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# An Alternative Proof of the Largest Number of Maximal Independent Sets in Connected Graphs Having at Most Two Cycles

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## Abstract

G. C. Ying, Y. Y. Meng, B. E. Sagan, and V. R. Vatter [1] found the maximum number of maximal independent sets in connected graphs which contain at most two cycles. In this paper, we give an alternative proof to determine the largest number of maximal independent sets among all connected graphs of order $n \geq 12$, which contain at most two cycles. We also characterize the extremal graph achieving this maximum value.

## Keywords

Maximal Independent Set, Connected Graph Having at Most Two Cycles

## 1. Introduction

Let $G=(V,E)$ be a simple undirected graph. An *independent set* is a subset $S$ of $V$ such that no two vertices in $S$ are adjacent. A *maximal independent set* is an independent set that is not a proper subset of any other independent set. The set of all maximal independent sets of a graph $G$ is denoted by $MI(G)$ and its cardinality by $mi(G)$.

The problem of determining the largest value of $mi(G)$ in a general graph of order $n$ and those graphs achieving the largest number was proposed by Erdös and Moser, and solved by Moon and Moser [2]. It was then extensively studied for various classes of graphs in the literature, including trees, forests, (connected) graphs with at most one cycle, bipartite graphs, connected graphs, $k$-connected graphs, (connected) triangle-free graphs; for a survey see [3]. Recently, Jin and Li [4] determined the second largest number of maximal independent sets among all graphs of order $n$.

There are results on independent sets in graphs from a different point of view. The *Fibonacci number* of a graph is the number of independent vertex subsets. The concept of the Fibonacci number of a graph was introduced in [5] and discussed in several papers [6] [7]. In addition, Jou and Chang [8] showed a linear-time algorithm for counting the number of maximal independent sets in a tree.

Jou and Chang [9] determined the largest number of maximal independent sets among all graphs and connected graphs of order n, which contain at most one cycle. Later B. E. Sagan and V. R. Vatter [10] found the largest number of maximal independent sets among all graphs of order n, which contain at most *r* cycles. In 2012, Jou [11] settled the second largest number of maximal independent sets in graphs with at most one cycle. G. C. Ying, Y. Y. Meng, B. E. Sagan, and V. R. Vatter [1] found the maximum number of maximal independent sets in connected graphs which contain at most two cycles. In this paper, we give an alternative proof to determine the largest number of maximal independent sets among all connected graphs of order $n \geq 12$, which contain at most two cycles. We also characterize the extremal graph achieving this maximum value.

For a graph $G=(V,E)$, the cardinality of $V(G)$ is called the *order*, and it is denoted by $|G|$. The *neighborhood* $N(x)$ of a vertex $x \in V(G)$ is the set of vertices adjacent to *x* in *G* and the *closed neighborhood* $N[x]$ is $\{x\} \cup N(x)$. The *degree* of *x* is the cardinality of $N(x)$, and it is denoted by $\deg(x)$. A vertex *x* is said to be a *leaf* if $\deg(x)=1$. For a set $A \subseteq V(G)$, the *deletion* of *A* from *G* is the graph $G-A$ obtained from *G* by removing all vertices in *A* and their incident edges. Two graphs $G_1$ and $G_2$ are *disjoint* if $V(G_1) \cap V(G_2) = \varnothing$. The *union* of two disjoint graphs $G_1$ and $G_2$ is the graph $G_1 \cup G_2$ with vertex set $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and edge set $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$. If a graph *G* is isomorphic to another graph *H*, we denote $G=H$. Denote $K_n$ a complete graph of order *n* and $C_n$ a cycle of order *n*. The *join* of two disjoint graphs $G_1$ and $G_2$ is the graph $G_1+G_2$ with vertex set $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and edge set $E(G_1 \cup G_2) = E(G_1) \cup E(G_2) \cup \{uv : u \in V(G_1) \text{ and } v \in V(G_2)\}$. The *star-product* of two disjoint graphs $G_1$ and $G_2$ is the graph $G_1 \times G_2$ with vertex set $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and edge set $E(G_1 \cup G_2) = E(G_1) \cup E(G_2) \cup \{v_1 v_2\}$, where $v_i$ is a vertex with maximum degree in $G_i$ for $i=1,2$.

