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Roughly *B*-invex Multi-Objective Programming Problems

Tarek Emam
Department of Mathematics, Faculty of Science, Suez Canal University, Suez, Egypt
Email: drtema@yahoo.com

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ABSTRACT

In this paper, we shall be interested in characterization of efficient solutions for special classes of problems. These classes consider roughly *B*-invexity of involved functions. Sufficient and necessary conditions for a feasible solution to be an efficient or properly efficient solution are obtained.

Keywords: Multi-Objective Programming Problems; Roughly *B*-invex; Efficient Solutions; Properly Efficient Solutions

1. Introduction

The study of multi-objective programming problems was very active in recent years. The minimum (efficient, Pareto) solution is an important concept in mathematical models, economics, decision theory, optimal control and game theory (see, for example, [1]). In most works, an assumption of convexity was made for the objective functions. Very recently, some generalized convexity has received more attention (see, for example, [2-6]). A significant generalization of convex functions is invex function introduced first by Hanson [7], which has greatly been applied in nonlinear optimization and other branches of pure and applied sciences.

The concept of *B*-invex functions was proposed by [8] as generalization of convex functions; these functions were extended to quasi *B*-invex, and pseudo *B*-invex functions. Many functions seem to be *B*-invex, but they are not, and many non *B*-invex functions are able to get *B*-invex by choosing a suitable condition. Based on the previous discussion, Tarek [9] introduced a new class of *B*-invex functions, this class called roughly *B*-invex functions.

Inspired and motivated by above works, the purpose of this paper is to formulate a multi-objective programming problem which it involves roughly *B*-invex functions. An efficient solution for considered problem is characterized by weighting and ε-constraint approaches. In the end of the paper, we obtain sufficient and necessary conditions for a feasible solution to be an efficient or properly efficient solution for this kind of problems. Let us survey, briefly, the definitions and some results of roughly *B*-invexity.

Definition 1 [10]

Let $y \in M \subset R^n$. The set *M* is said to be *B*-invex with respect to $\eta : M \times M \to R^n$ at $y \in M$ if there exists

$$b(x, y, \lambda) : M \times M \times [0,1] \to R_+,$$

such that $y + \lambda b\eta(x, y) \in M$, for each $x \in M$, and $0 \le \lambda \le 1$.

M is said to be *B*-invex set with respect to η if *M* is *B*-invex at each $y \in M$ with respect to the same η.

Note that, as in convex set, the intersection of finite (or infinite) family of *B*-invex sets is *B*-invex but the union is not necessarily *B*-invex set. Also, the sum of *B*-invex sets and the multiplying a *B*-invex set by a real number are again *B*-invex sets. Every *B*-invex set with respect to $\eta : M \times M \to R^n$ is an invex set when $b = 1$; but the con- verse is not necessarily true.

Definition 2 [9]

A numerical function *f*, defined on a *B*-invex subset *M* of R^n, is said to be roughly *B*-invex with respect to $\eta : M \times M \to R^n$ with roughness degree r at $y \in M$ if there exists $b(x, y, \lambda) : M \times M \times [0,1] \to R_+$, such that

$$f(y + \lambda b\eta(x, y)) \le \lambda bf(x) + (1 - \lambda b) f(y),$$

for each $x \in M$, and $0 \le \lambda \le 1$ such that $\|x - y\| \ge r$.

f is said to be roughly *B*-invex on *M* with respect to $\eta(x, y)$ if it is roughly *B-invex* at each $y \in M$ with respect to the same $\eta(x, y)$.

Every invex function, with respect to η is roughly *B*-invex function with respect to same η, where *b* (*x*, *y*) = 1; but the converse is not necessarily true. If the functions $f_i : R^n \to R$ are all roughly *B*-invex with respect to $\eta : M \times M \to R^n$ with roughness degree $r_i \forall i$ on a *B*-invex set $M \subseteq R^n$, then the function

