The Number of Maximal Independent Sets in Quasi-Tree Graphs and Quasi-Forest Graphs

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Abstract

A maximal independent set is an independent set that is not a proper subset of any other independent set. A connected graph (respectively, graph) $G$ with vertex set $V(G)$ is called a quasi-tree graph (respectively, quasi-forest graph), if there exists a vertex $x \in V(G)$ such that $G - x$ is a tree (respectively, forest). In this paper, we survey on the large numbers of maximal independent sets among all trees, forests, quasi-trees and quasi-forests. In addition, we further look into the problem of determining the third largest number of maximal independent sets among all quasi-trees and quasi-forests. Extremal graphs achieving these values are also given.

Keywords

Maximal Independent Set, Quasi-Tree Graph, Quasi-Forest Graph, Extremal Graph

1. Introduction and Preliminary

Let $G = (V, E)$ be a simple undirected graph. An independent set is a subset $S$ of $V$ such that no two vertices in $S$ are adjacent. A maximal independent set is an independent set that is not a proper subset of any other independent set. The set of all maximal independent sets of a graph $G$ is denoted by $\text{MI}(G)$ and its cardinality by $\text{mi}(G)$.

The problem of determining the largest value of $\text{mi}(G)$ in a general graph of order $n$ and those graphs achieving the largest number was proposed by Erdős and Moser, and solved by Moon and Moser [1]. It was then studied for various families of graphs, including trees, forests, (connected) graphs with at most one cycle, (connected) triangle-free graphs, ($k$-)connected graphs, bipartite graphs; for a survey see [2]. Jin and Li [3] investigated the second largest number of $\text{mi}(G)$ among all graphs of order $n$; Jou and Lin [4] further explored the same problem for trees and forests; Jin and Yan [5] solved the third largest number of...
Among all trees of order \( n \). A connected graph (respectively, graph) \( G \) with vertex set \( V(G) \) is called a quasi-tree graph (respectively, quasi-forest graph), if there exists a vertex \( x \in V(G) \) such that \( G - x \) is a tree (respectively, forest). The concept of quasi-tree graphs was mentioned by Liu and Lu in [6]. Recently, the problem of determining the largest and the second largest numbers of \( mi(G) \) among all quasi-tree graphs and quasi-forest graphs of order \( n \) was solved by Lin [7] [8].

In this paper, we survey on the large numbers of maximal independent sets among all trees, forests, quasi-trees and quasi-forests. In addition, we further look into the problem of determining the third largest number of maximal independent sets among all quasi-trees and quasi-forests. Extremal graphs achieving these values are also given.

For a graph \( G = (V,E) \), the neighborhood \( N_G(x) \) of a vertex \( x \) is the set of vertices adjacent to \( x \) in \( G \) and the closed neighborhood \( N_G[x] \) is \( \{x\} \cup N_G(x) \). The degree of \( x \) is the cardinality of \( N_G(x) \), denoted by \( \deg_G(x) \). For a set \( A \subseteq V(G) \), the deletion of \( A \) from \( G \) is the graph \( G - A \) obtained from \( G \) by removing all vertices in \( A \) and their incident edges. Two graphs \( G_1 \) and \( G_2 \) are disjoint if \( V(G_1) \cap V(G_2) = \emptyset \). The union of two disjoint graphs \( G_1 \) and \( G_2 \) is the graph \( G_1 \cup G_2 \) with vertex set \( V(G_1 \cup G_2) = V(G_1) \cup V(G_2) \) and edge set \( E(G_1 \cup G_2) = E(G_1) \cup E(G_2) \).

\( nG \) is the short notation for the union of \( n \) copies of disjoint graphs isomorphic to \( G \). Denote by \( C_n \) a cycle with \( n \) vertices and \( P_n \) a path with \( n \) vertices.

Throughout this paper, for simplicity, let \( r = \sqrt{2} \).

**Lemma 1.1** ([9]) For any vertex \( x \) in a graph \( G \), \( mi(G) \leq mi(G-x) + mi(G-N_G[x]) \).

**Lemma 1.2** ([10]) If \( G \) is the union of two disjoint graphs \( G_1 \) and \( G_2 \), then \( \mi(G) = \mi(G_1)\mi(G_2) \).

