Complete Convergence and Weak Law of Large Numbers for $\bar{\rho}$-Mixing Sequences of Random Variables

Qunying Wu$^{1,2}$

$^1$College of Science, Guilin University of Technology, Guilin, China
$^2$Guangxi Key Laboratory of Spatial Information and Geomatics, Guilin, China

Received September 5, 2012; revised October 7, 2012; accepted October 20, 2012

ABSTRACT

In this paper, the complete convergence and weak law of large numbers are established for $\bar{\rho}$-mixing sequences of random variables. Our results extend and improve the Baum and Katz complete convergence theorem and the classical weak law of large numbers, etc. from independent sequences of random variables to $\bar{\rho}$-mixing sequences of random variables without necessarily adding any extra conditions.

Keywords: $\bar{\rho}$-Mixing Sequence of Random Variables; Complete Convergence; Weak Law of Large Number

1. Introduction

Let $(\Omega, \mathcal{F}, P)$ be a probability space. The random variables we deal with are all defined on $(\Omega, \mathcal{F}, P)$. Let $\{X_n; n \geq 1\}$ be a sequence of random variables. For each nonempty set $S \subset N$, write $\mathcal{F}_S = \sigma(X_i; i \in S)$. Given $\sigma$-algebras $\mathcal{B}, \mathcal{R}$ in $\mathcal{F}$, let

$$\rho(\mathcal{B}, \mathcal{R}) = \sup \left\{ \text{corr}(X, Y); X \in L_2(\mathcal{B}), Y \in L_2(\mathcal{R}) \right\},$$

where $\text{corr}(X, Y) = \frac{EXY - EXEY}{\sqrt{\text{Var}X \text{Var}Y}}$. Define the $\bar{\rho}$-mixing coefficients by

$$\bar{\rho}(n) = \sup \rho(\mathcal{F}_S, \mathcal{F}_T),$$

(1.1)

where (for a given positive integer $n$) this sup is taken over all pairs of nonempty finite subsets $S, T$ of $N$ such that $\text{dist}(S, T) \geq n$.

Obviously $0 \leq \bar{\rho}(n+1) \leq \bar{\rho}(n) \leq 1, n \geq 0$, and $\bar{\rho}(0) = 1$ except in the trivial case where all of the random variables $X_i$ are degenerate.

Definition 1.1. A sequence of random variables $\{X_n; n \geq 1\}$ is said to be a $\bar{\rho}$-mixing sequence of random variables if there exists $k \in N$ such that $\bar{\rho}(k) < 1$.

Without loss of generality we may assume that $\{X_n; n \geq 1\}$ is such that $\bar{\rho}(1) < 1$ (see [1]). Here we give two examples of the practical application of $\bar{\rho}$-mixing.

Example 1.1. According to the proof of Theorem 2 in [2] and Remark 3 in [1], if $\{X_i; i \geq 1\}$ is a strictly stationary Gaussian sequence which has a bounded positive spectral density $f(t)$, then the sequence $\{f(X_i); i \geq 1\}$ has the property that $\bar{\rho}(1) < 1$. Therefore, instantaneous functions $\{f(X_i); i \geq 1\}$ of such a sequence provides a class of examples for $\bar{\rho}$-mixing sequences.

Example 1.2. If $\{X_n; n \geq 1\}$ has a bounded positive spectral density $f(t)$, i.e., $0 < m < f(t) < M$ for every $t$, then $\bar{\rho}(1) < 1 - m/M < 1$. Thus, $\{X_n; n \geq 1\}$ is a $\bar{\rho}$-mixing sequence.

$\bar{\rho}$-mixing is similar to $\rho$-mixing, but both are quite different. $\rho(k)$ is defined by (1.1) with index sets restricted to subsets $S$ of $[1, n]$ and subsets $T$ of $[n + k, \infty)$, $n, k \in N$. On the other hand, $\rho$-mixing sequence assume condition $\rho(k) \rightarrow 0$, but $\bar{\rho}$-mixing sequence assume condition that there exists $k \in N$ such that $\bar{\rho}(k) < 1$, from this point of view, $\bar{\rho}$-mixing is weaker than $\rho$-mixing.

