Asymmetry Index on Marginal Homogeneity for Square Contingency Tables with Ordered Categories

Kouji Tahata, Kanau Kawasaki, Sadao Tomizawa
Department of Information Sciences, Faculty of Science and Technology, Tokyo University of Science, Chiba, Japan
Email: {kouji_tahata, tomizawa}@is.noda.tus.ac.jp, k.kawasaki123@gmail.com
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ABSTRACT
For square contingency tables with ordered categories, the present paper considers two kinds of weak marginal homogeneity and gives measures to represent the degree of departure from weak marginal homogeneity. The proposed measures lie between –1 to 1. When the marginal cumulative logistic model or the extended marginal homogeneity model holds, the proposed measures represent the degree of departure from marginal homogeneity. Using these measures, three kinds of unaided distance vision data are analyzed.

Keywords: Marginal Homogeneity; Marginal Cumulative Logistic Model; Measure; Square Contingency Table

1. Introduction
Consider an \( R \times R \) square contingency table with ordered categories. Let \( p_{ij} \) denote the probability that an observation will fall in the \( i \)th row and \( j \)th column of the table ( \( i = 1, \ldots, R ; j = 1, \ldots, R \) ). Also let \( X \) and \( Y \) denote the row and column variables, respectively. The marginal homogeneity (MH) model ([1]) is defined by

\[
Y_{ii} \sim F_i(1, \ldots, 1) \quad \text{for} \quad i = 1, \ldots, R - 1
\]

with

\[
F_i(1, \ldots, 1) = \sum_{k=1}^{R} p_{ik} \quad \text{and} \quad F_i(1, \ldots, 1) = \sum_{k=1}^{R} p_{ki}.
\]

When the MH model does not hold, we are interested in applying the model that has weaker restriction than the MH model. As such a model, for example, there are the marginal cumulative logistic (ML) model ([2]) and the extended marginal homogeneity (EMH) model ([3-5]). We are also interested in considering the other structure of weak MH. The measures to represent the degree of departure from weak MH.

2. Weak Marginal Homogeneity I and Measure
2.1. Submeasure I
Let

\[
\Delta_i = \sum_{r=1}^{g-1} \left( F_{r+1}^X + F_{r+1}^Y \right),
\]

and

\[
F_{ij}^* = F_{ij}^X / \Delta_i, \quad F_{2r}^* = F_{2r}^Y / \Delta_i, \quad \text{for} \quad i = 1, \ldots, R - 1.
\]

Note that \( \sum_{r=1}^{g-1} \left( F_{r+1}^* + F_{2r}^* \right) = 1 \). Assuming that \( \left\{ F_{i+1}^X + F_{i+1}^Y \neq 0 \right\} \), consider the submeasure defined by

\[
\Psi_i = \frac{4}{\pi} \sum_{r=1}^{g-1} \left( F_{r+1}^* + F_{2r}^* \right) \left( \theta_{ij} - \frac{\pi}{4} \right),
\]

where

\[
\theta_{ij} = \sin^{-1} \left( \frac{F_{ij}^X}{\sqrt{\left( F_{ij}^X \right)^2 + \left( F_{ij}^Y \right)^2}} \right).
\]

Noting that \( 0 \leq \theta_{ij} \leq \pi/2 \), we see that
1) \( -1 \leq \Psi_i \leq 1 \),
2) \( \Psi_i = -1 \) if and only if \( F_{i+1}^X = 0 \) and \( F_{i+1}^Y > 0 \) \( (i = 1, \ldots, R - 1) \),
3) \( \Psi_i = 1 \) if and only if \( F_{i+1}^X = 0 \) and \( F_{i+1}^Y > 0 \) \( (i = 1, \ldots, R - 1) \). When the MH model holds, \( \Psi_i \) equals zero.
2.2. Submeasure II

Let
\[ S_i^X = 1 - F_i^X, \quad S_i^Y = 1 - F_i^Y \quad \text{for} \quad i = 1, \cdots, R - 1. \]

The MH model may be expressed as
\[ S_i^X = S_i^Y \quad \text{for} \quad i = 1, \cdots, R - 1. \]

