Estimation Using Censored Data from Exponentiated Burr Type XII Population

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Abstract

Maximum likelihood and Bayes estimators of the parameters, survival function (SF) and hazard rate function (HRF) are obtained for the three-parameter exponentiated Burr type XII distribution when sample is available from type II censored scheme. Bayes estimators have been developed using the standard Bayes and MCMC methods under square error and LINEX loss functions, using informative type of priors for the parameters. Simulation comparison of various estimation methods is made when \( n = 20, 40, 60 \) and censored data. The Bayes estimates are found to be, generally, better than the maximum likelihood estimates against the proposed prior, in the sense of having smaller mean square errors. This is found to be true whether the data are complete or censored. Estimates improve by increasing sample size. Analysis is also carried out for real life data.

Keywords: Exponentiated Distribution, Proportional Reversed Hazard Rate Model, Lehmann Alternatives, Maximum Likelihood and Bayes Estimation, Burr Type XII Distribution, Subjective Prior, SE and LINEX Loss Functions, MCMC

1. Introduction

Analogous to the Pearson system of distributions, Burr [1] introduced a system that includes twelve types of cumulative distribution functions (CDF) which yield a variety of density shapes. This system is obtained by considering CDF’s satisfying a differential equation which has a solution, given by:

\[
F(x) = \left[1 + \exp \left\{- \eta(x) \right\} \right]^{-1},
\]

where \( \eta(x) \) is chosen such that \( F(x) \) is a CDF on the real line. Twelve choices for \( \eta(x) \), made by Burr, resulted in twelve distributions from which types III, X and XII have been frequently used. The flexibilities of Burr XII distribution were investigated by Hatke [2], Burr [3], Rodrigues [4] and Tadikamalla [5].

In a different direction, it was Takahasi[6] who first noticed that the 3-parameter Burr XII probability density function (PDF) can be obtained by compounding a Weibull PDF with a gamma PDF. That is, if \( x|\theta \sim \text{Weibull}(\theta, \beta) \) and \( \theta \sim \text{gamma}(\gamma, \delta) \) then the compound PDF, say \( g(x|\beta, \gamma, \delta) \), is given by

\[
g(x|\beta, \gamma, \delta) = \int_0^\infty \left[ \theta \beta \beta^\theta e^{-\theta x^\beta} \right] \left[ \frac{1}{\Gamma(\gamma)} \theta^{\gamma-1} e^{-\theta x^\gamma} \right] d\theta
\]

which is the 3-parameter Burr XII (\( \beta, \gamma, \delta \)) PDF.

If \( \delta = 1 \), this PDF reduces to the 2-parameter Burr XII (\( \beta, \gamma \)), whose PDF, CDF, SF, and HRF are given, for \( x > 0 \), \( \beta, \gamma > 0 \), by:

\[
g(x|\beta, \gamma) = \beta \gamma x^{\beta-1} (1 + x^\beta)^{-\gamma-1}, \quad (1.1)
\]

\[
G(x|\beta, \gamma) = 1 - (1 + x^\beta)^{-\gamma}, \quad (1.2)
\]

\[
R_o(x|\beta, \gamma) = 1 - G(x|\beta, \gamma) = (1 + x^\beta)^{-\gamma}, \quad (1.3)
\]

\[
\lambda_0(x|\beta, \gamma) = g(x|\beta, \gamma) / \frac{R_o(x|\beta, \gamma)}{1 + x^\beta}. \quad (1.4)
\]

The Burr XII and its reciprocal Burr III distributions have been used in many applications such as actuarial science, as in Embrechts et al. [7] and Klugman[8], quantal bioassay as in Drane et al. [9], economics, as in McDonald and Richards [10], Morrison and Schmittlein...
Exponentiated distributions are also known as proportional reversed hazard rate models (PRHRM) with constant of proportionality $\alpha$. Reversed hazard rate function (RHRF) is defined by

$$\lambda_H^*(x) = \frac{h(x)}{H(x)}.$$ 

If $\lambda_H^*(x) = \frac{g(x)}{G(x)}$, where $H(x) = [G(x)]^\alpha$, then

$$\lambda_H^*(x) = \frac{\alpha [G(x)]^{\alpha-1} g(x)}{[G(x)]^\alpha} = \alpha \lambda_G^*(x).$$

That is, $\lambda_H^*(x)$ is proportional to $\lambda_G^*(x)$ with constant of proportionality $\alpha$. This is why the exponentiated distribution $H(x) = [G(x)]^\alpha$ is called PRHRM. See Gupta and Gupta [46]. Exponentiated distributions are also known as Lehmann alternatives, due to Lehmann [47], who defined the model, when $\alpha$ is a positive integer, as a non-parametric class of alternatives.