### 2. Survey on the Large Numbers of Maximal Independent Sets

In this section, we survey on the large numbers of maximal independent sets among all trees, forests, quasi-trees and quasi-forests. The results of the largest numbers of maximal independent sets among all trees and forests are described in Theorems 2.1 and 2.2, respectively.

**Theorem 2.1** ([10] [11]) If \( T \) is a tree with \( n \geq 1 \) vertices, then \( \mi(T) \leq t_i(n) \), where

\[
t_i(n) = \begin{cases} r^{n-2} + 1, & \text{if } n \text{ is even}, \\ r^{n-1}, & \text{if } n \text{ is odd}. \end{cases}
\]

Furthermore, \( \mi(T) = t_i(n) \) if and only if \( T \in T_i(n) \), where

\[
T_i(n) = \begin{cases} B \left( 2, \frac{n-2}{2} \right) \text{ or } B \left( 4, \frac{n-4}{2} \right), & \text{if } n \text{ is even}, \\ B \left( 1, \frac{n-1}{2} \right), & \text{if } n \text{ is odd}, \end{cases}
\]
where \( B(i, j) \) is the set of batons, which are the graphs obtained from the basic path \( P \) of \( i \geq 1 \) vertices by attaching \( j \geq 0 \) paths of length two to the end-points of \( P \) in all possible ways (see Figure 1).

**Theorem 2.2** ([10] [11]) If \( F \) is a forest with \( n \geq 1 \) vertices, then \( \text{mi}(F) \leq f_1(n) \), where

\[
f_1(n) = \begin{cases} r^n, & \text{if } n \text{ is even}, \\ r^{n-1}, & \text{if } n \text{ is odd}. \end{cases}
\]

Furthermore, \( \text{mi}(F) = f_1(n) \) if and only if \( F \in F_1(n) \), where

\[
F_1(n) = \begin{cases} \frac{n}{2} P_2, & \text{if } n \text{ is even}, \\ B \left( 1 + \frac{n - 1 - 2s}{2} \right) \cup sP_2 \text{ for some } s \text{ with } 0 \leq s \leq \frac{n - 1}{2}, & \text{if } n \text{ is odd}. \end{cases}
\]

The results of the second largest numbers of maximal independent sets among all trees and forests are described in Theorems 2.3 and 2.4, respectively.

**Theorem 2.3** ([4]) If \( T \) is a tree with \( n \geq 4 \) vertices having \( T \not\in T_1(n) \), then \( \text{mi}(T) \leq t_2(n) \), where

\[
t_2(n) = \begin{cases} r^{n-2}, & \text{if } n \geq 4 \text{ is even}, \\ 3, & \text{if } n = 5, \\ 3r^{n-3} + 1, & \text{if } n \geq 7 \text{ is odd}. \end{cases}
\]

Furthermore, \( \text{mi}(T) = t_2(n) \) if and only if \( T = T_2'(8), T_2'(8), P_{10} \) or \( T \in T_2(n) \), where \( T_2(n) \) and \( T_2'(8) \) are shown in Figure 2 and Figure 3, respectively.

**Theorem 2.4** ([4]) If \( F \) is a forest with \( n \geq 4 \) vertices having \( F \not\in F_1(n) \), then \( \text{mi}(F) \leq f_2(n) \), where

\[
\text{Figure 1. The baton } B(i, j) \text{ with } j = j_1 + j_2.
\]

\[
\text{Figure 2. The trees } T_2(n).
\]

\[
\text{Figure 3. The trees } T_2'(8) \text{ and } T_2''(8).
\]
The results of the third largest numbers of maximal independent sets among all trees and forests are described in Theorems 2.5 and 2.6, respectively.

**Theorem 2.5** ([15]) If $T$ is a tree with $n \geq 7$ vertices having $T \not\in T_i(n)$, $i = 1, 2$, then $mi(T) \leq t_5(n)$, where

$$t_5(n) = \begin{cases} 3r^{n-5}, & \text{if } n \geq 7 \text{ is odd}, \\ 7r^{n-7}, & \text{if } n \geq 7 \text{ is even}. \end{cases}$$

Furthermore, $mi(T) = t_5(n)$ if and only if $T \in T_5(n)$, where $T_5(n)$ are shown in Figure 4 and Figure 5, respectively.