$$h(x) = \sum_{i=1}^{k} a_i f_i(x)$$

is roughly *B*-invex with respect to same η with roughness degree

$$r = \min_{1 \le i \le k} r_i$$

on M for $a_i \ge 0$. If $f: R^n \to R$ is roughly B-invex with respect to $\eta: M \times M \to R^n$ with roughness degree r, on B-invex set $M \subseteq R^n$, then for any real number $\gamma \in R$ the level set $K_\gamma = \{x : x \in M, f(x) \le \gamma\}$ is B-invex set. A numerical function f defined on a B-invex set $M \subseteq R^n$ is roughly B-invex function with respect to $\eta: M \times M \to R^n$ with roughness degree r if and only if *epi*(f) is a B-invex set. If $(f_i)_{i \in I}$ is a family of numerical func- tions, which are roughly B-invex with respect to $\eta: M \times M \to R^n$ with roughness degree r and bounded from above on a B-invex set $M \subseteq R^n$, then the numerical function

$$f(x) = \operatorname*{Sup}_{i \in I} f_i(x)$$

is a roughly B-invex with respect to same η with roughness degree r on M. If $f: R^n \to R$ is a differentiable roughly B-invex function with respect to $\eta: M \times M \to R^n$ with roughness degree r, at $y \in M$, then there exists a function $\bar{b}(x, y)$, such that

$$\bar{b}(x, y)\eta(x, y)^T \nabla_x f(y) \le \bar{b}(x, y)[f(x) - f(y)],$$

for each $x \in M$ such that $\|x - y\| \ge r$.

Definition 3 [9]

A numerical function f, defined on a B-invex subset M of R^n, is said to be quasi roughly B-invex with respect to $\eta: M \times M \to R^n$ with roughness degree r at $y \in M$, if there exists $b(x, y, \lambda): M \times M \times [0,1] \to R_+$, such that

$$f(x) \le f(y) \Rightarrow f(y + \lambda b \eta(x, y)) \le f(y),$$

for each $x \in M$, and $0 \le \lambda \le 1$ such that $\|x - y\| \ge r$.

f is said to be quasi roughly B-invex on M with respect to $\eta(x, y)$ if it is roughly B-invex at each $y \in M$ with respect to the same $\eta(x, y)$.

A $f: R^n \to R$ is quasi roughly B-invex with respect to η: $M \times M \to R^n$ with roughness degree r, on $M \subseteq R^n$, if and only if the level set

$$K_\gamma = \{x : x \in M, f(x) \le \gamma\}$$

is B-invex set. A roughly B-invex function, with respect to $\eta: M \times M \to R^n$ with roughness degree r is quasi roughly B-invex function with respect to same η with roughness degree r. Let $M \subseteq R^n$ be B-invex set, if $f: R^n \to R$ is differentiable quasi roughly B-invex with respect to $\eta: M \times M \to R^n$ with roughness degree r, at $y \in M$, then there exists a function $\bar{b}(x, y)$, such that

$$\bar{b}(x, y)\eta(x, y)^T \nabla_x f(y) \le 0,$$

for each $x \in M$ such that $\|x - y\| \ge r$.

Definition 4 [9]

A numerical function f, defined on a B-invex subset M of R^n, is said to be pseudo roughly B-invex with respect to $\eta: M \times M \to R^n$ with roughness degree r at $y \in M$, if there exists $b(x, y, \lambda): M \times M \times [0,1] \to R_+$, and, there exists a strictly positive function a: $R^n \times R^n \to R$ such that

$$f(x) < f(y) \Rightarrow f(y + \lambda b \eta(x, y)) \le f(y) + \lambda(\lambda - 1)a(x, y)$$

for each $x \in M$ such that $\|x - y\| \ge r$.

If $f: R^n \to R$ is roughly B-invex function with respect to $\eta: M \times M \to R^n$ with roughness degree r on B-invex set $M \subseteq R^n$, then f is pseudo roughly B-invex function with respect to same η with roughness degree r on M. Let $M \subseteq R^n$ be B-invex set and $f: R^n \to R$ be a differentiable pseudo roughly B-invex with respect to $\eta: M \times M \to R^n$ with roughness degree r, at $y \in M$, then there exists a function $\bar{b}(x, y)$, such that

$$\bar{b}(x, y)\eta(x, y)^T \nabla_x f(y) \ge 0 \Rightarrow f(x) \ge f(y),$$

for each $x \in M$ such that $\|x - y\| \ge r$.