In general, the PDF, SF and HRF of the exponentiated CDF (1.5) are given by:

$$h(x) = \alpha [G(x)]^{\alpha-1} g(x), \quad (1.6)$$

$$R_H^*(x) = 1 - [G(x)]^\alpha, \quad (1.7)$$

$$\lambda_H^*(x) = \frac{h(x)}{R_H^*(x)} = \frac{\alpha [G(x)]^{\alpha-1} g(x)}{1 - [G(x)]^\alpha}, \quad (1.8)$$

**Relation between the HRF $\lambda_H^*$ of H and the HRF $\lambda_G^*$ of G**

If $0 < \alpha < 1$, then $\lambda_H^*(x) \geq \lambda_G^*(x)$ for all $x$ and if $\alpha \geq 1$ then $0 \leq \lambda_H^*(x) \leq \lambda_G^*(x)$, for all $x$. This follows by observing that

$$\lambda_H^*(x) = \alpha [1 - e_\alpha(x)] \lambda_G^*(x), \quad (1.9)$$

where

$$e_\alpha(x) = \frac{1 - [G(x)]^{\alpha-1}}{1 - [G(x)]^\alpha}. \quad (1.10)$$

If $0 < \alpha < 1$, then

$$-\infty < e_\alpha(x) \cdot \frac{\alpha}{\alpha - 1} \Rightarrow 1 - e_\alpha(x) \geq \frac{1}{\alpha} \Rightarrow \lambda_H^*(x) \geq \lambda_G^*(x).$$

If $\alpha \geq 1$, then

$$\frac{\alpha}{\alpha - 1} \leq e_\alpha(x) \leq 1 \Rightarrow 0 \leq \alpha [1 - e_\alpha(x)] \leq 1 \Rightarrow 0 \leq \lambda_H^*(x) \leq \lambda_G^*(x).$$

Notice that, for $\alpha \geq 0$, $e_\alpha(0) = 1$, $\lim_{x \to \infty} e_\alpha(x) = \frac{\alpha - 1}{\alpha}$, by using L’Hopital’s rule. It is clear, from (1.9), that $\lambda_H^*(x)$ is not proportional to $\lambda_G^*(x)$.

It can be shown that the CDF $H(x)$ is related to the HRF $\lambda_H^*(x)$ and RHRF $\lambda_H^*(x)$ by the relation...
\[ H(x) = \frac{\hat{\lambda}_n(x)}{\lambda_n(x) + \lambda'_n(x)}. \] (1.11)

2. Estimation of Parameters, SF and HRF

Suppose that n items, whose life times follow a CDF \( H(x) = \left[ G(x) \right]^r \), where the CDF \( G(x) \) may depend on a vector of parameters \( \beta \), are put on test and that the test is terminated at the \( r \)th failure (type II censoring). Suppose that the life times of the first \( r \) failed items \( x_1, \ldots, x_r \) have been observed. The likelihood function (LF) is then given by

\[
L(\theta; x) \propto \left[ \prod_{i=1}^{r} h(x_i; \theta) \left[ R_n(x_i; \theta) \right]^{a-r} \right],
\] (2.1)

where \( h() \) and \( R_n() \) are the PDF and SF corresponding to \( H() \), \( \theta = (x_1, \ldots, x_r) \) and \( \theta = (\alpha, \beta) \). In the EBBurr XII \((\alpha, \beta, \gamma)\), the CDF \( G() \) is Burr XII \((\beta, \gamma)\), given by (1.2), where \( \beta = (\beta, \gamma) \).

2.1. All Parameters of \( H \) Are Unknown

In this section, we consider the case in which the CDF \( G() \) is Burr XII \((\beta, \gamma)\), where \( \beta = (\beta, \gamma) \) and all parameters of \( H \) are unknown. In this case we have a vector of unknown parameters \( \theta = (\alpha, \beta) \) of \( H \).

2.1.1. Maximum Likelihood Estimation

The LF is given, in terms of \( G() \) and \( g() \), as:

\[
\ell(\theta; x) \propto \left[ \prod_{i=1}^{r} h(x_i; \theta) \left[ 1 - G(x_i; \theta) \right]^{a-r} \right] \left[ \prod_{i=1}^{r} g(x_i; \theta) \right]^{a-r}.
\] (2.2)

The log-LF is then given by

\[
\ell(\theta; x) \propto a \ln(\alpha) + \left( \alpha - 1 \right) \sum_{i=1}^{r} \ln G(x_i; \beta) + \sum_{i=1}^{r} \ln g(x_i; \theta) + (n-r) \ln \left( 1 - \left[ G(x_i; \beta) \right]^a \right).
\] (2.3)

Differentiate (2.3) partially with respect to \( \alpha \) and \( \beta = (\beta, \gamma) \) then equate to zero, to get

\[
\frac{r}{\alpha} + \sum_{i=1}^{r} \ln G(x_i; \beta) - \frac{(n-r) \left[ G(x_i; \beta) \right]^a \ln G(x_i; \beta)}{1 - \left[ G(x_i; \beta) \right]^a} = 0,
\] (2.4)

\[
(\alpha - 1) \sum_{i=1}^{r} A_{i, \beta}(\beta) + \sum_{i=1}^{r} B_i(\beta) - K_r(\theta) = 0,
\] (2.5)

where

\[
A_{i, \beta}(\beta) = \frac{1}{G(x_i; \beta)} \frac{\partial G(x_i; \beta)}{\partial \beta},
\]

\[
A_{i, \beta}(\beta) = \frac{1}{G(x_i; \beta)} \frac{\partial G(x_i; \beta)}{\partial \beta},
\]

\[
B_i(\beta) = \frac{1}{g(x_i; \beta)} \frac{\partial g(x_i; \beta)}{\partial \beta},
\]

\[
B_i(\beta) = \frac{1}{g(x_i; \beta)} \frac{\partial g(x_i; \beta)}{\partial \beta},
\]

and \( K_r(\theta) = \frac{(n-r)\alpha \left[ G(x_i; \beta) \right]^{a-1}}{1 - \left[ G(x_i; \beta) \right]^a} \).