**Theorem 2.6** ([12]) If $F$ is a forest with $n \geq 8$ vertices having $F \not\in F_i(n)$, $i = 1, 2$, then $mi(F) \leq f_5(n)$, where

$$f_5(n) = \begin{cases} 3r^{n-4}, & \text{if } n \geq 4 \text{ is even}, \\ 3, & \text{if } n = 5, \\ 7r^{n-7}, & \text{if } n \geq 7 \text{ is odd}. \end{cases}$$

Furthermore, $mi(F) = f_5(n)$ if and only if $F \in F_5(n)$, where $F_5(n)$ are shown in Figure 4 and Figure 5, respectively.

**Figure 4.** The trees $T_5(8), T_5(10), T_5'(10)$.

**Figure 5.** The trees $T_5(10), T_5'(10)$.
The results of the largest numbers of maximal independent sets among all quasi-tree graphs and quasi-forest graphs are described in Theorems 2.7 and 2.8, respectively.

**Theorem 2.7** ([7]) If $Q$ is a quasi-tree graph with $n \geq 5$ vertices, then $\text{mi}(Q) \leq q_1(n)$, where

$$ q_1(n) = \begin{cases} 3r^{n-4} & \text{if } n \text{ is even}, \\ r^{n-1} + 1 & \text{if } n \text{ is odd}. \end{cases} $$

Furthermore, $\text{mi}(Q) = q_1(n)$ if and only if $Q = C_5$ or $Q \in Q_1(n)$, where $Q_1(n)$ is shown in **Figure 6**.

**Theorem 2.8** ([7]) If $Q$ is a quasi-forest graph with $n \geq 2$ vertices, then $\text{mi}(Q) \leq \overline{q}_1(n)$, where

$$ \overline{q}_1(n) = \begin{cases} r^n, & \text{if } n \text{ is even}, \\ 3r^{n-3}, & \text{if } n \text{ is odd}. \end{cases} $$

Furthermore, $\text{mi}(Q) = \overline{q}_1(n)$ if and only if $Q \in \overline{Q}_1(n)$, where

$$ \overline{Q}_1(n) = \begin{cases} n \frac{P_2}{2}, & \text{if } n \text{ is even}, \\ C_3 \cup \frac{n-3}{2} P_2, & \text{if } n \text{ is odd}. \end{cases} $$

The results of the second largest numbers of maximal independent sets among all quasi-tree graphs and quasi-forest graphs are described in Theorems 2.9 and 2.10, respectively.

**Theorem 2.9** ([8]) If $Q$ is a quasi-tree graph with $n \geq 6$ vertices having $Q \notin Q_1(n)$, then $\text{mi}(Q) \leq q_2(n)$, where

$$ q_2(n) = \begin{cases} 5r^{n-6} + 1 & \text{if } n \text{ is even}, \\ r^{n-3}, & \text{if } n \text{ is odd}. \end{cases} $$

Furthermore, $\text{mi}(Q) = q_2(n)$ if and only if $Q \in Q_2(n)$, where

$$ Q_2(n) = \begin{cases} Q_1(n), Q_2^{(2)}(n), Q_3^{(2)}(n), Q_4^{(2)}(n), & \text{if } n \text{ is even}, \\ B\left(1, \frac{n-1}{2}\right), Q_2^{(3)}(7), Q_2^{(3)}(7), Q_2^{(4)}(7), & \text{if } n \text{ is odd}, \end{cases} $$

where $Q_1(n)$ is shown in **Figure 7** and **Figure 8**.

**Theorem 2.10** ([8]) If $Q$ is a quasi-forest graph with $n \geq 4$ vertices having $Q \notin \overline{Q}_1(n)$, then $\text{mi}(Q) \leq \overline{q}_2(n)$, where

$$ \overline{q}_2(n) = \begin{cases} 3r^{n-4}, & \text{if } n \text{ is even}, \\ 5r^{n-5}, & \text{if } n \text{ is odd}. \end{cases} $$

**Figure 6.** The graph $Q_1(n)$. 
Figure 7. The graphs $Q_{2i}^{(i)}(n)$, $1 \leq i \leq 4$.

Figure 8. The graphs $Q_{2i_o}^{(i)}(7)$, $1 \leq i \leq 4$.

