqrt{x}}{\pi^2} \right) \times Y_i^{-1} \right]^2 \right\} \]
\[ - E\sup_{\text{acf}} \left[ -\log \left[ \left( 1-\frac{2}{\pi} \right) \times y \right] \right] \]
\[ \geq E\sup_{\text{acf}} \left[ \left( 1-\frac{2}{\pi} \right) \times y \right] \]
\[ - E\sup_{\text{acf}} \log \left( \left[ 1-\frac{2}{\pi} \right] \times y \right) \] 
> \(-\infty\)

On the other hand,
\[ Eg_i(u) = E\left[ \left( 1-\frac{2}{\pi} \right) \times y \right] \]
\[ + E\log \left( \left( 1-\frac{2}{\pi} \right) \times y \right) \]

But
\[ \left[ Y_i - \left( s + \frac{2\sqrt{x}}{\pi^2} \right) \times Y_i^{-1} \right]^2 \]
\[ = \left( \phi - 1 \right)^2 \times Y_i^{-1} + \left( \phi - 1 \right) \times Y_i^{-1} \]
\[ + 2\left( \phi - s \right) \left( \phi - 1 \right) \times Y_i^{-1} + \sigma_i^2 \]
\[ E\left[ \left( \phi - 1 \right) \times Y_i^{-1} \right] \]
\[ = \frac{\left( \phi - 1 \right) \times Y_i^{-1} + \sigma_i^2}{\left( \phi - 1 \right) \times Y_i^{-1} + \sigma_i^2} \]
\[ E\left[ \left( \phi - 1 \right) \times Y_i^{-1} \right] \]
\[ = \frac{\left( \phi - 1 \right) \times Y_i^{-1} + \sigma_i^2}{\left( \phi - 1 \right) \times Y_i^{-1} + \sigma_i^2} \]
\[ \left( \phi - 1 \right) \times Y_i^{-1} + \sigma_i^2 \]
\[ E\left[ \left( \phi - 1 \right) \times Y_i^{-1} \right] \]
\[ = \frac{\left( \phi - 1 \right) \times Y_i^{-1} + \sigma_i^2}{\left( \phi - 1 \right) \times Y_i^{-1} + \sigma_i^2} \]
\[ \left( \phi - 1 \right) \times Y_i^{-1} + \sigma_i^2 \]
\[ E\left[ \left( \phi - 1 \right) \times Y_i^{-1} \right] \]
\[ = \frac{\left( \phi - 1 \right) \times Y_i^{-1} + \sigma_i^2}{\left( \phi - 1 \right) \times Y_i^{-1} + \sigma_i^2} \]
\[ \left( \phi - 1 \right) \times Y_i^{-1} + \sigma_i^2 \]
\[ E\left[ \left( \phi - 1 \right) \times Y_i^{-1} \right] \]
\[ = \frac{\left( \phi - 1 \right) \times Y_i^{-1} + \sigma_i^2}{\left( \phi - 1 \right) \times Y_i^{-1} + \sigma_i^2} \]
\[ \left( \phi - 1 \right) \times Y_i^{-1} + \sigma_i^2 \]
\[ E\left[ \left( \phi - 1 \right) \times Y_i^{-1} \right] \]
\[ = \frac{\left( \phi - 1 \right) \times Y_i^{-1} + \sigma_i^2}{\left( \phi - 1 \right) \times Y_i^{-1} + \sigma_i^2} \]
\[ \left( \phi - 1 \right) \times Y_i^{-1} + \sigma_i^2 \]
\[ E_{g_1}(u) = E \left( \frac{Y_0^2}{(1 - \frac{2}{\pi}) xY_0^2 + y} \right) \]

\[ \cdot \left( \sqrt{\frac{\delta}{\pi}} (\sigma - x) \right) \cdot E \log \left( \frac{1 - \frac{2}{\pi}}{xY_0^2 + y} \right) \]

\[ + E \left( \frac{\delta^2}{1 - \frac{2}{\pi}} xY_0^2 + y \right) + E \log \left( \frac{1 - \frac{2}{\pi}}{xY_0^2 + y} \right) \]

\[ + E \left( \frac{\delta^2}{1 - \frac{2}{\pi}} xY_0^2 + y \right) + E \log \left( \frac{1 - \frac{2}{\pi}}{xY_0^2 + y} \right) \]

\[ = \phi + \frac{\sqrt{\frac{\delta}{\pi}} (\sigma - x)}{1 - \frac{2}{\pi}} E \left( \frac{Y_0^2}{xY_0^2 + y} \right) \]