The ML estimator of \( \alpha \) can be written, using (2.4) and (2.5), in terms of \( \beta \), as

\[
\hat{\alpha}_{ML}(\beta) = 1 - \frac{\sum_{i=1}^{r} \left[ B_i(\beta) - A_i(\beta) \right]}{\sum_{i=1}^{r} \left[ A_{i, \beta}(\beta) - A_i(\beta) \right]},
\] (2.7)

The MLEs of \( \beta \) and \( \gamma \), say \( \hat{\beta}_{ML} \) and \( \hat{\gamma}_{ML} \), can be obtained by maximizing the log-likelihood function with respect to \( \beta \) and \( \gamma \). Once \( \hat{\beta}_{ML} \) and \( \hat{\gamma}_{ML} \) are obtained, the ML estimator of \( \alpha \), say \( \hat{\alpha}_{ML} \), can be obtained from (2.7).

The MLEs are used in determining the vector of hyper-parameters in the Bayes case (see Section 4).

2.1.2. Standard Bayes Method

We assume that \( \alpha \) is independent of \( (\beta, \gamma) \) and that \( \alpha \sim \text{Gamma}(b_1, b_2) \), \( \gamma \sim \text{Beta}(b_3, b_4) \) and \( \beta \sim \text{Gamma}(b_5, b_6) \) so that the prior PDF of \( \theta = (\alpha, \beta, \gamma) \) is given by

\[
\pi(\theta) = \pi_1(\alpha) \pi_2(\beta) \pi_3(\gamma),
\]

\[
\pi_1(\alpha) \propto \alpha^{b_1-1} e^{-\alpha b_2}, \quad \alpha > 0, (b_1, b_2 > 0),
\]

\[
\pi_2(\beta) = \pi_5(\beta) \pi_4(\gamma | \beta) \propto \beta^{b_3-1} e^{-\beta b_4}, \quad \beta, \gamma > 0, (b_3, b_4, b_5, b_6 > 0),
\]

LF (2.2) can be rewritten in the form

\[
L(\theta; x) \propto \sum_{j=0}^{\infty} \sum_{i=1}^{r} C_{ij} \exp \left[ -\alpha T_{ij}(\beta, \gamma) - T_0(\beta, \gamma) \right],
\] (2.8)
where,

\[ C_h = (-1)^h \binom{n-r}{j_1}, \]

\[ T_h (\beta, \gamma) = - \sum_{i=1}^r \ln G(x_i | \beta, \gamma) + j_1 \ln G(x_0 | \beta, \gamma), \quad (2.9) \]

\[ T_0 (\beta, \gamma) = \sum_{i=1}^n \ln G(x_i | \beta, \gamma) - \sum_{i=1}^r \ln g(x_i | \beta, \gamma). \quad (2.10) \]

The posterior PDF is then given, from LF (2.8) and the prior \( \pi(\theta) \) by

\[
\pi(\theta | x) = \left[ \frac{\pi(\theta)}{C_h} \right]^{-1} \left[ \sum_{j=0}^n C_j \alpha^{-h-j} \beta^{b_j+h_j} \gamma^{-h_j-1} e^{-\delta T_0 + \delta T_h(\beta, \gamma)} \right].
\]

\[ \left(2.11\right) \]

where \( S_0^* = \sum_{j=0}^n C_j I_{0, h}, \]

\[
I_{0, h} = \int_0^\infty \int_0^\infty \frac{\beta^{b_h+h_b-1} \gamma^{b_j+h_j} e^{-\delta T_h(\beta, \gamma)}}{T_{0, h}^*(\beta, \gamma)} d\beta d\gamma,
\]

\[ T_h (\beta, \gamma) = \beta (b_0 + \gamma) + T_0 (\beta, \gamma), \quad T_0 (\beta, \gamma) = b_0 + T_0 (\beta, \gamma) \]

\[ T_h (\beta, \gamma) \quad \text{and} \quad T_0 (\beta, \gamma) \quad \text{are given by (2.9) and (2.10).} \]

2.1.2.1. Bayes Estimators under SEL Function

AL-Hussaini [45] showed that, under squared error loss, the Bayes estimators of \( \alpha, \beta, \gamma, R_{\ell_2} (x_0) \) and \( \lambda_{\ell_2} (x_0) \) are given, for any \( G \) (which has parameters \( \beta \) and \( \gamma \), by

\[ \hat{\alpha} = \frac{(r+b_0) S_h^*}{S_0^*}, \hat{\beta} = \frac{S_0^*}{S_0^*}, \hat{\gamma} = \frac{S_0^*}{S_0^*}. \quad (2.13) \]

\[ \hat{R}_{\ell_2} (x_0) = 1 - \frac{S_j}{S_0}, \hat{\lambda}_{\ell_2} (x_0) = \frac{(r+b_1) S_j^*}{S_0^*}. \quad (2.14) \]

where

\[ S_h^* = \sum_{j_1=0}^n C_j I_{0, j_1}, S_0^* = \sum_{j_1=0}^n C_j I_{0, j_1}, C_h = (-1)^h \binom{n-r}{j_1}, \]

\[ I_{3, h} = \int_0^\infty \int_0^\infty \frac{\beta^{b_h+h_b-1} \gamma^{b_j+h_j} e^{-\delta T_h(\beta, \gamma)}}{T_{0, h}^*(\beta, \gamma)} d\beta d\gamma, \]

\[ I_{4, h} = \int_0^\infty \int_0^\infty \frac{\beta^{b_h+h_b-1} \gamma^{b_j+h_j} e^{-\delta T_h(\beta, \gamma)}}{T_{0, h}^*(\beta, \gamma)} d\beta d\gamma. \]