Furthermore, $mi(Q) = \overline{u}_2(n)$ if and only if $Q \in \overline{Q}_2(n)$, where

$$
\overline{Q}_2(n) = \begin{cases} 
\overline{P}_2 \cup \frac{n-4}{2} \overline{P}_2, & \text{if } n \text{ is even}, \\
\overline{Q}_2(6) \cup \frac{n-6}{2} \overline{P}_2, C_3 \cup B \left( \frac{n-4-2s}{2} \right) \cup sP_2, & \text{if } n \text{ is odd}, \\
\overline{Q}_2(5) \cup \frac{n-5}{2} \overline{P}_2, W \cup \frac{n-5}{2} \overline{P}_2, C_3 \cup \frac{n-5}{2} \overline{P}_2, & \text{if } n \text{ is odd},
\end{cases}
$$

where $W$ is a bow, that is, two triangles $C_3$ having one common vertex.

A graph is said to be unicyclic if it contains exactly one cycle. The result of the second largest number of maximal independent sets among all connected unicyclic graphs are described in Theorems 2.11.

**Theorem 2.11** ([13]) If $U$ is a connected unicyclic graph of order $n \geq 6$ with $U \neq C_5$ and $Q \notin Q(n)$, then $mi(G) \leq u_z(n)$, where

$$
u_z(n) = \begin{cases} 
5r^{n-6} + 1, & \text{if } n \text{ is even}, \\
3r^{n-5} + 2, & \text{if } n \text{ is odd}.
\end{cases}
$$

Furthermore, $mi(G) = \overline{u}_2(n)$ if and only if $U \in U_2(n)$, where

$$U_2(n) = \begin{cases} 
Q_{2e}^{(i)}(n), & \text{if } n \text{ is even}, \\
U_{2o}^{(i)}(n), U_{2o}^{(2)}(n), U_{2o}^{(3)}(n), U_{2o}^{(4)}(n), U_{2o}^{(5)}(n), U_{2o}^{(6)}(n), & \text{if } n \text{ is odd},
\end{cases}
$$

where $U_{2o}^{(i)}(n)$ is shown in Figure 9.

**3. Main Results**

In this section, we determine the third largest values of $mi(G)$ among all quasi-tree graphs and quasi-forest graphs of order $n \geq 7$, respectively. Moreover, the extremal graphs achieving these values are also determined.

**Theorem 3.1** If $Q$ is a quasi-tree graph of odd order $n \geq 7$ having $Q \notin Q_1(n), Q_2(n)$, then $mi(Q) \leq 3r^{n-5} + 2$. Furthermore, the equality holds if and only if $Q = U_{2o}^{(i)}$, $1 \leq i \leq 6$, where $U_{2o}^{(i)}(n)$ is shown in Figure 9.

**Proof:** It is straightforward to check that $mi(U_{2o}^{(i)}(n)) = 3r^{n-5} + 2$, $1 \leq i \leq 6$. Let $Q$ be a quasi-tree graph of odd order $n \geq 7$ having $Q \notin Q_1(n), Q_2(n)$ such that $mi(Q)$ is as large as possible. Then $mi(Q) \geq 3r^{n-5} + 2$. If $Q$ is a tree, by Theorems 2.1, 2.3 and $Q \notin Q_2(n)$, we have that
$3r^{\alpha-5} + 2 \leq mi(Q) \leq t_3(n) = 3r^{\alpha-5} + 1$. This is a contradiction.

Suppose that $Q$ contains at least two cycles and $x$ is the vertex such that $Q - x$ is a tree. Then $\deg_Q(x) \geq 3$. By Lemma 1.1, Theorems 2.1 and 2.2,

$$3r^{\alpha-5} + 2 \leq mi(Q) \leq mi(Q - x) + mi(Q - N_Q[x]) \leq r^{(\alpha-1)-1} + r^{(\alpha-3)-1} = 3r^{\alpha-5} + 1$$

which is a contradiction. We obtain that $Q$ is a connected unicyclic graph, thus the result follows from Theorem 2.11.