## 2. Preliminary

The following results are needed.

**Lemma 1.** ([12] [13]) *For any vertex x in a graph G, the following hold.*

1) $mi(G) \leq mi(G-x) + mi(G-N_G[x])$.

2) If *x* is a leaf adjacent to *y*, then $mi(G) = mi(G-N_G[x]) + mi(G-N_G[y])$.

**Lemma 2.** ([13]) *If G is the union of two disjoint graphs* $G_1$ *and* $G_2$*, then* $mi(G) = mi(G_1)\,mi(G_2)$.

**Lemma 3.** *Let* $a,b,n$ *be integers such that* $0 < a < \frac{n}{2} < b$ *and let* $f(x) = 2^x + 2^{n-x}$. *Then* $f(x) \leq \max\{f(a), f(b)\}$.

*Proof.* The derivative of $f(x)$ is

$$f'(x) = 2^x(\ln 2) - (2^{n-x})(\ln 2) = (\ln 2)\left[2^x - 2^{n-x}\right].$$

So $f'(x) < 0$ for any $x \in \left(a, \frac{n}{2}\right)$ and $f'(x) > 0$ for any $x \in \left(\frac{n}{2}, b\right)$. Then $f(x)$ is decreasing on $\left(a, \frac{n}{2}\right)$ and $f(x)$ is increasing on $\left(\frac{n}{2}, b\right)$. Hence $f(x) \le \max\{f(a), f(b)\}$. □

**Theorem 1.** ([9]) *If T is a tree with* $n \ge 1$ *vertices, then* $mi(G) \le t(n)$, *where*

$$t(n) = \begin{cases} 2^{\frac{n-1}{2}}, & \text{if } n \text{ is odd;} \\ 2^{\frac{n-2}{2}} + 1, & \text{if } n \text{ is even.} \end{cases}$$

*Furthermore,* $mi(T) = t(n)$ *if and only if* $T \in T(n)$, *where*

$$T(n) = \begin{cases} B\left(1, \frac{n-1}{2}\right), & \text{if } n \text{ is odd;} \\ B\left(2, \frac{n-2}{2}\right) \text{ or } B\left(4, \frac{n-4}{2}\right), & \text{if } n \text{ is even,} \end{cases}$$

*where* $B(i, j)$ *is the set of batons, which are the graphs obtained from a basic path P of* $i \ge 1$ *vertices by attaching* $j \ge 0$ *paths of length two to the endpoints of P in all possible ways* (*see* Figure 1).

**Theorem 2.** ([9]) *If F is a forest with* $n \ge 1$ *vertices, then* $mi(G) \le f(n)$, *where*

$$f(n) = \begin{cases} 2^{\frac{n-1}{2}}, & \text{if } n \text{ is odd ;} \\ 2^{\frac{n}{2}}, & \text{if } n \text{ is even.} \end{cases}$$

*Furthermore,* $mi(F) = f(n)$ *if and only if* $F \in F(n)$, *where*

$$F(n) = \begin{cases} B\left(1, \frac{n-1-2s}{2}\right) \cup sP_2, & \text{if } n \text{ is odd ;} \\ \frac{n}{2} P_2, & \text{if } n \text{ is even,} \end{cases}$$

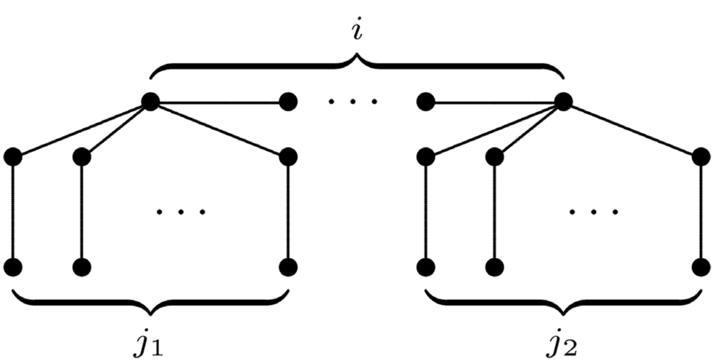


Figure 1. The baton $B(i, j)$ with $j = j_1 + j_2$.