2.1.2.2. Bayes Estimators under LINEX Loss Function

The SEL function has probably been the most popular loss function used in literature. The symmetric nature of SEL function gives equal weight to over- and under-estimation of the parameters under consideration. However, in life testing, over estimation may be more serious than under estimation or vice versa. Research has been directed towards asymmetric loss functions. Varian [48] suggested the use of linear-exponential (LINEX) loss function to be of the form

\[ \xi (\Delta) = b [e^\Delta - c \Delta - 1], \quad (2.15) \]

where \( c \neq 0, b \geq 0 \) and \( \Delta = \hat{u}(\theta) - u(\theta) \).

Thompson and Basu [49] generalized the LINEX loss function to the squared-exponential (SQUAREX) loss function to be of the form

\[ \xi^* (\Delta) = b [e^\Delta - d \Delta^2 - c \Delta - 1], \]

where \( d \neq 0, b, c \) and \( \Delta \) are as before.

The SQUAREX loss function reduces to the LINEX loss function if \( d = 0 \). If \( c = 0 \), the SQUAREX loss function reduces to the SEL function.

In this paper, we shall use the LINEX loss function, as the SQUAREX loss function could be similarly treated.

Using the LINEX loss function, the Bayes estimator of \( u(\theta) \), a function of the (vector) of parameter(s) \( \theta \), is given by

\[ \hat{u}_L (\theta) = - \frac{1}{c} \ln E \left[ e^{-c u(\theta)} | x \right], \quad (2.16) \]

\[ = - \frac{1}{c} \ln \left[ \int e^{-c u(\theta)} \pi (\theta | x) d\theta_1 \cdots d\theta_n \right], \]

where \( \pi (\theta | x) \) is the posterior PDF of the vector of parameters \( \theta \), given the set of data \( x \). In general, the integrals are taken over the n-dimensional space \( R_\theta \).

2.2. Theorem

The Bayes estimators of \( \alpha, \hat{\alpha}, \gamma, R_{\ell_2} (x_0) \) and \( \lambda_{\ell_2} (x_0) \)
under LINEX loss are
\[
\hat{\alpha} = -\frac{1}{c} \ln \left( \frac{S'_{\kappa}}{S_{\kappa}} \right), \quad \hat{\beta} = -\frac{1}{c} \ln \left( \frac{S'_{\beta}}{S_{\beta}} \right), \quad \hat{\gamma} = -\frac{1}{c} \ln \left( \frac{S'_{\gamma}}{S_{\gamma}} \right),
\]
\[
\hat{R}_c(x_0) = 1 - \frac{1}{c} \ln \left( \frac{S'_{x_0}}{S_{x_0}} \right), \quad \hat{\lambda}_c(x_0) = -\frac{1}{c} \ln \left( \frac{S'_{x_0}}{S_{x_0}} \right),
\]
(2.17)

where
\[
S'_{\kappa} = \sum_{j=0}^{\infty} C_j I_{0,j}, \quad S'_{\beta} = \sum_{j=0}^{\infty} C_j I_{1,j}, \quad \nu = 1, 2, 3,
\]
\[
S'_{4L} = \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} K_{j_1,j_2} I_{1,j_1,j_2}, \quad S'_{5L} = \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_3=0}^{\infty} K_{j_1,j_2,j_3} I_{1,j_1,j_2,j_3},
\]
(2.18)
\[
C_j = \frac{e^{1/j} C_j}{j^2!}
\]
\[
K_{j_1,j_2} = (-1)^{j_1} d_{j_2} a_{j_2} K_{j_1,j_2},
\]
\[
d_{j_2} = \frac{\Gamma (r + h_j)}{\Gamma (r + b_j)}, \quad a_{j_2,j_3} = j_2 (j_2 + 1) \cdots (j_2 + j_3 - 1),
\]
(2.19)
\[
I_{1,j} = \int_0^\infty \frac{e^{-\kappa \beta \gamma - c} \beta^\gamma \gamma^c e^{-\gamma \beta} \gamma^\beta \beta^\gamma} {\Gamma (r + b_j) \Gamma (r + h_j)} d\beta dy, 
\]
(2.20)

The proof is given in the Appendix 1.

3. Numerical Results and Comparisons

The estimates of \( \alpha, \hat{\alpha}, \gamma, \hat{\gamma} \), and \( R_{nL}(x_0) \) and \( \hat{\lambda}_{nL}(x_0) \) and their mean square errors (MSE) are computed by using the MLE, standard Bayes method (SB) and MCMC algorithm (SEL and LINEX loss). Such an algorithm is given in Appendix 2.

To see the effect of sample size on the performance of MLE and Bayes estimates, a comparison of different estimation methods is made when 1000 samples of size \( n = 20, 40, 60 \) each, are drawn from the population distribution, in the complete sample case and when data are censored at the 10% and 25% levels, for each sample size.