**Theorem 3.2** If $Q$ is a quasi-tree graph of even order $n \geq 8$ having $Q \not\in Q_2(n), Q_3(n)$, then $mi(Q) \leq 5r^{\alpha-6}$. Furthermore, the equality holds if and only if $Q = \mathcal{Q}^*(8)$, $\mathcal{Q}^*(10)$, $Q_{10}^{(i)}(n)$, $1 \leq i \leq 12$, where $\mathcal{Q}^*(8)$, $\mathcal{Q}^*(10)$ and $Q_{10}^{(i)}(n)$ are shown in Figure 10.

**Proof.** It is straightforward to check that $mi(\mathcal{Q}^*(8)) = mi(\mathcal{Q}^*(8)) = 10$, $mi(\mathcal{Q}^*(10)) = 20$ and $mi(Q_{10}^{(i)}(n)) = 5r^{\alpha-6}$, $1 \leq i \leq 12$. Let $Q$ be a quasi-tree graph of even order $n \geq 8$ having $Q \not\in Q_2(n), Q_3(n)$ such that $mi(Q)$ is as large as possible. Then $mi(Q) \geq 5r^{\alpha-6}$. If $Q$ is a tree, by Theorem 2.1, we have that $5r^{\alpha-6} \leq mi(Q) \leq t_3(n) = r^{\alpha-2} + 1$. This is a contradiction, so $Q$ contains at least one cycle. Let $x$ be the vertex such that $Q - x$ is a tree. Then $x$ is on some cycle of $Q$, it follows that $\deg_Q(x) \geq 2$. In addition, by Lemma 1.1, Theorems 2.1 and 2.2, $mi(Q - x) \leq 5r^{\alpha-6} - r^{(\alpha-3)-1} = 3r^{\alpha-6} = t_3(n-1)$. We consider the following three cases.

Case 1. $Q - x \in T(n-1)$. If $\deg_Q(x) \geq 6$ then $Q - N_Q[x]$ is a forest with at most $n - 7$ vertices, by Lemma 1.1, Theorems 2.1 and 2.2,

$$5r^{\alpha-6} \leq mi(Q) \leq mi(Q - x) + mi(Q - N_Q[x]) \leq r^{(\alpha-1)-1} + r^{(\alpha-3)-1} = 9r^{\alpha-8}.$$ 

This is a contradiction. So we assume that $2 \leq \deg_Q(x) \leq 5$.

- $\deg x = 2$. There are 6 possibilities for graph $Q$. See Figure 11. Note that $Q_1 = Q_1(n)$. By simple calculation, we have that $mi(Q_1) \leq r^{\alpha-2} + 1$ for $2 \leq i \leq 6$, a contradiction to $mi(Q) \geq 5r^{\alpha-6}$.

- $\deg x = 3$. Suppose that there exists an isolated vertex $y$ in $Q - N_Q[x]$ and $\mathcal{Q}^*_x = \mathcal{Q}^*_x(n)$, then

$$mi(Q) \leq mi(Q - x) + mi(Q - N_Q[x]) < r^{(\alpha-1)-1} + r^{(\alpha-4)-1} = 5r^{\alpha-6}.$$ 

Hence there are 4 possibilities for graph $Q$. See Figure 12.

Note that $Q_6 = Q_6^2(n)$, $Q_6 = Q_6^{(7)}(n)$ and $Q_{10} = Q_{10}^{(i)}(n)$. By simple calculation, we have $mi(Q_6) = r^{\alpha-2} + 1$, a contradiction to $mi(Q) \geq 5r^{\alpha-6}$.

- $4 \leq \deg x \leq 5$. Since $Q - N_Q[x]$ is a forest of odd order $n - 5$ or even or-

![Figure 9. The graphs $U_{2o}^{(i)}(n)$, $1 \leq i \leq 6$.](image)
under $n - 6$, by Lemma 1.1, Theorems 2.1 and 2.2, we have $5r^{n-6} \leq mi(Q) \leq mi(Q - x) + mi(Q - N_Q[x]) \leq r^{n-2} + r^{n-6} = 5r^{n-6}$. The equalities holding imply that $Q - x = T_i(n-1)$ and $Q - N_Q[x] = F_i(n-5)$ or $F_i(n-6)$. Hence we obtain that $Q = Q_{Q_1}^{(i)}(n), 1 \leq i \leq 4$.