*where* $0 \le s \le \frac{n-1}{2}$.

**Theorem 3.** ([9]) *If G is a graph of order* $n \ge 2$ *vertices with at most one cycle, then* $mi(G) \le g(n,1)$, *where*

$$g(n,1) = \begin{cases} 2^{\frac{n}{2}}, & \text{if } n \text{ is even;} \\ 3 \cdot 2^{\frac{n-3}{2}}, & \text{if } n \text{ is odd.} \end{cases}$$

*Furthermore,* $mi(G) = g(n,1)$ *if and only if* $G = G(n,1)$, *where*

$$G(n,1) = \begin{cases} \frac{n}{2} K_2, & \text{if } n \text{ is even;} \\ K_3 \cup \frac{n-3}{2} K_2, & \text{if } n \text{ is odd.} \end{cases}$$

**Theorem 4.** ([11]) *If G is a graph of order* $n \ge 4$ *with at most one cycle such that* $G \ne G(n,1)$, *then* $mi(G) \le g'(n,1)$, *where*

$$g'(n,1) = \begin{cases} 3 \cdot 2^{\frac{n-4}{2}}, & \text{if } n \ge 4 \text{ is even;} \\ 5 \cdot 2^{\frac{n-5}{2}}, & \text{if } n \ge 5 \text{ is odd.} \end{cases}$$

*Furthermore,* $mi(G) = g'(n,1)$ *if and only if* $G \in G'(n,1)$, *where*

$$G'(n,1) = \begin{cases} P_4 \cup \frac{n-4}{2} P_2, C_3 \cup T(2s-3) \cup \frac{n-2s}{2} P_2 \text{ or } C(2s,1) \cup \frac{n-2s}{2} P_2, & \text{if } n \ge 4 \text{ is even } (s \ge 2); \\ C_5 \cup \frac{n-5}{2} P_2 \text{ or } C(5,1) \cup \frac{n-5}{2} P_2, & \text{if } n \ge 5 \text{ is odd.} \end{cases}$$

**Theorem 5.** ([9]) *If H is a connected graph of order* $n \ge 3$ *with at most one cycle, then* $mi(H) \le h(n,1)$, *where*

$$h(n,1) = \begin{cases} 2^{\frac{n-1}{2}} + 1, & \text{if } n \ge 3 \text{ is odd;} \\ 3 \cdot 2^{\frac{n-4}{2}}, & \text{if } n \ge 4 \text{ is even.} \end{cases}$$

*Furthermore,* $mi(H) = h(n,1)$ *if and only if* $H \in H(n,1)$ (see Figure 2), *where*

$$H(n,1) = \begin{cases} K_3 * \frac{n-3}{2} K_2 \text{ or } C_5, & \text{if } n \ge 3 \text{ is odd;} \\ K_1 * \left( K_3 \cup \frac{n-4}{2} K_2 \right) \text{ or } P_4, & \text{if } n \ge 4 \text{ is even.} \end{cases}$$

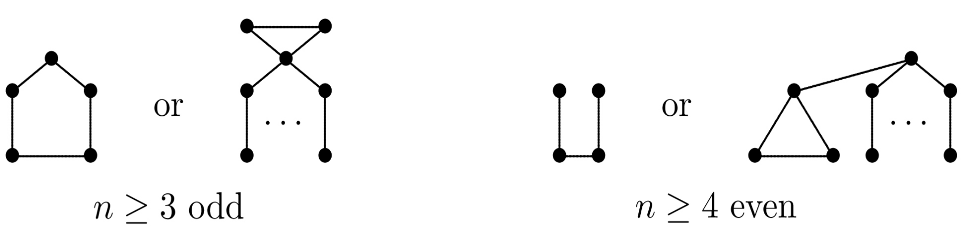


Figure 2. The extremal graphs $H(n,1)$ for $n \ge 3$.

## 3. The Alternative Proof

In this section, we give an alternative proof to determine the largest number of maximal independent sets among all connected graphs of order $n \ge 12$, which contain at most two cycles. We also characterize the extremal graph achieving this maximum value.