Let
\[ \Delta_2 = \sum_{i=1}^{R-1} \left( S_i^X + S_i^Y \right), \]
and for \( i = 1, \cdots, R - 1, \)
\[ \theta^{(i)} = \sin^{-1} \left( \frac{S_i^X}{\sqrt{S_i^X^2 + S_i^Y^2}} \right), \]
\[ S_{(i)}^X = \frac{S_i^X}{\Delta_2}, \quad S_{(i)}^Y = \frac{S_i^Y}{\Delta_2}. \]

Note that \( \sum_{i=1}^{R-1} \left( S_{(i)}^X + S_{(i)}^Y \right) = 1. \) Assuming that \( \{ S_i^X + S_i^Y \neq 0 \}, \) we shall define the submeasure \( \Psi_2 \) as follows;
\[ \Psi_2 = \frac{4}{\pi} \sum_{i=1}^{R-1} \left( S_{(i)}^X + S_{(i)}^Y \right) \left( \theta^{(i)} - \frac{\pi}{4} \right). \]

Noting that \( 0 \leq \theta^{(i)} \leq \pi/2, \) we see that \( i = 1 \) \( -1 \leq \Psi_2 \leq 1, \) \( 2 \) \( \Psi_2 = -1 \) if and only if \( S_i^X = 0 \) and \( S_i^Y > 0 \) \( (i = 1, \cdots, R - 1); \) and \( 3 \) \( \Psi_2 = 1 \) if and only if \( S_i^Y = 0 \) and \( S_i^X > 0 \) \( (i = 1, \cdots, R - 1). \) When the MH model holds, \( \Psi_2 \) equals zero.

2.3. Complete Measure

Assume that \( \{ F_i^X + F_i^Y \neq 0 \} \) and \( \{ S_i^X + S_i^Y \neq 0 \}. \) Consider a measure defined by
\[ \Psi = \frac{1}{2} \left( \Psi_1 + \Psi_2 \right). \]

We see that \( i = 1 \) \( -1 \leq \Psi \leq 1, \) \( 2 \) \( \Psi = -1 \) if and only if \( F_i^X = 1 \) \( (\text{then} \quad S_i^X = 0 \) and \( F_i^Y = 0 \) \( (\text{then} \quad S_i^Y = 1) \) for all \( i = 1, \cdots, R - 1); \) and \( 3 \) \( \Psi = 1 \) if and only if \( F_i^X = 0 \) \( (\text{then} \quad S_i^X = 1) \) and \( F_i^Y = 1 \) \( (\text{then} \quad S_i^Y = 0) \) for all \( i = 1, \cdots, R - 1). \) Thus, \( \Psi = -1 \) indicates that \( p_{1R} = 1 \) and the other cell probabilities are zero (say, upper-right-marginal inhomogeneity), and \( \Psi = 1 \) indicates that \( p_{R1} = 1 \) and the other cell probabilities are zero (say, lower-left-marginal inhomogeneity). When \( \Psi = 0, \) we shall refer to this structure as the weak marginal homogeneity I (WMH-I). We note that if the MH model holds then the structure of WMH-I holds, but the converse does not hold.

Therefore, using the measure \( \Psi, \) we can see whether the structure of WMH-I departs toward the upper-right-marginal inhomogeneity or toward the lower-left-marginal inhomogeneity. As the measure \( \Psi \) approaches \(-1, \) the departure from WMH-I becomes greater toward the upper-right-marginal inhomogeneity. While as the \( \Psi \) approaches \( 1, \) it becomes greater toward the lower-left-marginal inhomogeneity.