A number of writers have studied $\bar{\rho}$-mixing sequences of random variables and a series of useful results have been established. We refer to [2] for the central limit theorem [1,3], for moment inequalities and the strong law of large numbers [4-9], for almost sure convergence, and [10] for maximal inequalities and the invariance principle. When these are compared with the corresponding results for sequences of independent random variables, there still remains much to be desired.

The main purpose of this paper is to study the complete convergence and weak law of large numbers of partial sums of $\bar{\rho}$-mixing sequences of random variables and try to obtain some new results. We establish the
complete convergence theorems and the weak law of large numbers. Our results in this paper extend and improve the corresponding results of Feller [11] and Baum and Katz [12].

Lemma 1.1. ([10], Theorem 2.1) Suppose $K$ is a positive integer, $0 \leq r < 1$, and $q \geq 2$. Then there exists a positive constant $D = D(K, r, q)$ such that the following statement holds:

If $\{X_i; i \geq 1\}$ is a sequence of random variables such that $\rho(K) \leq r$ and $E X_i = 0$ and $E \left| X_i \right|^q < \infty$ for all $i \geq 1$, then suppose also that $\sum_{i=1}^{n} P(A_i) \leq c(1 - \alpha_n)$, i.e.,

$$
\left(1 - P\left(\max_{1 \leq i \leq n} |X_i| > x\right)\right)^2 \sum_{i=1}^{n} P\left(|X_i| > x\right) \leq cP\left(\max_{1 \leq i \leq n} |X_i| > x\right).
$$

2. Complete Convergence

In the following, let $a(x) - b(x)$ denote $a(x)/b(x) \to 1, x \to \infty$, and $a_n \ll b_n$ $(a_n \gg b_n)$ denote that there exists a constant $c > 0$ such that $a_n \leq c b_n$ $(a_n \geq c b_n)$ for sufficiently large $n$, log, mean $\ln(\max(x, e))$, and $S_n = \sum_{i=1}^{n} X_i$.

Definition 2.1. A measurable function $I(x) > 0 (x > 0)$ is said to be a slowly varying function at $\infty$ if for any $c > 0$, $\lim_{x \to \infty} \frac{I(cx)}{I(x)} = 1$.

Lemma 2.1 ([13], Lemma 1). Let $I(x)$ be a slowly varying function at $\infty$. Then

i) $\lim_{k \to \infty} \sup_{2^k \leq x \leq 2^{k+1}} \frac{I(x)}{I(2^k)} = 1$.

ii) $\lim_{x \to \infty} x^\delta I(x) = \infty$, $\lim_{x \to \infty} x^{-\delta} I(x) = 0$, for any $\delta > 0$.

iii) For any $r > 0$ and $\eta > 0$, there exist positive constants $c_1$ and $c_2$ (depending only on $r, \eta$, and the function $I(\cdot)$) such that for any positive number $k$,

$$
c_1 2^{v_r} I(2^r \eta) \leq \sum_{j=1}^{k} 2^{v_r} I(2^j \eta) \leq c_2 2^{v_r} I(2^r \eta).
$$

iv) For any $r < 0$ and $\eta > 0$, there exist positive constants $d_1$ and $d_2$ (depending only on $r, \eta$, and the function $I(\cdot)$) such that for any positive number $k$,

$$
d_1 2^{v_r} I(2^r \eta) \leq \sum_{j=1}^{k} 2^{v_r} I(2^j \eta) \leq d_2 2^{v_r} I(2^r \eta).
$$