**Theorem 6.** *If H is a connected graph of order* $n \ge 12$ *with at most two cycles, then* $mi(H) \le h(n,2)$, *where*

$$h(n,2)=\begin{cases}3\cdot 2^{\frac{n-4}{2}}+2, & \text{if } n\ge 12 \text{ is even;}\\ 9\cdot 2^{\frac{n-7}{2}}, & \text{if } n\ge 13 \text{ is odd.}\end{cases}$$

*Furthermore,* $mi(H)=h(n,2)$ *if and only if* $H \in H(n,2)$, *where*

$$H(n,2)=\begin{cases}K_3\times\left(K_3\cup\frac{n-6}{2}K_2\right), & \text{if } n\ge 12 \text{ is even;}\\ K_1\times\left(2K_3\cup\frac{n-7}{2}K_2\right), & \text{if } n\ge 13 \text{ is odd.}\end{cases}$$

A *unicyclic graph* is a connected graph having one cycle. The order of a unicyclic graph is at least three. The following lemmas will be needed in the proof of main theorem.

**Lemma 4.** *Suppose that* $G=T\cup H$ *is the union of a tree T and a unicyclic graph H, where* $|G|=n\ge 6$. *Then* $mi(G)\le 2\cdot h(n-2,1)$. *The equality holds if and only if* $G=K_2\cup H(n-2,1)$ *or* $G=K_3\cup B\left(1,\frac{n-4}{2}\right)$.

*Proof.* Let $|T|=k$. Note that *H* has one cycle, then $|H|=n-k\ge 3$. We consider two cases.

Case 1. $n\ge 6$ is even.

By Lemma 2, Theorem 1 and Theorem 5, we have

$$\begin{aligned} mi(G) &= mi(T)\cdot mi(H)\le t(k)\cdot h(n-k,1)\\ &=\begin{cases}2^{\frac{k-1}{2}}\cdot\left(2^{\frac{n-k-1}{2}}+1\right), & \text{if } 1\le k\le n-3 \text{ is odd;}\\ \left(2^{\frac{k-2}{2}}+1\right)\cdot\left(3\cdot 2^{\frac{n-k-4}{2}}\right), & \text{if } 2\le k\le n-4 \text{ is even;}\end{cases}\\ &=\begin{cases}2^{\frac{n-2}{2}}+2^{\frac{k-1}{2}}, & \text{if } 1\le k\le n-3 \text{ is odd;}\\ 3\left(2^{\frac{n-6}{2}}+2^{\frac{n-k-4}{2}}\right), & \text{if } 2\le k\le n-4 \text{ is even;}\end{cases}\\ &\le\begin{cases}2^{\frac{n-2}{2}}+2^{\frac{(n-3)-1}{2}}, & \text{if } 1\le k\le n-3 \text{ is odd;}\\ 3\left(2^{\frac{n-6}{2}}+2^{\frac{n-(2)-4}{2}}\right), & \text{if } 2\le k\le n-4 \text{ is even;}\end{cases}\\ &=\begin{cases}3\cdot 2^{\frac{n-4}{2}}, & \text{if } 1\le k\le n-3 \text{ is odd;}\\ 3\cdot 2^{\frac{n-4}{2}}, & \text{if } 2\le k\le n-4 \text{ is even;}\end{cases} =\begin{cases}t(n-3)\cdot 3, & \text{if } 1\le k\le n-3 \text{ is odd;}\\ 2\cdot h(n-2,1), & \text{if } 2\le k\le n-4 \text{ is even;}\end{cases}\\ &=2\cdot h(n-2,1)\end{aligned}$$

If the equality holds, then $mi(G)=t(n-3)\cdot h(3,1)$ or $mi(G)=t(2)\cdot h(n-2,1)$. Hence the equality holds if and only if $G=K_2\cup H(n-2,1)$ or $G=K_3\cup B\left(1,\frac{n-4}{2}\right)$.

Case 2. $n\ge 7$ is odd.