3. Weak Marginal Homogeneity II and Measure

Let
\[ T_i^X = \Pr \left( X \leq i \mid X \neq Y \right) = \sum_{k=1}^{R} p_k^i, \]
\[ T_i^Y = \Pr \left( Y \leq i \mid X \neq Y \right) = \sum_{k=1}^{R} p_k^i, \]
for \( i = 1, \cdots, R - 1, \) where
\[ p_k^i = \frac{1}{\delta} (p_k - p_{ki}), \quad p_k^i = \frac{1}{\delta} (p_k - p_{ki}), \quad \delta = \sum_{i=1}^{R} p_{ki}. \]

The MH model may be expressed by
\[ T_i^X = T_i^Y \quad \text{for} \quad i = 1, \cdots, R - 1. \]

We shall consider the submeasure \( \Psi_1 \) which is defined by the submeasure \( \Psi_2 \) replaced \( \{ F_i^X \} \) and \( \{ F_i^Y \} \) by \( \{ T_i^X \} \) and \( \{ T_i^Y \} \), respectively.

Let
\[ U_i^X = 1 - T_i^X, \quad U_i^Y = 1 - T_i^Y \quad \text{for} \quad i = 1, \cdots, R - 1. \]

The MH model may be expressed by
\[ U_i^X = U_i^Y \quad \text{for} \quad i = 1, \cdots, R - 1. \]

We shall consider the submeasure \( \Psi_2 \) which is defined by the submeasure \( \Psi_2 \) replaced \( \{ S_i^X \} \) and \( \{ S_i^Y \} \) by \( \{ U_i^X \} \) and \( \{ U_i^Y \} \), respectively.

Assume that \( \{ T_i^X + T_i^Y \neq 0 \} \) and \( \{ U_i^X + U_i^Y \neq 0 \}. \) Consider a measure defined by
\[ \Upsilon = \frac{1}{2} \left( Y_1 + Y_2 \right). \]

We see that \( -1 \leq \Upsilon \leq 1. \) Let \( p_{ij}^* = p_{ij} / \delta \quad (i \neq j). \) In a similar way to \( \Psi, \) \( \Upsilon = -1 \) indicates that \( p_{1R}^* = 1 \) and the other \( p_{ji}^* \) are zero \( (i \neq j) \) (say, conditional upper-right-marginal inhomogeneity), and \( \Upsilon = 1 \) indicates that \( p_{R1}^* = 1 \) and the other \( p_{ji}^* \) are zero \( (i \neq j) \) (say, conditional lower-left-marginal inhomogeneity). When \( \Upsilon = 0, \) we shall refer to this structure as the weak marginal homogeneity II (WMH-II). We note that if the MH model holds then the structure of WMH-II holds, but the converse does not hold.
4. Relationships between Measures and Models

We shall consider the relationship between the measure \( \Psi \) (or \( Y \)) and the ML model. The ML model is given by

\[
L_i^X - L_i^Y = \Delta \quad \text{for} \quad i = 1, \ldots, R - 1,
\]

where

\[
L_i^X = \log \left( \frac{F_i^X}{1 - F_i^X} \right), \quad L_i^Y = \log \left( \frac{F_i^Y}{1 - F_i^Y} \right).
\]

A special case of this model obtained by putting \( \Delta = 0 \) is the MH model. The ML model may also be expressed as

\[
F_{ij}^X = \frac{\exp(\theta_i)}{1 + \exp(\theta_i)}, \quad F_{ij}^Y = \frac{\exp(\theta_i - \Delta)}{1 + \exp(\theta_i - \Delta)},
\]

for \( i = 1, \ldots, R - 1 \). Therefore, when the ML model holds,
1) \( \Delta > 0 \) if and only if \( \{F_{ij}^X > F_{ij}^Y\} \),
2) \( \Delta < 0 \) if and only if \( \{F_{ij}^X < F_{ij}^Y\} \),
3) \( \Delta = 0 \) if and only if \( \{F_{ij}^X = F_{ij}^Y\} \).

We obtain the following theorem.

**Theorem 1.** When the ML model holds,
1) \( \Delta > 0 \) if and only if \( \Psi < 0 \) (\( Y < 0 \)),
2) \( \Delta < 0 \) if and only if \( \Psi > 0 \) (\( Y > 0 \)),
3) \( \Delta = 0 \) (i.e., the MH model holds) if and only if \( \Psi = 0 \) (\( Y = 0 \)).