Under the LINEX loss, different values of \( c (0.5, 0.01, 0.1) \) are used, for different sample sizes and censoring values. The computational values are reported in Tables 1(a)-(c).

The actual population values are \( \alpha = 2.5, \beta = 1.5, \gamma = 2 \).

3.1. Remark

It can be numerically shown that the vector of parameters \( \theta = (\alpha, \beta, \gamma) \) satisfying the log-likelihood Equations (2.4)-(2.6) actually maximizes the likelihood function (2.3). This is done by applying Theorem (7-9) on p. 152 of Apostol [50].

3.2. Simulation Comparisons

Simulation comparisons of various estimation methods made when \( n = 20, 40, 60 \) and censored data. From Tables 1(a-c), below, it may be observed that the Bayes estimates are, generally, better than the MLEs against the proposed prior in the sense of having smaller MSEs. Even for sample size as small as \( n = 20 \) good Bayes estimates (with smaller MSEs), are obtained under the LINEX loss function as well as SEL with the same censoring level. All estimates improve by increasing sample size. Analysis is also carried out for real life data, in Section 4.

4. Real Life Data

In this section we analyze real life data set to demonstrate how the proposed methods can be used in practice. To check the validity of the fitted model, we use Kolmogorov-Smirnov goodness of fit test (KS) to test “the fitted distribution function is \( H(x) \)”. We plot the fitted distribution function \( H(x) \) using the three methods (ML, SBM, MCMC) and the empirical distribution function in each case.

The breaking strengths of 64 (= \( n \)) single carbon fibers of length 10 (Lawless [51],p. 573) are:

\[1.901, 2.132, 2.203, 2.228, 2.257, 2.350, 2.361, 2.396, 2.397, 2.445, 2.454, 2.454, 2.474, 2.518, 2.522, 2.525, 2.532, 2.575, 2.614, 2.616, 2.618, 2.624, 2.625, 2.659, 2.675,\]
Table 1. (a) Complete sample \((r = 20)\); (b) Censored sample \((r = 18)\); (c) Censored sample \((r = 15)\).

(a)

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<th>LINEX</th>
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(b)

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<td>MCMC</td>
</tr>
<tr>
<td>1.0808</td>
<td>0.9685</td>
<td>0.9767</td>
</tr>
<tr>
<td>(0.0993)</td>
<td>(0.0113)</td>
<td>(0.0117)</td>
</tr>
</tbody>
</table>

(c)

<table>
<thead>
<tr>
<th>Estimate (MSE)</th>
<th>SEL</th>
<th>LINEX</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\hat{a})</td>
<td>SB</td>
<td>MCMC</td>
</tr>
<tr>
<td>3.7674</td>
<td>2.5148</td>
<td>2.5536</td>
</tr>
<tr>
<td>(2.5013)</td>
<td>(0.0684)</td>
<td>(0.0848)</td>
</tr>
<tr>
<td>(\hat{b})</td>
<td>SB</td>
<td>MCMC</td>
</tr>
<tr>
<td>1.3723</td>
<td>1.4797</td>
<td>1.4880</td>
</tr>
<tr>
<td>(0.0416)</td>
<td>(0.0268)</td>
<td>(0.0241)</td>
</tr>
<tr>
<td>(\hat{c})</td>
<td>SB</td>
<td>MCMC</td>
</tr>
<tr>
<td>2.5580</td>
<td>1.9171</td>
<td>1.9167</td>
</tr>
<tr>
<td>(0.5274)</td>
<td>(0.0263)</td>
<td>(0.0410)</td>
</tr>
<tr>
<td>(\hat{R}_n(x_n))</td>
<td>SB</td>
<td>MCMC</td>
</tr>
<tr>
<td>0.7214</td>
<td>0.6802</td>
<td>0.6794</td>
</tr>
<tr>
<td>(0.0074)</td>
<td>(0.0035)</td>
<td>(0.0036)</td>
</tr>
<tr>
<td>(\hat{\lambda}_n(x_n))</td>
<td>SB</td>
<td>MCMC</td>
</tr>
<tr>
<td>1.1170</td>
<td>0.9783</td>
<td>0.9874</td>
</tr>
<tr>
<td>(0.1336)</td>
<td>(0.0178)</td>
<td>(0.0201)</td>
</tr>
</tbody>
</table>
In the complete sample case \((r = n)\), the estimates of the parameters, SF, HRF at \(x_0 = 3\) and the corresponding p-value of KS goodness of fit test are given in Table 2(a). The Bayes estimates (SB and MCMC) are calculated for the hyper-parameters \(b_1 = 180, b_2 = 0.6, b_3 = 2, b_4 = 3, b_5 = 2\). We have used the same values of \(b_2, b_3, b_4, b_5\) as in the simulation study. To give a value for \(b_1\), we noticed that MLE of \(\alpha\) is quite large. In the Bayes case, the mean of the gamma \((b_1, b_2)\) prior depends on \(b_1, b_2\). For fixed \(b_2\) at 0.6, this mean is large if \(b_1\) is large. After some fitting trials we found that \(b_1 = 180\) gives a good fit. See Figure 1.