By Lemma 3 and since $n\ge 7$,

$$2^{\frac{k-2}{2}}+2^{\frac{n-k-1}{2}}\le \max\left\{2^{\frac{(2)-2}{2}}+2^{\frac{n-(2)-1}{2}},2^{\frac{(n-3)-2}{2}}+2^{\frac{n-(n-3)-1}{2}}\right\}=2^{\frac{n-3}{2}}+1 \text{ for } 2\le k\le n-3.$$

By Theorem 1 and Theorem 5, we have

$$\begin{aligned} mi(G) &= mi(T)\cdot mi(H)\le t(k)\cdot h(n-k,1)\\ &=\begin{cases} 2^{\frac{k-1}{2}}\cdot\left(3\cdot 2^{\frac{n-k-4}{2}}\right), & \text{if } 1\le k\le n-4 \text{ is odd};\\ \left(2^{\frac{k-2}{2}}+1\right)\cdot\left(2^{\frac{n-k-1}{2}}+1\right), & \text{if } 2\le k\le n-3 \text{ is even};\end{cases}\\ &=\begin{cases} 3\cdot 2^{\frac{n-5}{2}}, & \text{if } 1\le k\le n-4 \text{ is odd};\\ 2^{\frac{n-3}{2}}+2^{\frac{k-2}{2}}+2^{\frac{n-k-1}{2}}+1, & \text{if } 2\le k\le n-3 \text{ is even};\end{cases}\\ &\le\begin{cases} 3\cdot 2^{\frac{n-5}{2}}, & \text{if } 1\le k\le n-4 \text{ is odd};\\ 2^{\frac{n-3}{2}}+1+\left(2^{\frac{n-3}{2}}+1\right), & \text{if } 2\le k\le n-3 \text{ is even};\end{cases}\\ &\le 2^{\frac{n-1}{2}}+2=2\cdot h(n-2,1). \end{aligned}$$

If the equality holds, then $mi(G)=t(2)\cdot h(n-2,1)$. Hence the equality holds if and only if $G=K_2\cup H(n-2,1)$.

By case 1 and case 2, we have that $mi(G)\le 2\cdot h(n-2,1)$. The equality holds if and only if $G=K_2\cup H(n-2,1)$ or $G=K_3\cup B\left(1,\frac{n-4}{2}\right)$. □

**Lemma 5.** *Suppose that F is a forest of order* $n\ge 4$ *having at most two components. Then* $mi(F)\le 2\cdot t(n-2)$ *and the equality holds if and only if* $F=K_2\cup T(n-2)$ *or* $F=B\left(1,\frac{n-1}{2}\right)$.

*Proof.* Let $F$ be a forest of order $n\ge 4$ having at most two components such that $mi(F)$ as large as possible. Then $mi(F)\ge mi(K_2\cup T(n-2))=2\cdot t(n-2)$. If $F$ has one component, then $F$ is a tree and, by Theorem 1, $2\cdot t(n-2)\le mi(F)\le t(n)$. Then $n$ is odd and $mi(F)=t(n)$. By Theorem 1, $F=B\left(1,\frac{n-1}{2}\right)$. Now we assume that $F$ have two components. Let $T_1$ and $T_2$ be the components of $F$, where $|T_1|=s$. We consider two cases.

Case 1. $n\ge 4$ is even.

By Lemma 3, $2^{\frac{s-2}{2}}+2^{\frac{n-s-2}{2}}\le \max\left\{2^{\frac{(2)-2}{2}}+2^{\frac{n-(2)-2}{2}},2^{\frac{(n-2)-2}{2}}+2^{\frac{n-(n-2)-2}{2}}\right\}=2^{\frac{n-4}{2}}+1$ for