2. Problem Formulation

Let $f_j: R^n \to R$, $j = 1, 2, \cdots, k$, and $g_i: R^n \to R$, $i = 1, 2, \cdots, m$ are real valued roughly B-invex functions on R^n. A roughly B-invex multi-objective programming problem is formulated as follows:

$$\text{(P)} \qquad \begin{array}{l} Min\ f_j(x), \\ \text{subject to} \\ x \in M = \{x \in R^n : g_i(x) \le 0, i = 1, 2, \cdots m\}. \end{array}$$

Definition 5 [11]

A feasible solution x for (P) is said to be an efficient solution for (P) if and only if there is no other feasible x for (P) such that, for some $i \in \{1, 2, \cdots, k\}$,

$$f_i(x) < f_i(x^*), f_j(x) \le f_j(x^*), \text{for all } j \ne i.$$

Definition 6 [11]

An efficient solution $x \in M$ for (P) is a properly efficient solution for (P) if there exists a scalar $\mu > 0$ such that for each i, $i = 1, 2, \cdots, k$, and each $x \in M$ satisfying $f_i(x) < f_i(x^*)$, there exists at least one $j \ne i$ with $f_j(x) > f_j(x^*)$, and

$$\left[f_i(x) - f_i(x^*)\right] / \left[f_j(x^*) - f_j(x)\right] \le \mu.$$

Lemma 1 [9]

If $g_i: R^n \to R$ is roughly B-invex with respect to $\eta: M \times M \to R^n$ with roughness degree r, on R^n, $i = 1, 2, \cdots, m$, then the set

$$M = \{x \in R^n : g_i(x) \le 0, i = 1, 2, \cdots, m\}$$

is B-invex set.

Lemma 2 [9]

If $g_i: R^n \to R$ is quasi roughly B-invex with respect to $\eta: M \times M \to R^n$ with roughness degree r, on R^n, $i = 1, 2, \cdots, m$, then the set

$$M = \{x \in R^n : g_i(x) \le 0, i = 1, 2, \cdots, m\}$$

is B-invex set.

Lemma 3

Let $\bar{b}(x, y, \lambda): M \times M \times [0,1] \to [0,1]$. If $f: R^n \to R$ is a roughly B-invex function with respect to $\eta: M \times M \to R^n$ with roughness degree r on a B-invex set $M \subseteq R^n$, then the set

$$A = \bigcup_{x \in M} A(x) \quad \text{is convex,}$$

where $A(x) = \{z : z \in R^k, z > f(x) - f(x^*)\}, x \in M$.

Proof. Let $z^1, z^2 \in A$, then for $x^1, x^2 \in M$ and $0 \le \lambda \le 1$, we have

$$\begin{aligned} &\lambda z^1 + (1-\lambda) z^2 \\ &> \lambda\left[f(x^1) - f(x^*)\right] + (1-\lambda)\left[f(x^2) - f(x^*)\right] \\ &= \lambda f(x^1) + (1-\lambda) f(x^2) - f(x^*) \\ &\ge \lambda \bar{b} f(x^1) + (1 - \lambda\bar{b}) f(x^2) - f(x^*) \\ &\ge f\left(x^2 + \lambda \bar{b} \eta(x^1, x^2)\right) - f(x^*). \end{aligned}$$

Since f is a roughly B-invex function on a B-invex set M. Then $\lambda z^1 + (1-\lambda) z^2 \in A$, and hence A is convex set.

For a feasible point $x^* \in M$, we denote $I(x^*)$ as the index set for binding constraints at x^*, *i.e.*, $I(x^*) = \{i : g_i(x^*) = 0\}$.

3. Characterizing Efficient Solutions by Weighting Approach

To characterizing an efficient solution for problem (P) by weighting approach [11] let us scalarize problem (P) to become in the form.

$$(\mathrm{P}_w) \qquad Min \sum_{j=1}^{k} w_j f_j(x), \text{ s.t. } x \in M,$$

where $\quad w_j \ge 0,\ j = 1, 2, \cdots, k,\ \sum_{j=1}^{k} w_j = 1$

and f_j, $j = 1, 2, \cdots, k$ are roughly B-invex with respect to $\eta: M \times M \to R^n$ with roughness degree r_j on B-invex set M.

Theorem 1

If $\bar{x} \in M$ is an efficient solution for problem (P), then there exist

$$w_j \ge 0,\ j = 1, 2, \cdots, k,\ \sum_{j=1}^{k} w_j = 1$$

such that $\bar{x}$ is an optimal solution for problem (P_w).