Suppose that, this test is terminated after the first 55 \((= r)\) failures, the estimates of the parameters, SF, HRF at \(x_0 = 3\) and the corresponding p value of Kolmogorov-Smirnov goodness of fit test are given in Table 2(b).

### Table 2. (a) Complete Sample \((r = 64)\); (b) Censored Sample \((r = 55)\).

<table>
<thead>
<tr>
<th>Estimate</th>
<th>MLE</th>
<th>SEL</th>
<th>LINEX</th>
<th>MLE</th>
<th>SEL</th>
<th>LINEX</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>(c = -0.1)</td>
<td>(c = 0.01)</td>
<td>(c = 0.1)</td>
<td>(c = -0.1)</td>
</tr>
<tr>
<td>(\hat{\alpha})</td>
<td>414.57</td>
<td>302.12</td>
<td>302.20</td>
<td>324.82</td>
<td>331.37</td>
<td>299.99</td>
</tr>
<tr>
<td>(\hat{\beta})</td>
<td>2.5019</td>
<td>2.1284</td>
<td>2.1000</td>
<td>2.1321</td>
<td>2.1833</td>
<td>2.1280</td>
</tr>
<tr>
<td>(\hat{\gamma})</td>
<td>2.3222</td>
<td>2.5581</td>
<td>2.5610</td>
<td>2.5617</td>
<td>2.5619</td>
<td>2.5577</td>
</tr>
<tr>
<td>(\hat{\lambda}_0(x_0))</td>
<td>0.4458</td>
<td>0.4751</td>
<td>0.4719</td>
<td>0.4752</td>
<td>0.4689</td>
<td>0.4751</td>
</tr>
<tr>
<td>(\hat{\lambda}_n(x_n))</td>
<td>1.3422</td>
<td>1.1342</td>
<td>1.1640</td>
<td>1.1315</td>
<td>1.1689</td>
<td>1.1345</td>
</tr>
<tr>
<td>p-value</td>
<td>0.6190</td>
<td>0.7906</td>
<td>0.6545</td>
<td>0.7208</td>
<td>0.5318</td>
<td>0.7706</td>
</tr>
</tbody>
</table>

5. Concluding Remarks

Estimation of the parameters, survival and hazard rate functions are obtained when data are drawn from the three-parameter exponentiated Burr type XII distribution. Type II censoring is imposed on data. The maximum likelihood and Bayes methods are used in estimation. In the Bayes case, the estimators are obtained under squared-error and LINEX loss functions. The methods are compared by computing the mean squared errors (overall Bayes risks, in the Bayes case).

Kolmogorov-Smirnov goodness of fit test shows that the exponentiated Burr type XII distribution fits the data of the breaking strengths of 64 \((= n)\) single carbon fibers of length 10, given in Lawless, in all cases.

From Tables 1(a)-(c), it may be noticed that the Bayes estimates are, generally, better than the MLEs against the
Figure 1. Empirical and fitted CDF using different methods of estimation, (a) MLE; (b) SEL (SB); (c) SEL (MCMC); (d) LINEX (SB), $c = -0.1$; (e) LINEX (SB), $c = 0.01$; (f) LINEX (SB), $c = 0.1$. 

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proposed prior in the sense of having smaller MSEs. Even for sample size as small as \( n = 20 \), good Bayes estimates (with smaller MSEs), are obtained under LINEX loss function as well as SEL with the same censoring level. All estimates improve by increasing sample size.

6. References


Appendix 1

Proof of Theorem

From (2.16) we have,

\[
\hat{\alpha}_L = -\frac{1}{c} \ln E \left( e^{\alpha \theta} \big| x \right) = -\frac{1}{c} \ln \left\{ \int_0^\infty \int_0^\infty e^{\alpha \theta} \pi(\theta \big| x) \, d\alpha d\theta \right\}.
\]

Using the joint posterior \( \pi(\theta \big| x) \), given by (2.8),

\[
\hat{\alpha}_L = -\frac{1}{c} \ln \left\{ \sum_{\gamma=0}^{\infty} C_\gamma \int_0^\infty \int_0^\infty \alpha^{c+\lambda_h} e^{-\alpha \left[ a_T \gamma (\beta,\gamma) \right]} \beta^{\lambda_h} e^{-\alpha \gamma (\beta,\gamma)} \, d\alpha d\gamma \right\} = \frac{1}{c} \ln \left\{ \sum_{\gamma=0}^{\infty} C_\gamma \int_0^\infty \int_0^\infty \frac{\beta^{\lambda_h} e^{-\alpha \gamma (\beta,\gamma)}}{[T_\gamma(\beta,\gamma)]^{\gamma+\lambda_h}} \, d\beta d\gamma \right\} = \frac{1}{c} \ln \left\{ S_{1L} / S_0^* \right\}.
\]