Theorem 2.1. Let $\{X_n; n \geq 1\}$ be a $\rho$-mixing sequence of identically distributed random variables. Suppose that $I(x) > 0$ is a slowly varying function at $\infty$, and also assume that for each $a > 0$, the function $I(x)$ is bounded on the interval $(0, a)$. Suppose $0 < p < 2$ and $\alpha p > 2$; and if $\alpha \leq 1$ then suppose also that $EX_i = 0$. Then

$$
E\left[\left|X_i\right|^\alpha \left|I\left(\left|X_i\right|^\alpha\right)\right|\right] < \infty
$$

and
\[ \sum_{n=1}^{\infty} n^{\alpha p - 2} P \left( \max_{j \leq \infty} |S_j| > \varepsilon n^\alpha \right) < \infty, \quad \forall \varepsilon > 0 \] (2.2)
are equivalent.

For \( \alpha p = 1 \), we also have the following theorem under adding the condition that \( l(x) \) is a monotone non-decreasing function.

**Theorem 2.2.** Let \( \{X_n; n \geq 1\} \) be a \( \rho \)-mixing sequence of identically distributed random variables. Let \( l(x) > 0 \) is a slowly varying function at \( \infty \) and monotone non-decreasing function. Suppose \( \alpha > 1/2 \); and if \( \alpha \leq 1 \) then suppose also that \( EX_1 = 0 \). Then
\[ E \left( |X_1|^{\alpha p} l \left( |X_1|^{\alpha} \right) \right) < \infty \] (2.3) and
\[ \sum_{n=1}^{\infty} n^{\alpha p - 2} l \left( \max_{j \leq \infty} |S_j| > \varepsilon n^\alpha \right) < \infty, \quad \forall \varepsilon > 0 \] (2.4)
are equivalent.

Taking \( l(x) = 1 \) and \( l(x) = \log x \) respectively in Theorems 2.1 and 2.2 we can immediately obtain the following corollaries.

**Corollary 2.1.** Let \( \{X_n; n \geq 1\} \) be a \( \rho \)-mixing sequence of identically distributed random variables. Suppose \( 0 < p < 2 \) and \( \alpha p > 1 \); and if \( \alpha \leq 1 \) then suppose also that \( EX_1 = 0 \). Then
\[ E \left| X_1 \right|^p < \infty \] and
\[ \sum_{n=1}^{\infty} n^{\alpha p - 2} P \left( \max_{j \leq \infty} |S_j| > \varepsilon n^\alpha \right) < \infty, \quad \forall \varepsilon > 0 \] are equivalent.

**Corollary 2.2.** Let \( \{X_n; n \geq 1\} \) be a \( \rho \)-mixing sequence of identically distributed random variables. Suppose \( 0 < p < 2 \) and \( \alpha p > 1 \); and if \( \alpha \leq 1 \) then suppose also that \( EX_1 = 0 \). Then
\[ E \left( |X_1|^p \log |X_1| \right) < \infty \] and
\[ \sum_{n=1}^{\infty} n^{\alpha p - 2} \log n P \left( \max_{j \leq \infty} |S_j| > \varepsilon n^\alpha \right) < \infty, \quad \forall \varepsilon > 0 \] are equivalent.

**Remark 2.1.** When \( \{X_n; n \geq 1\} \) i.i.d., Corollary 2.5 becomes the Baum and Katz [12] complete convergence theorem. So Theorems 2.1 and 2.2 extend and improve the Baum and Katz complete convergence theorem from the i.i.d. case to \( \rho \)-mixing sequences.
\[
\sum_{i=1}^{\alpha} E|X_i|I_{[(i-1)^a, i^a]} \\
\leq \sum_{i=1}^{\alpha} I_{[(i-1)^a, i^a]}^a E|X_i|^a \\
\leq \sum_{i=1}^{\alpha} E|X_i|^a < \infty.
\]

By \( i_1 \to \infty \) and the Kronecker lemma,

\[
n_{i_0} \sum_{i=1}^{\alpha} E|X_i|I_{[(i-1)^a, i^a]} \to 0, n \to \infty.
\]