Proof. Let $\bar{x} \in M$ be an efficient solution for problem (P), then the system $f_j(x) - f_j(\bar{x}) < 0$, $j = 1, 2, \cdots, k$ has no solution $x \in M$. Upon Lemma 3 and by applying the generalized Gordan theorem [12], there exist $p_j \ge 0$, $j = 1, 2, \cdots, k$ such that

$$p_j\left[f_j(x) - f_j(\bar{x})\right] \ge 0,\ j = 1, 2, \cdots, k,$$

and $$\frac{p_j}{\sum_{j=1}^{k} p_j} f_j(x) \ge \frac{p_j}{\sum_{j=1}^{k} p_j} f_j(\bar{x}).$$

Denote $$w_j = \frac{p_j}{\sum_{j=1}^{k} p_j},$$

then $$w_j \ge 0,\ j = 1, 2, \cdots, k,\ \sum_{j=1}^{k} w_j = 1,$$

and $$\sum_{j=1}^{k} w_j f_j(\bar{x}) \le \sum_{j=1}^{k} w_j f_j(x).$$

Hence $\bar{x}$ is an optimal solution for problem (P_w).

Theorem 2

If $\bar{x} \in M$ is an optimal solution for $(\mathrm{P}_{\bar{w}})$ corresponding to $\bar{w}_j$, then $\bar{x}$ is an efficient solution for problem (P) if either one of the following two conditions holds:

(i) $\bar{w}_j > 0$, for all $j = 1, 2, \cdots, k$; or (ii) $\bar{x}$ is the unique solution of $(\mathrm{P}_{\bar{w}})$.

Proof. To proof see V. Chankong, Y. Y. Haimes [11].

4. Characterizing Efficient Solutions by ε-Constraint Approach

An ε-constraint approach is one of the common approaches for characterizing efficient solutions of multiobjective programming problems [11]. In the following we shall characterizing an efficient solution for multiobjective roughly B-invex programming problem (P) in term of an optimal solution of the following scalar problem.

$$\mathrm{P}_q(\varepsilon) \qquad \begin{aligned} &Min\ f_q(x), \\ &\text{subiect to } x \in M, \\ &f_j(x) \le \varepsilon_j,\ j = 1, 2, \cdots, k,\ j \ne q. \end{aligned}$$

where f_j, $j = 1, 2, \cdots, k$ are roughly B-invex with respect to $\eta: M \times M \to R^n$ with roughness degree r_j on B-invex set M.

Theorem 3

If $\bar{x} \in M$ is an efficient solution for problem (P), then $\bar{x}$ is an optimal solution for problem $\mathrm{P}_q(\bar{\varepsilon})$ corresponding to $\bar{\varepsilon}_j = f_j(\bar{x})$.

Proof.

Let $\bar{x}$ be not optimal solution for $\mathrm{P}_q(\bar{\varepsilon})$ where $\bar{\varepsilon}_j = f_j(\bar{x}), j = 1, 2, \cdots, k,\ j \ne q$. So there exists $x \in M$ such that $$f_q(x) < f_q(\bar{x}),$$

$$f_j(x) \le \bar{\varepsilon}_j = f_j(\bar{x}),\ j = 1, 2, \cdots, k,\ j \ne q.$$

Thus, $\bar{x}$ is inefficient solution for problem (P) which is a contradiction. Hence $\bar{x}$ is an optimal solution for problem $P_q(\bar{\varepsilon})$.

Theorem 4

Let $\bar{x} \in M$ is an optimal solution of $P_q(\bar{\varepsilon})$ for all $q = 1, 2, \cdots, k$, where $\bar{\varepsilon}_j = f_j(\bar{x})$, $j = 1, 2, \cdots, k$. Then $\bar{x}$ is an efficient solution for problem (P).

Proof.

Since $\bar{x} \in M$ is an optimal solution for $P_q(\bar{\varepsilon})$, for all $q = 1, 2, \cdots, k$. So, for each $x \in M$, we get $f_q(\bar{x}) \le f_q(x)$, $q = 1, 2, \cdots, k$.