Similarly,

\[
\hat{\beta}_L = -\frac{1}{c} \ln E \left( e^{\beta \theta} \big| x \right) = -\frac{1}{c} \ln \left\{ \int_0^\infty \int_0^\infty e^{\beta \theta} \pi(\theta \big| x) \, d\alpha d\theta \right\}
\]

\[
\hat{\beta}_L = -\frac{1}{c} \ln \left\{ \sum_{\gamma=0}^{\infty} C_\gamma \int_0^\infty \int_0^\infty \alpha^{c+\lambda_h} e^{-\alpha \left[ a_T \gamma (\beta,\gamma) \right]} \beta^{\lambda_h} e^{-\alpha \gamma (\beta,\gamma)} \, d\alpha d\gamma \right\} = \frac{1}{c} \ln \left\{ \sum_{\gamma=0}^{\infty} C_\gamma \int_0^\infty \int_0^\infty \frac{\beta^{\lambda_h} e^{-\alpha \gamma (\beta,\gamma)}}{[T_\gamma(\beta,\gamma)]^{\gamma+\lambda_h}} \, d\beta d\gamma \right\} = \frac{1}{c} \ln \left\{ S_{1L} / S_0^* \right\},
\]

and

\[
\hat{\gamma}_L = -\frac{1}{c} \ln \left( e^{-\gamma \theta} \big| x \right) = -\frac{1}{c} \ln \left\{ \int_0^\infty \int_0^\infty e^{-\gamma \theta} \pi(\theta \big| x) \, d\alpha d\gamma \right\}
\]

\[
\hat{\gamma}_L = -\frac{1}{c} \ln \left\{ \sum_{\gamma=0}^{\infty} C_\gamma \int_0^\infty \int_0^\infty \alpha^{c+\lambda_h} e^{-\alpha \left[ a_T \gamma (\beta,\gamma) \right]} \beta^{\lambda_h} e^{-\alpha \gamma (\beta,\gamma)} \, d\alpha d\gamma \right\} = \frac{1}{c} \ln \left\{ \sum_{\gamma=0}^{\infty} C_\gamma \int_0^\infty \int_0^\infty \frac{\beta^{\lambda_h} e^{-\alpha \gamma (\beta,\gamma)}}{[T_\gamma(\beta,\gamma)]^{\gamma+\lambda_h}} \, d\beta d\gamma \right\} = \frac{1}{c} \ln \left\{ S_{1L} / S_0^* \right\}.
\]

\[
\hat{K}_L (x_0) = 1 - \ln \left\{ \int_0^\infty \int_0^\infty \left[ 1 + \sum_{\alpha=1}^\infty \left. \pi(\theta \big| x) \right| d\alpha d\beta \right] \right\}
\]

\[
\hat{K}_L (x_0) = 1 - \ln \left\{ \sum_{\gamma=0}^{\infty} \sum_{\beta=0}^{\infty} K_{\gamma,\beta} \int_0^\infty \int_0^\infty \alpha^{c+\lambda_h} e^{-\alpha \left[ a_T \gamma (\beta,\gamma) \right]} \beta^{\lambda_h} e^{-\alpha \gamma (\beta,\gamma)} \, d\alpha d\gamma \right\} = 1 - \ln \left\{ \sum_{\gamma=0}^{\infty} \sum_{\beta=0}^{\infty} \frac{\beta^{\lambda_h} e^{-\gamma (\beta,\gamma)}}{[T_\gamma(\beta,\gamma)]^{\gamma+\lambda_h}} \, d\beta d\gamma \right\} = 1 - \ln \left\{ S_{1L} / S_0^* \right\}
\]

and
\[
\hat{\lambda}_t(x_0) = -\frac{1}{c} \ln \left[ \sum_{j_2=1}^{N} \sum_{j_1=0}^{j_2!} \frac{a_{j_2,j_1}}{j_2!} \left( \frac{g(x_0)}{G(x_0)} \right)^{j_2} \int_0^\infty \int_0^\infty \alpha^{j_2} \left[ G(x_0)^{(j_2+j_1+1)} \pi(\theta|x) \right] d\alpha d\beta d\gamma \right]
\]

\[
= -\frac{1}{c} \ln \left[ \sum_{j_2=1}^{N} \sum_{j_1=0}^{j_2!} K_{j_2,j_1} \left. \int_0^\infty \int_0^\infty \alpha^{j_2} \left[ G(x_0)^{(j_2+j_1+1)} \pi(\theta|x) \right] \frac{\beta^{j_2+h-1} \gamma^{j_1+1} e^{-\tau(\beta,\gamma)}}{T_{j_2}(\beta,\gamma)-(j_2+j_1) \ln G(x_0)^{j_2+h+1}} d\beta d\gamma \right] \Gamma(r+h) S_0^* \right]
\]

\[
S^*_0, \quad S^*_v, \quad v = 0, 1, \ldots, 5 \quad \text{are given by (2.18) and (2.19).}
\]

**Appendix 2**

Implementation of MCMC method

To use the MCMC method in computing Bayes estimates of \( \alpha, \beta, \gamma, R(x_0), \hat{\lambda}(x_0) \), at specific value of \( x_0 \), we first notice that the general problem is in evaluating the integral

\[
E[x_\theta(\theta)] = \int \phi(\theta) \pi(\theta|x) \theta + \frac{\theta \pi(\theta|x)}{\int \phi(\theta) \pi(\theta|x) \theta < \infty}. \]