Hence (2.5) holds. So to prove (2.2) it suffices to prove that

\[
\sum_{n=1}^{\alpha} n^{a^{p-2}}l(n) P\left( \max_{\alpha \leq 0} \left| \sum_{i=1}^{\alpha} (Y_i - EZ_i) \right| > \epsilon n^a \right) < \infty, \quad (2.7)
\]

and \( \forall \epsilon > 0 \),

\[
\sum_{n=1}^{\alpha} n^{a^{p-2}}l(n) P\left( \max_{\alpha \leq 0} \left| \sum_{i=1}^{\alpha} (Y_i - EZ_i) \right| > \epsilon n^a \right) < \infty, \quad (2.8)
\]

By Lemmas 2.1 (i), (iii), (2.1), and for each \( a > 0 \), the function \( l(x) \) is bounded on the interval \( (0, a) \),

\[
\sum_{n=1}^{\alpha} n^{a^{p-2}}l(n) P\left( \max_{\alpha \leq 0} \left| \sum_{i=1}^{\alpha} (Y_i - EZ_i) \right| > \epsilon n^a \right) < \infty, \quad (2.9)
\]

Noting \( \alpha p - 2 > -1 \), by Lemma 2.1 (ii), we have

\[
\sum_{n=1}^{\alpha} P\left( \max_{\alpha \leq 0} |X_i| \geq \epsilon n^a \right) < \infty.
\]

Thus,

\[
\sum_{n=1}^{\alpha} n^{a^{p-2}}l(n) P\left( \max_{\alpha \leq 0} |X_i| \geq \epsilon n^a \right) < \infty.
\]

Therefore, for sufficiently large \( n \),

\[
\sum_{i=1}^{\alpha} P\left( |X_i| \geq \epsilon 2^{a^2} n^a \right) \leq 4eP\left( |X_i| \geq \epsilon 2^{a^2} n^a \right) < \frac{1}{2},
\]

which, in conjunction with Lemma 1.2, gives

\[
\sum_{i=1}^{\alpha} P\left( |X_i| \geq \epsilon 2^{a^2} n^a \right) \leq 4eP\left( \max_{\alpha \leq 0} |X_i| \geq \epsilon 2^{a^2} n^a \right).
\]

Q. Y. WU 487
Putting this one into (2.9), we get furthermore
\[ \sum_{n=1}^{\infty} n^{-p} I(-(n) P(|X_i| \geq 2^{a}En^{a}) < \infty, \quad \forall \varepsilon > 0. \]
Thus, by Lemmas 2.1 (i), (iii),
\[ \infty \geq \sum_{n=1}^{\infty} \sum_{j=1}^{n} n^{-p} I(-(n) P(|X_i| \geq 2^{a}En^{a}) \geq \sum_{j=1}^{\infty} 2^{a} I\left(|X_i| \geq \varepsilon 2^{a(j+1)} \right) \leq \sum_{k=1}^{\infty} n^{a} P\left(|X_i| \geq \varepsilon 2^{a(k+1)} \right) \]
\[ = \sum_{k=1}^{\infty} n^{a} P\left(|X_i| \geq \varepsilon 2^{a(k+1)} \right) \]
\[ \geq \sum_{k=1}^{\infty} n^{a} P\left(|X_i| < \varepsilon 2^{a(k+1)} \right) \]
\[ = \sum_{k=1}^{\infty} P\left(|X_i| < \varepsilon 2^{a(k+1)} \right) \]
\[ \geq \sum_{k=1}^{\infty} \frac{1}{k} \sum_{l=1}^{k} \left|X_i\right|^{\alpha l} \left|\left(X_i\right)^{\alpha l}\right| \frac{1}{l(l+1)} \]
\[ \leq 1 + \sum_{k=1}^{\infty} \frac{1}{k} \sum_{l=1}^{k} \left|X_i\right|^{\alpha l} \left|\left(X_i\right)^{\alpha l}\right| \frac{1}{l(l+1)} \]
Hence, by (2.3),
\[ E\left|X_i\right|^{\alpha l} < \infty. \quad (2.10) \]
i) For \( \alpha \leq 1 \), by \( EX_i = 0 \) and (2.10),
\[ n^{-a} \max_{1 \leq j \leq n} \left|\sum_{i=1}^{n} E|X_i| I_{\left|\left(X_i\right)\right|>\varepsilon n^{a}}\right| \leq n^{-a} E\left|X_i\right| \left|\left(X_i\right)^{\alpha l}\right| \frac{1}{\left(\left(X_i\right)\right)^{\alpha l}} \left|\left(X_i\right)^{\alpha l}\right| \frac{1}{l(l+1)} \]
\[ = E\left|X_i\right|^{\alpha l} \rightarrow 0, \quad (2.11) \]
ii) For \( \alpha > 1 \), i.e., \( 1/\alpha < 1 \),
\[ n^{-a} \max_{1 \leq j \leq n} \left|\sum_{i=1}^{n} E|X_i| I_{\left|\left(X_i\right)\right|>\varepsilon n^{a}}\right| \leq n^{-a} E\left|X_i\right| \left|\left(X_i\right)^{\alpha l}\right| \frac{1}{\left(\left(X_i\right)\right)^{\alpha l}} \left|\left(X_i\right)^{\alpha l}\right| \frac{1}{l(l+1)} \]
\[ = E\left|X_i\right|^{\alpha l} \rightarrow 0, \quad (2.12) \]