\[ + E \left( \frac{\delta^2}{1 - \frac{2}{\pi}} xY_0^2 + y \right) + E \log \left( \frac{1 - \frac{2}{\pi}}{xY_0^2 + y} \right) \]

\[ + E \left( \frac{\delta^2}{1 - \frac{2}{\pi}} xY_0^2 + y \right) + E \log \left( \frac{1 - \frac{2}{\pi}}{xY_0^2 + y} \right) \]

\[ = \phi + \frac{\sqrt{\frac{\delta}{\pi}} (\sigma - x)}{1 - \frac{2}{\pi}} E \left( \frac{Y_0^2}{xY_0^2 + y} \right) \]

\[ + E \left( \frac{\delta^2}{1 - \frac{2}{\pi}} xY_0^2 + y \right) + E \log \left( \frac{1 - \frac{2}{\pi}}{xY_0^2 + y} \right) \]

\[ + E \left( \frac{\delta^2}{1 - \frac{2}{\pi}} xY_0^2 + y \right) + E \log \left( \frac{1 - \frac{2}{\pi}}{xY_0^2 + y} \right) \]

where

\[ h(x) = x - \ln x, \ x > 0 \]

\[ h'(x) = 1 - \frac{1}{x}, \ h'(x) = 0 \ \Rightarrow \ x = 1 \]

\[ \Rightarrow h(x) > h(1) = 1 \ \forall x > 0 \]

\[ \Rightarrow E_{g_1}(u) \geq E_{g_1}(\theta) \]

and \( E_{g_1}(u) \geq E_{g_1}(\theta) \) if and only if

\[ \phi + \frac{\sqrt{\frac{\delta}{\pi}} (\sigma - x)}{1 - \frac{2}{\pi}} E \left( \frac{Y_0^2}{xY_0^2 + y} \right) = 1 \]

\[ \Rightarrow x = \sigma^2 \], \( \sigma^2, s = \phi \)

If

\[ \left( \frac{1 - \frac{2}{\pi}}{\sigma^2} \right) xY_0^2 + y = a \text{ for } \ a \in \mathbb{R} \]

\[ \text{and } y = \sigma^2 \] a.s

If \( \sigma^2 = x \) or \( y = \sigma^2 \), \( P(Y_0^2 = c) = 1 \)

But \( Y_0^2 = (\phi + h_1)Y_0^2 + c + 2(\phi + h_1)cY_0^2 \) a.s

But \( \{Y_k, k \in \mathbb{Z}\} \) is a stationary series

\[ \Rightarrow P(Y_0^2 = c) = 1 \]

\[ \Rightarrow c = (\phi + h_1)c + c^2 \]

\[ \Rightarrow E_{Y_0^2} = 0 \]

\[ \text{and } \inf_{a \in \mathbb{R}} \sup_{n \to \infty} l_n(u, A) \leq \inf_{n \to \infty} \sup_{a \in \mathbb{R}} l_n(u, A) \]

\[ \text{with each positive integer } n, l_n(u) \text{ is a continuous function in compact set } G, \]

\[ \Rightarrow \lim_{n \to \infty} \sup_{a \in \mathbb{R}} l_n(u, A) \leq \lim_{n \to \infty} \sup_{a \in \mathbb{R}} l_n(u, A) \]

\[ \Rightarrow \lim_{n \to \infty} \inf_{a \in \mathbb{R}} l_n(u, A) \leq \lim_{n \to \infty} \inf_{a \in \mathbb{R}} l_n(u, A) \]

\[ \Rightarrow \lim_{n \to \infty} \inf_{a \in \mathbb{R}} l_n(u, A) \leq \lim_{n \to \infty} \inf_{a \in \mathbb{R}} l_n(u, A) \]

\[ \Rightarrow \lim_{n \to \infty} \inf_{a \in \mathbb{R}} l_n(u, A) \leq \lim_{n \to \infty} \inf_{a \in \mathbb{R}} l_n(u, A) \]

\[ \Rightarrow \lim_{n \to \infty} \inf_{a \in \mathbb{R}} l_n(u, A) \leq \lim_{n \to \infty} \inf_{a \in \mathbb{R}} l_n(u, A) \]
such that:
\[ r < E \inf_{t \in U_{n_i}} g_i(t). \]