This implies that the system $f_j(x) - f_j(\bar{x}) \le 0$, $j = 1, 2, \cdots, k$ has no solution $x \in M$, *i.e.* $\bar{x}$ is an efficient solution for problem (P).

5. Sufficient and Necessary Conditions for Efficiency

In this section, we discuss the sufficient and necessary conditions for a feasible solution x^* to be efficient or properly efficient for problem (P) in the form of the following theorems.

Theorem 5

Suppose there exists a feasible solution x^* for (P), and scalars $w_j > 0$, $j = 1, 2, \cdots, k, u_i \ge 0$, $i \in I(x^*)$, such that

$$\sum_{j=1}^{k} w_j \nabla f_j(x^*) + \sum_{i \in I(x^*)} u_i \nabla g_i(x^*) = 0. \tag{1}$$

If f_j, $j = 1, 2, \cdots, k$ are roughly B-invex with respect to $\eta : M \times M \to R^n$ with roughness degree r at x^* and g_i, $i \in I(x^*)$ is roughly B-invex with respect to same η with roughness degree r_i at x^*. Then x^* is a properly efficient solution for problem (P).

Proof.

Since f_j, $j = 1, 2, \cdots, k$ and g_i, $i \in I(x^*)$ are roughly B-invex with respect to same η, then there exists a function $b^*(x, x^*)$ such that

$$\begin{aligned}
& b^*(x,x^*) \sum_{j=1}^{k} w_j \left[f_j(x) - f_j(x^*) \right] \\
& \ge \eta(x,x^*) b^*(x,x^*) \sum_{j=1}^{k} w_j \left[\nabla f_j(x^*) \right]^t \\
& = -\eta(x,x^*) b^*(x,x^*) \sum_{i \in I(x^*)} u_i \left[\nabla g_i(x^*) \right]^t \\
& \ge b^*(x,x^*) \left[\sum_{i \in I(x^*)} u_i g_i(x^*) - \sum_{i \in I(x^*)} u_i g_i(x) \right] \\
& = -b^*(x,x^*) \sum_{i \in I(x^*)} u_i g_i(x) \ge 0.
\end{aligned}$$

by (1) for each $x \in M$ such that

$$\left\| x - x^* \right\| \ge \max_{1 \le j \le k, i \in I(x^*)} (r_j, r_i).$$

Thus, $b^*(x,x^*) \sum_{j=1}^{k} w_j f_j(x) \ge b^*(x,x^*) \sum_{j=1}^{k} w_j f_j(x^*)$,

for all $x \in M$, which implies that x^* is the minimizer of

$$\sum_{j=1}^{k} \lambda_j f_j(x)$$

such that
$$\lambda_j = \frac{w_j}{b^*(x,x^*) \sum_{j=1}^{k} w_j}$$

under the constraint $g(x) \le 0$ where $\left\| x - x^* \right\| \ge r$. Hence, from Theorem (4.11) of [11], x^* is a properly efficient solution for problem (P).

Theorem 6 Let x^* be a feasible solution for (P). If there exist scalars $w_j \ge 0, j = 1, 2, \cdots, k$,

$$\sum_{j=1}^{k} w_j = 1, u_i \ge 0, i \in I(x^*),$$

such that the triplet (x^*, w_i, u_i) satisfies (1) of Theorem (5),

$$\sum_{j=1}^{k} w_j f_j$$

is strictly roughly B-invex with respect to η: $M \times M \to R^n$ with roughness degree r at x^* and g_i, $i \in I(x^*)$ is roughly B-invex with respect to same η with roughness degree r_i at x^*. Then x^* is an efficient solution for problem (P).

Proof.

Suppose that x^* is not an efficient solution for (P). Then, there exists a feasible $x \in M$, and index v such that $f_v(x) < f_v(x^*)$, $f_j(x) \le f_j(x^*)$, for all $j \ne v$.