If we can draw samples from \( \pi(\theta|x) \), then Monte Carlo integration allows us to estimate this expectation by the average:

\[
\hat{\phi}_N = \frac{1}{N} \sum_{i=1}^{N} \phi(\theta^{(i)}) \quad \text{if we generate samples using a Markov chain (aperiodic, irreducible and has a stationary distribution with PDF } \pi(\theta|x) \text{), then by the ergodic theorem } \hat{\phi}_N \rightarrow E_x[\phi(\theta)], \text{as } N \rightarrow \infty. \text{ The estimate } \hat{\phi}_N \text{ is called an ergodic average. Also for such chains, if the variance of } \phi(\theta) \text{ is finite, the central limit theorem holds and convergence occurs geometrically. Early iterations } \phi^{(1)}, \ldots, \phi^{(M)} \text{, reflect starting value } \theta^{(0)}. \text{ These iterations are called burn-in. After the burn-in, we say that the chain has ’converged’. The burn-in are omitted from the ergodic averages to end up with}
\]

\[
\hat{\phi} = \frac{1}{N-M} \sum_{i=M+1}^{N} \phi(\theta^{(i)}).
\]

Methods for determining M are called *convergence diagnostics*. For details on the MCMC, see Cowles and Carlin [52], Gelman and Rubin [53], Roberts et al. [54], Tierney [55] and Gamerman and Lopes [56].

Associated Bayesian methods based on MCMC tools and novel model diagnostic tools to perform inference based on fully specified models are discussed by Sinha et al [57].

The data set is analyzed by applying the provided Gibbs sampler and Metropolis-Hasting algorithm, using WinBugs 1.4 (http://www.mrcbsu.cam.ac.uk/bugs/winbugs/contents.shtml), which can be downloaded and used.

To implement the MCMC method, based on SEL function, we have

Step 0: Take some initial guess of \( \alpha, \beta, \gamma \) say \( \alpha^{(0)}, \beta^{(0)} \) and \( \gamma^{(0)} \).

Step 1: Generate \( \alpha^{(1)}, \beta^{(1)}, \gamma^{(1)} \) from the respective posteriors:

\[
\pi(\alpha|\beta,\gamma,x) = A_0 \alpha^{-h-1} \sum_{j=1}^{N} C_j e^{-\alpha \gamma_{j \in (\beta,\gamma)}},
\]

\[
\pi(\beta|\alpha,\gamma,x) = A_j \sum_{h=0}^{\infty} C_h \beta^{-h-1} e^{-\alpha \gamma_{h \in (\beta,\gamma)}},
\]

\[
\pi(\gamma|\alpha,\beta,x) = A_h \sum_{j=0}^{\infty} C_j \beta^{-h-1} e^{-\alpha \gamma_{j \in (\beta,\gamma)}}.
\]

with

\[
A_0^{-1} = \frac{1}{r+b} \sum_{h=0}^{\infty} C_h \left( \frac{1}{T_{h}(\beta,\gamma)} \right)^{r+h},
\]

\[
A_j^{-1} = \sum_{h=0}^{\infty} C_h \int_0^{\infty} \beta^{-h-1} e^{-\alpha \gamma_{h \in (\beta,\gamma)}} d\beta,
\]

\[
A_h^{-1} = \sum_{j=0}^{\infty} C_j \int_0^{\infty} \beta^{-h-1} e^{-\alpha \gamma_{j \in (\beta,\gamma)}} d\gamma.
\]

Step 2: From \( i = 1 \) to \( N-1 \), generate:

\( \alpha^{(i)} \) from \( \pi(\alpha|\beta^{(i)},\gamma^{(i)},x) \), \( \beta^{(i)} \) from \( \pi(\beta|\alpha^{(i)},\gamma^{(i)},x) \), \( \gamma^{(i)} \) from \( \pi(\gamma|\alpha^{(i)},\beta^{(i)},x) \).

Step 3: Calculate the Bayes estimators of \( \alpha, \beta, \gamma \) from:

\[
\hat{\alpha}_g = \frac{1}{N-M} \sum_{i=M+1}^{N} \alpha^{(i)},
\]

\[
\hat{\beta}_g = \frac{1}{N-M} \sum_{i=M+1}^{N} \beta^{(i)},
\]

\[
\hat{\gamma}_g = \frac{1}{N-M} \sum_{i=M+1}^{N} \gamma^{(i)}.
\]

For a given time \( x_0 \), the Bayes estimators of the SF and HRF are computed from

\[
\hat{R}_g(x_0) = \frac{1}{N-M} \sum_{i=M+1}^{N} \left[ 1 - \left( 1 + x_0^{(i)} \right)^{-\gamma^{(i)}} \right]^{\alpha^{(i)}}.
\]
These are the Bayes estimators based on SEL function. 
The Bayes estimators using MCMC method based on 
LINEX loss function are given by 

$$\hat{\alpha}_L (x_0) = \frac{1}{N-M} \sum_{i=M+1}^{N} \alpha^{(i)} \beta^{(i)} \gamma^{(i)} x_0^{\beta^{(i)}-1} \left[1 + x_0^{\beta^{(i)}} \right]^{-\gamma^{(i)}-1} \left[1 - \left(1 + x_0^{\beta^{(i)}} \right)^{-\gamma^{(i)}-1} \right].$$

where

$$R^{(i)} (x_0) = 1 - \left[1 + x_0^{\beta^{(i)}} \right]^{-\gamma^{(i)}-1},$$

$$\hat{\lambda}_L (x_0) = 1 - \left[1 + x_0^{\beta^{(i)}} \right]^{-\gamma^{(i)}-1} \left[1 - \left(1 + x_0^{\beta^{(i)}} \right)^{-\gamma^{(i)}-1} \right].$$