$2 \le s \le n-2$. By Lemma 2 and Theorem 1, then

$$2\cdot t(n-2) = 2\left(2^{\frac{n-4}{2}}+1\right) \le mi(F) = mi(T_1)\cdot mi(T_2) \le t(s)\cdot t(n-s)$$
$$= \begin{cases} 2^{\frac{s-1}{2}}\cdot\left(2^{\frac{n-s-1}{2}}\right), & \text{if } 1\le s\le n-1 \text{ is odd;} \\ \left(2^{\frac{s-2}{2}}+1\right)\cdot\left(2^{\frac{n-s-2}{2}}+1\right), & \text{if } 2\le s\le n-2 \text{ is even;}\end{cases}$$
$$= \begin{cases} 2^{\frac{n-2}{2}}, & \text{if } 1\le s\le n-1 \text{ is odd;} \\ 2^{\frac{n-4}{2}}+1+\left(2^{\frac{s-2}{2}}+2^{\frac{n-s-2}{2}}\right), & \text{if } 2\le s\le n-2 \text{ is even;}\end{cases}$$
$$\le \begin{cases} 2^{\frac{n-2}{2}}, & \text{if } 1\le s\le n-1 \text{ is odd;} \\ 2^{\frac{n-4}{2}}+1+\left(2^{\frac{n-4}{2}}+1\right), & \text{if } 2\le s\le n-2 \text{ is even;}\end{cases}$$
$$= \begin{cases} 2^{\frac{n-2}{2}}, & \text{if } 1\le s\le n-1 \text{ is odd;} \\ 2^{\frac{n-2}{2}}+2, & \text{if } 2\le s\le n-2 \text{ is even;}\end{cases}$$
$$\le 2\left(2^{\frac{n-4}{2}}+1\right) = 2\cdot t(n-2).$$

The equalities hold and $mi(F) = t(2)\cdot t(n-2)$. By Theorem 1, $F = K_2 \cup T(n-2)$. The equality holds if and only if $F = K_2 \cup T(n-2)$.

Case 2. $n \ge 5$ is odd.

Then $F$ has exactly one even component, we assume that $s \ge 2$ is even. By Theorem 1, then

$$2\cdot t(n-2) = 2^{\frac{n-1}{2}} \le mi(F) = mi(T_1)\cdot mi(T_2) \le t(s)\cdot t(n-s) = \left(2^{\frac{s-2}{2}}+1\right)\cdot\left(2^{\frac{n-s-1}{2}}\right)$$
$$= 2^{\frac{n-3}{2}} + 2^{\frac{n-s-1}{2}} \le 2^{\frac{n-3}{2}} + 2^{\frac{n-(2)-1}{2}} = 2^{\frac{n-1}{2}} = 2\cdot t(n-2)$$

. The equalities hold and $mi(F) = t(2)\cdot t(n-2)$. The equality holds if and only if $F = K_2 \cup T(n-2)$. □

The following is the proof of Theorem 6.

*Proof.* Let $H$ be a connected graph of order $n \ge 12$ with at most two cycles such that $mi(H)$ as large as possible. Then $mi(H) \ge mi(H(n,2)) = h(n,2)$. Since $mi(H) \ge h(n,2) > h(n,1)$, by Theorem 5, $H$ have at least two cycles. That means that $H$ have exactly two cycles and $H \ne C_n$. Let $v$ be a vertex lying on some cycle such that $deg(v)$ is as large as possible. Since $H \ne C_n$, we can see that $deg(v) = s \ge 3$. The graph $H - v$ is a graph of order $n-1$ with at most one cycle. We consider two cases.

Case 1. $H - v = G(n-1,1)$.

Then $H - v = K_3 \cup \frac{n-4}{2}P_2$ or $H - v = \frac{n-1}{2}P_2$. Since $H$ is connected, this means that $v$ connects to every component of $G(n-1,1)$. Then $mi(H-N[v])$ has at most one edge, then $mi(H-N[v])\le 2$. So we have

$$h(n,2)\le mi(H)\le mi(H-v)+mi(H-N[v])\le g(n-1,1)+2$$
$$=\begin{cases}3\cdot 2^{\frac{n-4}{2}}+2, & \text{if } n\ge 12 \text{ is even;}\\ 2^{\frac{n-1}{2}}+2, & \text{if } n\ge 13 \text{ is odd;}\end{cases}\le h(n,2).$$

Then $mi(G-N[v])=2$ and $n\ge 12$ is even. So $H-v=K_3\cup\frac{n-4}{2}P_2$. Note that $H$ has two cycles, hence $H=K_3\times\left(K_3\cup\frac{n-6}{2}K_2\right)=H(n,2)$ for even $n\ge 12$.

<u>Case 2.</u> $H-v\ne G(n-1,1)$.