Since
$$\sum_{j=1}^{k} w_j f_j$$

is strictly roughly B-invex with respect to η with roughness degree r at x^*, then there exists a function $b^*(x, x^*)$ such that

$$\begin{aligned}
& 0 \ge b^*(x,x^*) \left[\sum_{j=1}^{k} w_j f_j(x) - \sum_{j=1}^{k} w_j f_j(x^*) \right]. \\
& \Rightarrow 0 > \eta(x,x^*) b^*(x,x^*) \sum_{j=1}^{k} w_j \left[\nabla f_j(x^*) \right]^t
\end{aligned} \tag{2}$$

Also, roughly B-invexity of g_i, $i \in I(x^*)$ with respect to same η at x^* with roughness degree r_i implies

$$\eta(x,x^*) b^*(x,x^*) \nabla g_i(x^*) \le b^*(x,x^*) \left[g_i(x) - g_i(x^*) \right],$$

i.e.
$$\eta(x,x^*) b^*(x,x^*) \nabla g_i(x^*) \le 0, i \in I(x^*), \tag{3}$$

such that $\left\| x - x^* \right\| \ge \max(r, r_i)$. Adding (2) and (3), contradicts (1). Hence, x^* is an efficient solution for problem (P).

Remark 1

Similarly as in Theorem (5), it can be easily seen that x^* becomes properly efficient solution for (P), in the above theorem, if $w_j > 0$, for all $j = 1, 2, \cdots, k$.

Theorem 7

Suppose there exists a feasible solution x^* for (P), and scalars $w_i > 0, j = 1, 2, \cdots, k,\ u_i \geq 0,\ i \in I\left(x^*\right)$ such that (1) of Theorem (5) holds. If

$$\sum_{j=1}^{k} w_j f_j$$

is pseudo roughly B-invex with respect to $\eta : M \times M \to R^n$ with roughness degree r at x^* and g_I is quasi roughly B-invex with respect to same η with roughness degree r_I at x^*. Then x^* is a properly efficient solution for problem (P).

Proof.

Since $g_I\left(x\right) \leq g_I\left(x^*\right) = 0,\ u_i \geq 0$, and g_I are quasi roughly B-invex with respect to η with roughness degree r_I at x^*, then there exists a function $b^*\left(x, x^*\right)$ such that

$$\eta\left(x, x^*\right) b^*\left(x, x^*\right) \sum_{i \in I\left(x^*\right)} u_i \left[\nabla g_i\left(x^*\right)\right]^t \leq 0,$$

for all $x \in M$ such that $\left\|x - x^*\right\| \geq \max_i r_i$. By using (1), we have $\eta\left(x, x^*\right) b^*\left(x, x^*\right) \sum_{j=1}^{k} w_j \left[\nabla f_j\left(x^*\right)\right]^t \geq 0$

which implies

$$b^*\left(x, x^*\right) \sum_{i=1}^{k} w_j f_j\left(x\right) \geq b^*\left(x, x^*\right) \sum_{j=1}^{k} w_j f_j\left(x^*\right),$$

since $\sum_{j=1}^{k} w_j f_j$

is pseudo roughly B-invex with respect to same η with roughness degree r at x^* which implies that x^* is the minimizer of

$$\sum_{j=1}^{k} \lambda_j f_j\left(x\right)$$

such that $\lambda_j = \dfrac{w_j}{b^*\left(x, x^*\right) \sum_{j=1}^{k} w_j}$

under the constraint $g\left(x\right) \leq 0$ where $\left\|x - x^*\right\| \geq \max(r, r_i)$. Therefore, x^* is a properly efficient solution for problem (P).

Theorem 8

Suppose that there exist a feasible solution x^* for (P) and scalars $w_j \geq 0, j = 1, 2, \cdots, k$,

$$\sum_{j=1}^{k} w_j = 1, \quad u_i \geq 0, i \in I\left(x^*\right),$$

such that (1) of Theorem (5) holds. Let

$$\sum_{j=1}^{k} w_j f_j$$

be strictly pseudo roughly B-invex with respect to $\eta : M \times M \to R^n$ with roughness degree r at x^* and g_I be quasi roughly B-invex with respect to same η with roughness degree r_I at x^*. Then x^* is an efficient solution for problem (P).