3. Weak Law of Large Numbers

**Theorem 3.1.** Suppose \( p > 1/2 \). Let \( \{X_i:n \geq 1\} \) be a \( \rho \)-mixing sequence of identically distributed random variables satisfying
\[ \lim_{n \rightarrow \infty} nP\left(|X_i| > n^{p}\right) = 0. \quad (3.1) \]

Then
\[ \frac{S_n}{n^{p}} - n^{-p} \left|\sum_{i=1}^{n} E\left|X_i\right| I_{\left|\left(X_i\right)\right|>\varepsilon n^{a}}\right| \rightarrow 0. \quad (3.2) \]

**Remark 3.1.** When \( p = 1 \) and \( \{X_i:n \geq 1\} \) i.i.d., then Theorem 3.1 is the weak law of large numbers (WLLN) due to Feller [11]. So, Theorem 3.1 extends the sufficient part of the Feller’s WLLN from the i.i.d. case to a \( \rho \)-mixing setting.

**Proof of Theorem 3.1.** Let \( X_i' = X_i I_{\left|\left(X_i\right)\right|>n^{a}} \) for \( 1 \leq j \leq n \) and \( S_n' = \sum_{j=1}^{n} X_j' \). Then, for each \( n \geq 2 \),
\[ \{X_i'\} \text{ are } \rho \text{-mixing identically distributed random variables and for every } \varepsilon > 0, \]
\[ P\left(\left|X_j' - S_n'\right| > \varepsilon n^{p}\right) \leq nP\left(|X_i| > n^{p}\right) \]
\[ \leq \sum_{j=1}^{n} P\left(|X_j| > n^{p}\right) = nP\left(|X_i| > n^{p}\right) \rightarrow 0, \]
via (3.1). So that (3.1) entails
\[ \frac{S_n'}{n^{p}} - \frac{S_n}{n^{p}} \rightarrow 0. \]
Thus, to prove (3.2) it suffices to verify that
\[ \frac{S'_n}{n^p} - n^{1-p} E X_1 I_{[k_1,k_\alpha]} \xrightarrow{p} 0. \] (3.3)

By (3.1) and the Toeplitz lemma,
\[ \sum_{k=1}^n k^{2p-2} kP(|X_1| > k^p) \xrightarrow{p} 0, \quad n \to \infty. \]