Sets \( U(u), u \in C \) are open covers of \( C \), so \( C \) holds such finite open covers, are called \( U(u_1), U(u_2), \ldots, U(u_k) \) of \( C \). In accordance with Ergodic theorem, with every \( 1 \leq j \leq k \), we have:
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \inf_{t \in U_{u_i}} g_i(u) = E \inf_{t \in U_{[u_j]}} g_i(u) > r \text{ a.s}
\]

See that
\[
\inf_{u \in C} l_n(u) \geq \min_{i \in [j], u \in U_{u_i}} l_n(u) \geq \min_{i \in [j], u \in U_{u_i}} \inf_{t \in U_{u_i}} g_i(u)
\]
\[
\Rightarrow \lim_{n \to \infty} \inf_{u \in C} l_n(u) \geq \min_{i \in [j], u \in U_{u_i}} \inf_{t \in U_{u_i}} g_i(u) > r
\]

In out of events \( B_r \) with \( P(B_r) = 0 \) with \( r \) satisfying: \( r < \inf_{u \in C} Eg_i(u) \).

Therefore, \( \lim_{n \to \infty} \inf_{u \in C} l_n(u) \geq \inf_{u \in C} Eg_i(u) \text{ a.s} \)

But \( Eg_i(u) \) is continuous and \( \theta \in C \) is singly minimum of \( Eg_i(u) \)
\[
\Rightarrow \lim_{n \to \infty} \inf_{u \in C} l_n(u) > Eg_i(\theta) \text{ a.s}
\]

Let \( U \) is a open sphere with center \( \theta \) and enough small radius and \( U^* = U \cap \Gamma \). If \( \hat{\theta} \not\in U^* \), existing a random subseries \( n_k \) such that with \( C^* = \Gamma / U^* \), we have:
\[
\lim_{n \to \infty} \inf_{u \in C} l_n(u) = \lim_{k \to \infty} \inf_{u \in C} \hat{\theta}_n(u)
\]
\[
\leq \lim_{n \to \infty} \sup_{u \in C} l_n(\theta) \text{ a.s}
\]

But \( \limsup_{u \in C} \hat{\theta}_n(u) \leq Eg_i(\theta) < \liminf_{u \in C} l_n(u) \)

hence, with each above \( U^* \), existing random variable \( n_0 \) such that \( \hat{\theta}_n \in U^*, \forall n \geq n_0 \).

This completes the proof. \( \square \)

4. Simulation

In this section, we simulate series (1) with different values of \( \theta = (\phi, \sigma^2, \sigma^2) \). These simulations show stationary and non-stationary series cases.

We simulate series (1) with different values of \( \theta = (\phi, \sigma^2, \sigma^2) \) and in each case we can check the stationary conditions of the series (1) by Lemma 1. In Figure 1, we see that the series is not stationary with the negative slope \( \phi = -1.07 \) and in Figures 2 and 3 we simulate the not stationary series with positive slope \( \phi = 0.9 \) and \( \phi = 0.93 \). Figure 4 presents a stationary but clustering series. Figures 5-7 present stationary series with parameters are \( \phi = -0.7, \phi = 0 \) and \( \phi = 0.7 \).
Figure 5. Simulation for series \( Y_t \) defined by (1) with \( \phi = -0.7; \sigma_\phi = 0.1; \sigma_\epsilon = 0.1 \).

Figure 6. Simulation for series \( Y_t \) defined by (1) with \( \phi = 0; \sigma_\phi = 0.1; \sigma_\epsilon = 0.1 \).

Figure 7. Simulation for series \( Y_t \) defined by (1) with \( \phi = 0.7; \sigma_\phi = 0.1; \sigma_\epsilon = 0.1 \).

5. Application for Real-Time Series

In this section, we use model (1) for the model of return series of the price of gold on the free market in Hanoi, Vietnam. **Figure 8** show the Return series of Gold price \( r_t \).

From the data series we estimate for vector \( \theta = (\phi, \sigma_\phi^2, \sigma_\epsilon^2) \) is \( \hat{\theta} = (0.0004, 0.0002, 0.0069) \). So, we can use the following model to forecast the future value of gold price:

\[
r_t = (0.0004 + b_t) r_{t-1} + \epsilon_t\]

\( \epsilon_t \sim N(0, 0.0069), b_t \sim N(0, 0.0002) \).

6. Conclusion

This paper has solved some problems relating to a kind of first order time series with coefficient regression affected by non-negative random elements. In subsequent studies, the author will consider the asymptotic estimates of the parameters.

**REFERENCES**