Next, we shall consider the relationship between the measure \( \Psi \) (or \( Y \)) and the EMH model, defined by

\[
G(i) = rG_{2(i)} \quad \text{for} \quad i = 1, \ldots, R - 1,
\]

where

\[
G_{1(i)} = \sum_{j=1}^{R} p_{ij}, \quad G_{2(i)} = \sum_{j=1}^{R} p_{ij} \cdot
\]

A special case of this model obtained by putting \( r = 1 \) is the MH model. Noting that \( G_{1(i)} - G_{2(i)} = F_{ij}^X - F_{ij}^Y \)
(i = 1, \ldots, R - 1), we obtain the following theorem.

**Theorem 2.** When the EMH model holds,
1) \( r > 1 \) if and only if \( \Psi < 0 \) (\( Y < 0 \)),
2) \( r < 1 \) if and only if \( \Psi > 0 \) (\( Y > 0 \)),
3) \( r = 1 \) (i.e., the MH model holds) if and only if \( \Psi = 0 \) (\( Y = 0 \)).

Thus, when the ML (EMH) model holds, the measures \( \Psi \) and \( Y \) are adequate to represent the degree of departure from MH.

5. Approximate Confidence Interval for Measures

Let \( n_j \) denote the observed frequency in the \( j \)th row and \( j \)th column of the table \( (i = 1, \ldots, R; j = 1, \ldots, R) \). Assuming that a multinomial distribution applies to the \( R \times R \) table, we shall consider an approximate standard error and large-sample confidence interval for the measure \( \Psi \), using the delta method, as described by [8]. The sample version of \( \Psi \), i.e., \( \hat{\Psi} \), is given by \( \Psi \) with \( \{p_{ij}\} \) replaced by \( \{\hat{p}_{ij}\} \), where \( \hat{p}_{ij} = n_{ij}/n \) and \( n = \sum n_j \). Using the delta method, we obtain the following theorem.

**Theorem 3.** \( \sqrt{n}(\hat{\Psi} - \Psi) \) has asymptotically (as \( n \to \infty \)) a normal distribution with mean zero and variance \( \sigma^2[\hat{\Psi}] \), where

\[
\sigma^2[\hat{\Psi}] = \frac{1}{4} \sum_{i=1}^{R} \sum_{j=1}^{R} \left( a_{ij} + b_{ij} \right)^2 p_{ij},
\]

and \( \hat{I}() \) is the indicator function, \( \hat{I}(\cdot) = 1 \) if true, 0 if not.

Also, the sample version of \( \Psi \), i.e., \( \hat{\Psi} \), is given by \( \hat{\Psi} \) with \( \{p_{ij}\} \) by \( \{\hat{p}_{ij}\} \). We obtain the following theorem.

**Theorem 4.** \( \sqrt{n}(\hat{\Psi} - \Psi) \) has asymptotically (as \( n \to \infty \)) a normal distribution with mean zero and variance \( \sigma^2[\hat{\Psi}] \), where

\[
\sigma^2[\hat{\Psi}] = \frac{1}{4} \sum_{i=1}^{R} \sum_{j=1}^{R} \left( c_{ij} + d_{ij} \right)^2 p_{ij},
\]
in Table 1(a), the estimated value of the measure $\Psi$ is $-0.0130$ and all values in confidence interval for $\Psi$ are negative. Therefore, the structure of WMH-I for a woman’s right and left eyes departs toward the upper-right-marginal inhomogeneity. Also we see from Table 3 that for the data in Table 1(a), the estimated value of the measure $\Psi$ is $-0.0130$ and all values in confidence interval for $\Psi$ are negative. Therefore, the structure of WMH-II for a woman’s right and left eyes departs toward the conditional upper-right-marginal inhomogeneity.

Table 4 gives the values of likelihood ratio chi-squared statistic for testing goodness-of-fit of each of MH, ML, and EMH models. We see from Table 4 that each of ML and EMH models fits these data well. Thus the measures $\Psi$ and $\Psi'$ would indicate the degree of departure from MH. We can see from these measures that the degree of departure from MH for the vision data in Table 1(a) is estimated to be $1.30$ (4.36) percent of the maximum departure toward the (conditional) upper-right-marginal inhomogeneity which indicates that the grade of right eye for arbitrary woman is “Best” and the grade of her left eye is “Worst”.