Thus, together with \( \sum k^{2p-2} \ll n^{2p-1} \) for \( p > 1/2 \), we have
\[ n^{-2p+1} \sum k^{2p-2} P(|X_1| > k^p) \xrightarrow{p} 0, \quad n \to \infty, \]
which, in conjunction with Lemma 1.1, yields for every \( \varepsilon > 0 \).
\[ P\left( S'_n - ES'_n > \varepsilon n^p \right) \ll n^{-2p} E\left( S'_n - ES'_n \right)^2 \]
\[ = n^{-2p} E\left( \sum_{j=1}^n \left( X'_j - EX'_j \right)^2 \right) \]
\[ \ll n^{-2p} \sum_{j=1}^n E\left( X'_j - EX'_j \right)^2 \ll n^{-2p+1} EX_1^2 \]
\[ = n^{-2p+1} EX_1 I_{[k_1,k_\alpha]} \]
\[ = n^{-2p+1} \sum_{k=1}^n \left( k^{2p} - k^{2p-2} \right) P\left( |X_1| > (k-1)^p \right) - P\left( |X_1| > k^p \right) \]
\[ \leq n^{-2p+1} \sum_{k=1}^n k^{2p} P\left( |X_1| > (k-1)^p \right) - n^{-2p} P\left( |X_1| > n^p \right) \]
\[ \ll n^{-2p+1} \left( \sum_{k=1}^n k^{2p-1} P\left( |X_1| > k^p \right) + 1 \right) \xrightarrow{p} 0. \]

Thus
\[ \frac{S'_n}{n^p} - ES'_n \xrightarrow{p} 0, \quad n \to \infty. \]
\[ \frac{S'_n}{n^p} - n^{1-p} EX_1 I_{[k_1,k_\alpha]} \xrightarrow{p} 0. \]
\[ i.e. (3.3) \text{ holds.} \]

4. Examples

In this section, we give two examples to show our Theorems.

Example 4.1. Let \( \{ X_n; n \geq 1 \} \) be a \( \tilde{\rho} \)-mixing sequence of identically distributed random variables. Suppose \( 0 < p < 2 \) and \( \alpha p \geq 1 \); and if \( \alpha \leq 1 \) then suppose also that \( EX_1 = 0 \). Assume that \( I(x) = \log^r x, \ r > 0 \) and \( X_1 \) has a distribution with
\[ P\left( |X_1| > x \right) \sim \frac{1}{x^{1/p}}. \ \beta > r+1. \]

Is easy to verify that \( I(x) \) satisfies the conditions of Theorems 2.1 and 2.2, and
\[ E \left( X'_n I \left( |X_1|^{\alpha} \right) \right) < \infty. \]

Therefore, by Theorems 2.1 and 2.2,
\[ \sum_{n=1}^{\infty} n^{p-2} \log n P\left( \max_{|S| > \varepsilon n^p} \right) < \infty, \ \forall \varepsilon > 0. \]

Example 4.2. Suppose \( p > 1/2 \). Let \( \{ X_n; n \geq 1 \} \) be a \( \tilde{\rho} \)-mixing sequence of identically distributed random variables. Assume that \( X_1 \) has a distribution with
\[ P\left( |X_1| > x \right) = o\left( \frac{1}{x^{1/p}} \right), \]
then obviously,
\[ \lim_{n \to \infty} n P\left( |X_1| > n^p \right) = 0. \]

Thus, by Theorem 3.1,
\[ \frac{S'_n}{n^p} - n^{1-p} EX_1 I_{[k_1,k_\alpha]} \xrightarrow{p} 0. \]

5. Acknowledgements

The work is supported by the National Natural Science Foundation of China (11061012), project supported by Program to Sponsor Teams for Innovation in the Construction of Talent Highlands in Guangxi Institutions of Higher Learning ([2011] 47), the Guangxi China Science Construction of Talent Highlands in Guangxi Institutions of Higher Learning ([2011] 47), the Guangxi China Science Foundation (2012GXNSFAA053010), and the support program of Key Laboratory of Spatial Information and Geomatics (1103108-08).

REFERENCES