Case 2. $Q - x \in T_i(n-1)$. If $\deg_Q(x) \geq 4$ then $Q - N_Q[x]$ is a forest with at most $n - 5$ vertices, by Lemma 1.1, Theorems 2.2 and 2.3, we have that $5r^{n-6} \leq mi(Q) \leq mi(Q - x) + mi(Q - N_Q[x]) \leq 3r^{(n-1)-5} + 1 + r^{(n-5)-1} = 4r^{n-6} + 1$. This is a contradiction. So we assume that $2 \leq \deg_Q(x) \leq 3$.

- $\deg x = 2$. Suppose that $Q - N_Q[x] \notin F_i(n-3)$, by Lemma 1.1, Theorems 2.3 and 2.4, we have that

**Figure 10.** The graphs $Q'(8)$, $Q''(8)$, $Q'''(10)$ and $Q_{Q_1}^{(i)}(n), 1 \leq i \leq 12$.

**Figure 11.** The graphs $Q'_i(n), 1 \leq i \leq 6$.

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Figure 12. The graphs $Q^*_i, \ 7 \leq i \leq 10$.

$5r^{n-6} \leq mi(Q) \leq mi(Q - x) + mi(Q - N_Q[x]) \leq 3r^{(n-1)-5} + 1 + 7r^{(n-3)-7} = 19r^{n-10} + 1$.

The equalities holding imply that $n = 10$, that is, $Q - x = T_i(9)$ and $Q - N_Q[x] = F_i(7)$. Hence we obtain that $Q = Q^{*}(10)$. Now we assume that $Q - N_Q[x] \in F_i(n-3)$. There are 7 possibilities for graph $Q$. See Figure 13.

Note that $Q^*_{11} = Q^{(1)}_{13}(n), \ Q^*_{12} = Q^{(5)}_{13}(n)$ and $Q^*_{13} = Q^{(6)}_{13}(n)$. By simple calculation, we have $mi(Q^*_i) \leq r^{n-2} + 2$ for $14 \leq i \leq 17$, a contradiction to $mi(Q) \geq 5r^{n-6}$ when $n \geq 10$. In addition, $r^{n-2} + 2 = mi(Q^*_7) = 5r^{n-6}$ when $n = 8$, it follows that $Q = Q^{(6)}_{13}(8)$.

In the following, we will investigate the same problem for quasi-forest graphs.

Theorem 3.3 If $Q$ is a quasi-forest graph of odd order $n \geq 7$ having
$Q \notin \overline{Q}_2(n), \overline{Q}_3(n)$, then $mi(Q) \leq 9r^{n-7}$. Furthermore, the equality holds if and only if $Q = \overline{Q}_i^{(j)}(n)$, $1 \leq i \leq 4$, where $\overline{Q}_i^{(j)}(n)$ is shown in Figure 15.

Figure 13. The graphs $Q_i^*$, $11 \leq i \leq 17$.

Figure 14. The graphs $Q_i^*$, $18 \leq i \leq 24$.

Figure 15. The graphs $\overline{Q}_i^{(j)}(n)$, $1 \leq i \leq 4$. 
Proof. It is straightforward to check that \( mi(\overline{Q}_i(n)) = 9r^{n-7}, \) \( 1 \leq i \leq 4. \) Let \( Q \) be a quasi-forest graph of odd order \( n \geq 7 \) having \( Q \notin \overline{Q}_i(n), \overline{Q}_2(n) \) such that \( mi(Q) \) is as large as possible. Then \( mi(Q) \geq 9r^{n-7}. \) If \( Q \) is a forest, by Theorem 2.2, we have that \( 9r^{n-7} \leq mi(Q) \leq f_1(n) = r^{n-7}. \) This is a contradiction, so \( Q \) contains at least one cycle. Let \( x \) be a vertex such that \( Q - x \) is a forest. Then \( x \) is on some cycle of \( Q \), it follows that \( deg_Q(x) \geq 2 \) and \( Q - N_Q[x] \) is a forest with at most \( n - 3 \) vertices. By Lemma 1.1, Theorem 2.2 and 2.6, we obtain that \( mi(Q - x) \geq mi(Q) - mi(Q - N_Q[x]) \geq 9r^{n-7} - r^{n-3} = 5r^{n-7} = f_3(n - 1). \)

We consider the following three cases.