Proof. Suppose that x^* is not an efficient solution for (P).Then, there exists a feasible x for (P), and index v such that $f_v\left(x\right) < f_v\left(x^*\right), f_i\left(x\right) \leq f_i\left(x^*\right)$, for all $i \neq r$. then there exists a function $b^*\left(x, x^*\right)$ such that,

$$b^*\left(x, x^*\right) \sum_{j=1}^{k} w_j f_j\left(x\right) \leq b^*\left(x, x^*\right) \sum_{j=1}^{k} w_j f_j\left(x^*\right),$$

Strictly pseudo roughly B-invexity of $\sum_{j=1}^{k} w_j f_j$

implies that $\eta\left(x, x^*\right) b^*\left(x, x^*\right) \sum_{j=1}^{k} w_j \left[\nabla f_j\left(x^*\right)\right]^t < 0$

for all $x \in M$ such that $\left\|x - x^*\right\| \geq r$. Since g_I is quasi roughly B-invex with respect to same η with roughness degree r_I at x^* and $g_I\left(x\right) \leq g_I\left(x^*\right) = 0$,

then $\eta\left(x, x^*\right) b^*\left(x, x^*\right) \nabla g_I\left(x^*\right) \leq 0$,

for all $x \in M$ such that

$$\left\|x - x^*\right\| \geq \max_i r_i.$$

The proof now similar to the proof of Theorem (6).

Remark 2

Similarly as in Theorem (7), it can be easily seen that x^* becomes properly efficient solution for (P), in the above theorem, if $w_j > 0$, for all $j = 1, 2, \cdots, k$.

Theorem 9

Suppose there exists a feasible solution x^* for (P), and scalars $w_j > 0,\ j = 1, 2, \cdots, k, u_i \geq 0,\ i \in I\left(x^*\right)$ such that (1) of Theorem (5) holds. Let

$$\sum_{j=1}^{k} w_j f_j$$

be pseudo roughly B-invex with respect to $\eta : M \times M \to R^n$ with roughness degree r_I at x^* and $u_I\, g_I$ be quasi roughly B-invex with respect to same η with roughness degree r_I at x^*. Then x^* is a properly efficient solution for problem (P).

Proof. The proof is similar to the proof of Theorem (7).

Theorem 10

Suppose that there exist a feasible solution x^* for (P) and scalars $w_j \geq 0, j = 1, 2, \cdots, k$,

$$\sum_{j=1}^{k} w_j = 1, u_i \geq 0, i \in I\left(x^*\right),$$

such that (1) of Theorem (5) holds. If $I\left(x^*\right) \neq \varphi$,

$$\sum_{i=1}^{k} w_j f_j$$

is quasi roughly B-invex with respect to $\eta : M \times M \to R^n$ with roughness degree r at x^* and $u_I\, g_I$ is strictly pseudo roughly B-invex with respect to same η with roughness degree r_I at x^*. Then x^* is an efficient solution for problem (P).

Proof. The proof is similar to the proof of Theorem (8).

Remark 3

Similarly as in Theorem (7), it can be easily seen that x^* becomes properly efficient solution for (P), in the above theorem, if $w_j > 0$, for all $j = 1, 2, \cdots, k$.

Theorem 11 (Necessary Optimality Criteria)

Assume that x^* is a properly efficient solution for problem (P). Assume also that there exist a feasible point $\tilde{x}$ for (P) such that $g_i(\tilde{x}) < 0,\ i = 1, 2, \cdots, m$, and each $g_i,\ i \in I(x^*)$ is roughly B-invex with respect to η: $M \times M \to R^n$ with roughness degree r_i at x^*. Then, there exists scalars $w_j > 0,\ j = 1, 2, \cdots, k$ and $u_i \geq 0, i \in I(x^*)$, such that the triplet (x^*, w_j, u_i) satisfies

$$\sum_{j=1}^{k} w_j \nabla f_j(x^*) + \sum_{i \in I(x^*)} u_i \nabla g_i(x^*) = 0. \tag{4}$$

Proof. Let the following system

$$\begin{aligned} &\eta(x, x^*)^t \nabla f_q(x^*) < 0, \\ &\eta(x, x^*)^t \nabla f_j(x^*) \leq 0, \text{for all } j \neq q. \\ &\eta(x, x^*)^t \nabla g_i(x^*) \leq 0,\ i \in I(x^*) \end{aligned} \tag{5}$$

has a solution for every $q = 1, 2, \cdots, k$. Since by the assumed Slater-type condition,

$$g_i(\tilde{x}) - g_i(x^*) < 0, i \in I(x^*),$$

and then from roughly B_i-invexity of g_i at x^* with respect to η, there exists a function $b(\tilde{x}, x^*)$ such that

$$b(\tilde{x}, x^*)\eta(\tilde{x}, x^*)^t \nabla g_i(x^*) < 0, i \in I(x^*). \tag{6}$$