Let $deg(v)=d$. Then the subgraph $H-N[v]$ is a graph of order $n-d-1$ having at most one cycle. By Theorem 3, $mi(H-N[v])\le g(n-d-1)$. By Theorem 4, $mi(H-v)\le g'(n-1,1)$. By Lemma 1 and Theorem 4, then

$$g(n-d-1)\ge mi(H-N[v])\ge mi(H)-mi(H-v)\ge h(n,2)-g'(n-1,1)$$
$$=\begin{cases}\left(3\cdot 2^{\frac{n-4}{2}}+2\right)-\left(5\cdot 2^{\frac{n-6}{2}}\right), & \text{if } n\ge 12 \text{ is even;}\\ \left(9\cdot 2^{\frac{n-7}{2}}\right)-\left(3\cdot 2^{\frac{n-5}{2}}\right), & \text{if } n\ge 13 \text{ is odd;}\end{cases}$$
$$=\begin{cases}2^{\frac{n-6}{2}}+2, & \text{if } n\ge 12 \text{ is even;}\\ 3\cdot 2^{\frac{n-7}{2}}, & \text{if } n\ge 13 \text{ is odd;}\end{cases}=\begin{cases}g(n-6,1)+2, & \text{if } n\ge 12 \text{ is even;}\\ g(n-4,1), & \text{if } n\ge 13 \text{ is odd;}\end{cases}$$

Then

$$deg(v)=d\le\begin{cases}4, & \text{if } n\ge 12 \text{ is even;}\\ 3, & \text{if } n\ge 13 \text{ is odd.}\end{cases}$$

**Claim.** $deg(v)=3$.

Suppose that $deg(v)=4$, then $n$ is even. By Theorem 3, $mi(H-N[v])\le g(n-5,1)$ and

$$mi(H-v)\ge mi(H)-mi(H-N[v])\ge h(n,2)-g(n-5,1)=\left(3\cdot 2^{\frac{n-4}{2}}+2\right)-3\cdot 2^{\frac{n-8}{2}}$$
$$=9\cdot 2^{\frac{n-8}{2}}+2>2^{\frac{n-2}{2}}=f(n-1).$$

By Theorem 2, $H-v$ is not a forest and $H-v$ has one cycle. Let $H''$ be the component of $H-v$ having one cycle and $F=H-v-V(H'')$, where $|H''|=s\ge 3$. Note that $H$ has two cycles and $v$ is lying on some cycle. Thus $v$ has two edges incident to some component of $H-v$. Since $deg(v)=4$, the number of the components of $H-v$ is at most three. Thus $F$ is either a tree or the union of two trees. By Lemma 5, $mi(F)\le 2\cdot t(n-s-3)$. By Lemma 3,
$2^{\frac{s+1}{2}}+2^{\frac{n-s-3}{2}}\le\max\left\{2^{\frac{(3)+1}{2}}+2^{\frac{n-(3)-3}{2}},2^{\frac{(n-5)+1}{2}}+2^{\frac{n-(n-5)-3}{2}}\right\}\le 2^{\frac{n-4}{2}}+2$ for $3\le s\le n-5$. By

Theorem 5, $mi(H'') \le h(s,1)$. Note that $mi(H-v) \ge 9\cdot 2^{\frac{n-8}{2}}+2$. By Lemma 2 and Lemma 5,

$$9\cdot 2^{\frac{n-8}{2}}+2 \le (H-v) = mi(F)\cdot mi(H'') \le \begin{cases} 2\cdot h(n-3,1), & \text{if } s=n-3; \\ (2\cdot t(n-s-3))\cdot h(s,1), & \text{if } s\ne n-3; \end{cases}$$

$$= \begin{cases} 2\cdot(2^{\frac{n-4}{2}}+1), & \text{if } s=n-3; \\ \left(2\cdot\left(2^{\frac{n-s-5}{2}}+1\right)\right)\cdot\left(2^{\frac{s-1}{2}}+1\right), & \text{if } 3\le s\le n-5 \text{ is odd}; \\ \left(2\cdot 2^{\frac{n-s-4}{2}}\right)\cdot\left(3\cdot 2^{\frac{s-4}{2}}\right), & \text{if } 2\le s\le n-4 \text{ is even}; \end{cases}$$