Example 2: Consider the unaided vision data in Table 1(b), taken from [9]. We see from Table 2 that for the

Table 1. Unaided distance vision data of (a) 7477 women in Britain from [1]; (b) 3242 men in Britain from [9] and (c) 4746 students in Japan from [3].

<table>
<thead>
<tr>
<th>(a) Women in Britain</th>
<th>Right eye grade</th>
<th>Left eye grade</th>
</tr>
</thead>
<tbody>
<tr>
<td>grade</td>
<td>(1)</td>
<td>(2)</td>
</tr>
<tr>
<td>Best (1)</td>
<td>1520</td>
<td>266</td>
</tr>
<tr>
<td>Second (2)</td>
<td>234</td>
<td>1512</td>
</tr>
<tr>
<td>Third (3)</td>
<td>117</td>
<td>362</td>
</tr>
<tr>
<td>Worst (4)</td>
<td>36</td>
<td>82</td>
</tr>
<tr>
<td>Total</td>
<td>1907</td>
<td>2222</td>
</tr>
</tbody>
</table>

(b) Men in Britain

<table>
<thead>
<tr>
<th>Right eye grade</th>
<th>Left eye grade</th>
</tr>
</thead>
<tbody>
<tr>
<td>grade</td>
<td>(1)</td>
</tr>
<tr>
<td>Best (1)</td>
<td>821</td>
</tr>
<tr>
<td>Second (2)</td>
<td>116</td>
</tr>
<tr>
<td>Third (3)</td>
<td>72</td>
</tr>
<tr>
<td>Worst (4)</td>
<td>43</td>
</tr>
<tr>
<td>Total</td>
<td>1052</td>
</tr>
</tbody>
</table>

(c) Students in Japan

<table>
<thead>
<tr>
<th>Right eye grade</th>
<th>Left eye grade</th>
</tr>
</thead>
<tbody>
<tr>
<td>grade</td>
<td>(1)</td>
</tr>
<tr>
<td>Best (1)</td>
<td>1291</td>
</tr>
<tr>
<td>Second (2)</td>
<td>149</td>
</tr>
<tr>
<td>Third (3)</td>
<td>64</td>
</tr>
<tr>
<td>Worst (4)</td>
<td>20</td>
</tr>
<tr>
<td>Total</td>
<td>1524</td>
</tr>
</tbody>
</table>

Table 2. Estimates of $\hat{\Psi}$, estimated approximate standard errors for $\hat{\Psi}$, and approximate 95% confidence intervals for $\hat{\Psi}$, applied to Table 1.

<table>
<thead>
<tr>
<th>Table</th>
<th>$\hat{\Psi}$</th>
<th>S. E.</th>
<th>C. I.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1(a)</td>
<td>-0.0130</td>
<td>0.0037</td>
<td>(-0.0203, -0.0056)</td>
</tr>
<tr>
<td>1(b)</td>
<td>0.0055</td>
<td>0.0064</td>
<td>(-0.0071, 0.0181)</td>
</tr>
<tr>
<td>1(c)</td>
<td>0.0125</td>
<td>0.0040</td>
<td>(0.0048, 0.0203)</td>
</tr>
</tbody>
</table>
Table 3. Estimates of $\hat{\Psi}$, estimated approximate standard errors for $\hat{\Psi}$, and approximate 95% confidence intervals for $\hat{\Psi}$, applied to Table 1.

<table>
<thead>
<tr>
<th>Table</th>
<th>$\hat{\Psi}$</th>
<th>S. E.</th>
<th>C. I.</th>
</tr>
</thead>
<tbody>
<tr>
<td>t(a)</td>
<td>-0.0436</td>
<td>0.0126</td>
<td>(-0.0683, -0.0190)</td>
</tr>
<tr>
<td>t(b)</td>
<td>0.0172</td>
<td>0.0201</td>
<td>(-0.0222, 0.0566)</td>
</tr>
<tr>
<td>t(c)</td>
<td>0.0517</td>
<td>0.0163</td>
<td>(0.0198, 0.0836)</td>
</tr>
</tbody>
</table>

Table 4. Values of likelihood ratio chi-squared statistic for the MH, ML, and EMH models applied to Table 1.