Therefore from (5) and (6)

$$\left[\eta(x, x^*) + \rho b(\tilde{x}, x^*)\eta(\tilde{x}, x^*)\right]^t \nabla g_i(x^*) < 0, i \in I(x^*),$$

for all $\rho > 0$. Hence for some positive λ small enough

$$g_i\left(x^* + \lambda\left[\eta(x, x^*) + \rho b(\tilde{x}, x^*)\eta(\tilde{x}, x^*)\right]\right) \leq g_i(x^*) = 0,$$

$i \in I(x^*)$.

Similarly, for $i \notin I(x^*), g_i(x^*) < 0$, and for $\lambda > 0$ small enough

$$g_i\left(x^* + \lambda\left[\eta(x, x^*) + \rho b(\tilde{x}, x^*)\eta(\tilde{x}, x^*)\right]\right) \leq 0, i \notin I(x^*).$$

Thus, for λ sufficiently small and all

$$\rho > 0, x^* + \lambda\left[\eta(x, x^*) + \rho b(\tilde{x}, x^*)\eta(\tilde{x}, x^*)\right]$$

is feasible for problem (P). For sufficiently small $\rho > 0$, (5) gives

$$f_q\left(x^* + \lambda\left[\eta(x, x^*) + \rho b(\tilde{x}, x^*)\eta(\tilde{x}, x^*)\right]\right) < f_q(x^*). \tag{7}$$

Now, for all $j \neq q$ such that

$$f_j\left(x^* + \lambda\left[\eta(x, x^*) + \rho b(\tilde{x}, x^*)\eta(\tilde{x}, x^*)\right]\right) > f_j(x^*) \tag{8}$$

Consider the ratio (see Equation (9))

From (5), $N(\lambda, \rho) \to -(x - x^*)^t \nabla f_q(x^*) > 0$. Similarly, $D(\lambda, \rho) \to (x - x^*)^t \nabla f_j(x^*) \leq 0$; but, by (8) $D(\lambda, \rho) > 0$, so $D(\lambda, \rho) \to 0$.

Thus, the ratio in (9) becomes unbounded, contradicting the proper efficiency of x^* for (P). Hence, for each $q = 1, 2, \cdots, k$, the system (5) has no solution. The result then follows from an application of the Farkas Lemma as in [12], namely

$$\sum_{j=1}^{k} w_j \nabla f_j(x^*) + \sum_{i \in I(x^*)} u_i \nabla g_i(x^*) = 0.$$

Theorem 12

Assume that x^* is an efficient solution for problem (P) at which the Kuhn-Tucker constraint qualification is satisfied. Then, there exist scalars

$$w_j \geq 0, j = 1, 2, \cdots, k, \sum_{j=1}^{k} w_j = 1, u_i \geq 0, i = 1, 2, \cdots, m,$$

such that

$$\sum_{j=1}^{k} w_j \nabla f_j(x^*) + \sum_{i=1}^{m} u_i \nabla g_i(x^*) = 0, \sum_{i=1}^{m} u_i g_i(x^*) = 0.$$

Proof.

Since every efficient solution is a weak minimum, then by applying Theorem (2.2) of Weir and Mond [13] for x^*, we get, there exists $w \in R^k$, $u \in R^m$ such that

$$w^t \nabla f(x^*) + u^t \nabla g(x^*) = 0, u^t g(x^*) = 0,$$

$$u \geq 0, w \geq 0, w^t e = 0, \text{ where } e = (1, 1, \cdots, 1) \in R^k.$$

$$\frac{N(\lambda, \rho)}{D(\lambda, \rho)} = \frac{\left[f_q(x^*) - f_q\left(x^* + \lambda\left[\eta(x, x^*) + \rho b(\tilde{x}, x^*)\eta(\tilde{x}, x^*)\right]\right)\right]\Big/\lambda}{\left[f_j\left(x^* + \lambda\left[\eta(x, x^*) + \rho b(\tilde{x}, x^*)\eta(\tilde{x}, x^*)\right]\right) - f_j(x^*)\right]\Big/\lambda}. \tag{9}$$

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