Case 1. \( Q - x \notin F_1(n - 1). \) If \( deg_Q(x) \geq 7 \) then \( Q - N_Q[x] \) is a forest with at most \( n - 8 \) vertices, by Lemma 1.1 and Theorem 2.2, we have that \( 9r^{n-7} \leq mi(Q) \leq mi(Q - x) + mi(Q - N_Q[x]) \leq r^{n-7} + \rho(x; n)^{x-1} = 17r^{n-9}. \) This is a contradiction. So we assume that \( 2 \leq deg_Q(x) \leq 6. \) There are 9 possibilities for graph \( Q. \) See Figure 16.

Note that \( \overline{Q}_1 = \overline{Q}_1(n), \overline{Q}_2 = \overline{Q}_2(n), \overline{Q}_3 = \overline{Q}_3(n), \overline{Q}_4 = \overline{Q}_4(n), \overline{Q}_5 = \overline{Q}_5(n), \overline{Q}_6 = \overline{Q}_6(n). \) By simple calculation, we have \( mi(\overline{Q}_i) \leq 17r^{n-7}, \) \( i = 6, 8, 9, \) a contradiction to \( mi(Q) \geq 9r^{n-7}. \)

Case 2. \( Q - x = F_2(n - 1). \) If \( deg_Q(x) \geq 3 \) then \( Q - N_Q[x] \) is a forest with at most \( n - 4 \) vertices, by Lemma 1.1, Theorems 2.2 and 2.4, we have that \( 9r^{n-7} \leq mi(Q) \leq mi(Q - x) + mi(Q - N_Q[x]) \leq 3r^{n-4} + \rho(n; n)^{x-1} = 4r^{n-5}. \) This is a contradiction. So we assume that \( deg_Q(x) = 2. \) There are 5 possibilities for graph \( Q. \) See Figure 17.

Note that \( \overline{Q}_0 = \overline{Q}_1(n), \overline{Q}_2 = \overline{Q}_3(n), \overline{Q}_4 = \overline{Q}_4(n). \) By simple calculation, we have \( mi(\overline{Q}_i) \leq 3r^{n-5} + 1, \) \( i = 1, 11, 13, \) a contradiction to \( mi(Q) \geq 9r^{n-7}. \)

Case 3. \( Q - x \in F_3(n - 1). \) Since \( Q - N_Q[x] \) is a forest with at most \( n - 3 \) vertices, by Lemma 1.1, Theorems 2.2 and 2.6, we have that \( 9r^{n-7} \leq mi(Q) \leq mi(Q - x) + mi(Q - N_Q[x]) \leq 5r^{n-6} + r^{n-3} = 9r^{n-7}. \) The equali-

\[ \omega \]

**Figure 16.** The graphs \( \overline{Q}_i, 1 \leq i \leq 9. \)
ties holding imply that $Q-x \in F_i(n-1)$ and $Q-N_Q[x] \in F_i(n-3)$. There are 3 possibilities for graph $Q$. See Figure 18.

Note that $\overline{Q}_i = \overline{Q}_{3+i}(n)$. By simple calculation, we have $mi(\overline{Q}) = 8r^{n-7}$, $15 \leq i \leq 16$, a contradiction to $mi(Q) \geq 9r^{n-7}$.

**Theorem 3.4** If $Q$ is a quasi-forest graph of even order $n \geq 8$ having $Q \notin \overline{Q}_i(n), \overline{Q}_j(n)$, then $mi(Q) \leq 11r^{n-8}$. Furthermore, the equality holds if and only if $Q = Q_2(8) \cup \frac{n-8}{2}P_2$.

**Proof.** It is straightforward to check that $mi(Q_2(8) \cup \frac{n-8}{2}P_2) = 11r^{n-8}$. Let $Q$ be a quasi-forest graph of even order $n \geq 8$ having $Q \notin \overline{Q}_i(n), \overline{Q}_j(n)$ such that $mi(Q)$ is as large as possible. Then $mi(Q) \geq 11r^{n-8}$. If $Q$ is a forest, by Theorems 2.2, 2.4, 2.6, 2.8 and 2.10, we have that $11r^{n-8} \leq mi(Q) \leq f_3(n) = 5r^{n-6}$. This is a contradiction, so $Q$ contains a component $\hat{Q}$ with at least one cycle.