<table>
<thead>
<tr>
<th>Applied models</th>
<th>Degrees of freedom</th>
<th>Table</th>
<th>l(a)</th>
<th>l(b)</th>
<th>l(c)</th>
</tr>
</thead>
<tbody>
<tr>
<td>MH</td>
<td>3</td>
<td>11.99*</td>
<td>3.68</td>
<td>11.18*</td>
<td></td>
</tr>
<tr>
<td>ML</td>
<td>2</td>
<td>0.39</td>
<td>3.16</td>
<td>1.41</td>
<td></td>
</tr>
<tr>
<td>EMH</td>
<td>2</td>
<td>0.005</td>
<td>2.94</td>
<td>0.56</td>
<td></td>
</tr>
</tbody>
</table>

*Means significant at the 0.05 level.

data in Table 1(b) the estimated value of measure $\Psi$ is 0.0055 and the confidence interval for $\Psi$ includes zero. So this may indicate that there is a structure of WMH-I in the data in Table 1(b). Also we see from Table 3 that for the data in Table 1(c), the estimated value of measure $\hat{\Psi}$ is 0.0172 and the confidence interval for $\hat{\Psi}$ includes zero. So this may indicate that there is a structure of WMH-II in the data in Table 1(b).

Example 3: Consider the data in Table 1(c) taken from [3,10]. We see from Table 2 that for the data in Table 1(c), the estimated value of the measure $\Psi$ is 0.0125 and all values in confidence interval for $\Psi$ are positive. Therefore, the structure of WMH-I for a student’s right and left eyes departs toward the lower-left-marginal inhomogeneity. Also we see from Table 3 that for the data in Table 1(c), the estimated value of the measure $\hat{\Psi}$ is 0.0517 and all values in confidence interval for $\hat{\Psi}$ are positive. Therefore, the structure of WMH-II for a student’s right and left eyes departs toward the conditional lower-left-marginal inhomogeneity.

We see from Table 4 that each of ML and EMH models fits these data well. Thus the measures $\Psi$ and $\hat{\Psi}$ would indicate the degree of departure from MH. We can see from these measures that the degree of departure from MH for the vision data in Table 1(c) is estimated to be 1.25 (5.17) percent of the maximum departure toward the (conditional) lower-left-marginal inhomogeneity which indicates that the grade of right eye for arbitrary student is “Worst” and the grade of his/her left eye is “Best”.

7. Concluding Remarks

For the analysis of square contingency tables with ordered categories, when the ML model, or the EMH model, or other asymmetry models, for example, [11]’s conditional symmetry model (defined by $p_{ij}/p_{ji} = \theta$ for $i < j$) holds, the proposed measures $\Psi$ and $\hat{\Psi}$ are adequate to represent the degree of departure from the MH model toward two maximum departures, i.e., toward the (conditional) lower-left-marginal inhomogeneity or toward the (conditional) upper-right-marginal inhomogeneity.

8. Discussion

[6,7] considered the measures to represent the degree of departure from MH. The present paper has considered two types of maximum marginal inhomogeneity (i.e., the lower-left-marginal inhomogeneity and the upper-right-marginal inhomogeneity). The measures in [6,7] take the value 1 in two types of maximum marginal inhomogeneity. The measures $\Psi$ and $\hat{\Psi}$ in the present paper can distinguish these two kinds of maximum marginal inhomogeneity by the values $-1$ or $1$ although the measures in [6,7] cannot distinguish them. Also the proposed measures can represent the degree of departure from MH when the ML or the EMH models, or the other asymmetry models hold. Therefore for the ordinal data, the proposed measures rather than those in [6,7] may be useful to represent the degree of departure from MH.

9. Acknowledgements

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REFERENCES