$$= \begin{cases} 2^{\frac{n-2}{2}}+2, & \text{if } s=n-3; \\ 2^{\frac{n-4}{2}}+2+\left(2^{\frac{s+1}{2}}+2^{\frac{n-s-3}{2}}\right), & \text{if } 3\le s\le n-5 \text{ is odd}; \\ 3\cdot 2^{\frac{n-6}{2}}, & \text{if } 2\le s\le n-4 \text{ is even}; \end{cases}$$

$$\le \begin{cases} 2^{\frac{n-2}{2}}+2, & \text{if } s=n-3; \\ 2^{\frac{n-4}{2}}+2+\left(2^{\frac{n-4}{2}}+2\right), & \text{if } 3\le s\le n-5 \text{ is odd}; \\ 3\cdot 2^{\frac{n-6}{2}}, & \text{if } 2\le s\le n-4 \text{ is even}; \end{cases} \le 2^{\frac{n-2}{2}}+4 < 9\cdot 2^{\frac{n-8}{2}}+2,$$

where $n \ge 12$. This is a contradiction, so $deg(v)=3$.

By Claim, $deg(v)=3$. Note that $H$ has two cycles and $v$ is lying on some cycle. Thus $v$ has two edges incident to some component of $H-v$. Since $deg(v)=3$, the number of the components of $H-v$ is at most two. Thus $H-v = T\cup H'$, where $T$ is a tree and $H'$ is a unicyclic graph. By Lemma 4 and Theorem 3, then $mi(H-v) = mi(T)\cdot mi(H') \le 2\cdot h(n-3,1) = 2\cdot h(n-3,1)$ and $mi(H-N[v]) \le g(n-4,1)$. So

$$\begin{aligned} h(n,2) &\le mi(H) \le mi(H-v)+mi(H-N[v]) \\ &\le 2\cdot h(n-3,1)+g(n-4,1) \\ &= \begin{cases} 2\left(2^{\frac{n-4}{2}}+1\right)+2^{\frac{n-4}{2}}, & \text{if } n\ge 12 \text{ is even}; \\ 2\left(3\cdot 2^{\frac{n-7}{2}}\right)+3\cdot 2^{\frac{n-7}{2}}, & \text{if } n\ge 13 \text{ is odd}; \end{cases} \\ &= \begin{cases} 3\cdot 2^{\frac{n-4}{2}}+2, & \text{if } n\ge 12 \text{ is even}; \\ 9\cdot 2^{\frac{n-7}{2}}, & \text{if } n\ge 13 \text{ is odd}. \end{cases} \\ &= h(n,2). \end{aligned}$$

The equalities hold. Then $H-N[v]=G(n-4,1)$ and, by Lemma 4, $H-v=K_2\cup H(n-3,1)$ or $K_3\cup B\left(1,\frac{n-5}{2}\right)$. If $H-v=K_3\cup B\left(1,\frac{n-5}{2}\right)$, then $n$ is odd and $H-N[v]\neq K_3\cup\frac{n-7}{2}K_2$. That is $H-N[v]\neq G(n-4,1)$, this is a contradiction. Thus $H-v=K_2\cup H(n-3,1)$. If $n$ is even, where $n\geq 12$, then $H-v=K_2\cup H(n-3,1)=K_2\cup\left(K_3\times\cup\frac{n-6}{2}K_2\right)$ and $H-N[v]=G(n-4,1)=\frac{n-4}{2}K_2$. Then, there exists a vertex $u\in V(H-v)$ lying on some cycle such that $deg(u)\geq\frac{n}{2}>3$. This contradicts to the claim, so $n$ is odd. Thus $H-v=K_2\cup H(n-3,1)=K_2\cup\left(K_1*\left(K_3\cup\frac{n-7}{2}K_2\right)\right)$ and $H-N[v]=G(n-4,1)=K_3\cup\frac{n-7}{2}K_2$. Hence $H=K_1\times\left(2K_3\cup\frac{n-7}{2}K_2\right)=H(n,2)$ for odd $n\geq 13$. □

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