Let $|\hat{Q}| = s$. Suppose that $Q-\hat{Q} \notin \frac{n-S}{2}P_2$. Since $\hat{Q}$ is not a tree and $Q \notin \overline{Q}_i(n), \overline{Q}_j(n)$, by Lemma 1.2, Theorems 2.2, 2.4 and 2.7, we have that

$$mi(Q) = mi(\hat{Q}) \cdot mi(Q-\hat{Q})$$

$$\leq \begin{cases} 3r^{s-4} \cdot 3r^{(s-1)-4}, & \text{if } s \geq 4 \text{ is even}, \\ 3 \cdot 7r^{(s-3)-7}, & \text{if } s = 3, \\ (r^{s-1} + 1) \cdot r^{(s-3)-1}, & \text{if } s \geq 5 \text{ is odd}, \\ 9r^{s-8}, & \text{if } s \geq 4 \text{ is even}, \\ 21r^{n-10}, & \text{if } s = 3, \\ 5r^{n-6}, & \text{if } s \geq 5 \text{ is odd}, \\ <11r^{n-8}, & \end{cases}$$

![Figure 17](image17.png)

*Figure 17.* The graphs $\overline{Q}_i$, $10 \leq i \leq 14$.

![Figure 18](image18.png)

*Figure 18.* The graphs $\overline{Q}_i$, $15 \leq i \leq 17$. 
which is a contradiction. Hence we obtain that \( s \) is even and \( Q - \hat{Q} = \frac{n - s}{2} P_2 \).

Let \( x \) be the vertex in \( \hat{Q} \) such that \( \hat{Q} - x \) is a forest and \( w(\hat{Q} - x) \) be the number of components of \( \hat{Q} - x \). We consider the following two cases.

Case 1. \( w(\hat{Q} - x) = 1 \). Then \( \hat{Q} \) is a quasi-tree graph. Since \( Q \not\in \overline{Q}_i(n), \overline{Q}_i(n) \) it follows that \( s \geq 8 \). By Lemma 1.2 and Theorem 2.9, it follows that \( mi(Q) = (5r^{n-6} + 1) \cdot r^{n-6} = 5r^{n-6} + r^{n-8} \leq 11r^{n-8} \). The equality holding imply that \( s = 8 \). In conclusion, \( Q = Q_2(8) \cup \frac{n - 8}{2} P_2 \).

Case 2. \( w(\hat{Q} - x) \geq 2 \). Then \( \deg x \geq 3 \). In addition, suppose that \( Q - N_{Q}[x] \) has a isolated vertex or \( \deg_{Q}(x) \geq 4 \), by Lemma 1.1 and Theorem 2.2, we have that \( 11r^{n-8} \leq mi(Q) \leq mi(Q - x) + mi(Q - N_{\hat{Q}}[x]) \leq r^{(n-1)-1} + r^{(n-5)-1} = 5r^{n-6} \). This is a contradiction, hence, we have that \( \deg_{Q}(x) = 3 \) and \( Q - N_{Q}[x] \) has no isolated vertex. For the case that \( Q - x \not\in F_1(n-1) \), by Lemma 1.1, Theorems 2.2 and 2.4, we have that \( 11r^{n-8} \leq mi(Q) \leq mi(Q - x) + mi(Q - N_{\hat{Q}}[x]) \leq 7r^{(n-1)-7} + r^{n-6} = 11r^{n-8} \). The equalities holding imply that \( Q - x \in \overline{Q}_2(n-1) \) and \( Q - N_{\hat{Q}}[x] \not\in F_1(n-4) \). Since \( w(\hat{Q} - x) \geq 2 \), there no such graph \( Q \). For the other case that \( Q - x \in F_1(n-1) \), there are 2 possibilities for graph \( Q \). See Figure 19.

Note that \( \overline{Q}_{18} = \overline{Q}_2(n) \) and \( \overline{Q}_{19} = Q_2(8) \cup \frac{n - 8}{2} P_2 \) when \( s = 8 \). On the other hand, \( mi(Q_{19}) \leq 21r^{n-10} \) when \( s \geq 10 \), a contradiction to \( mi(Q) \geq 11r^{n-8} \).

References


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