Note on Cyclically Interval Edge Colorings of Simple Cycles

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Abstract

A proper edge $t$-coloring of a graph $G$ is a coloring of its edges with colors $1, 2, \ldots, t$ such that all colors are used, and no two adjacent edges receive the same color. A cyclically interval $t$-coloring of a graph $G$ is a proper edge $t$-coloring of $G$ such that for each vertex $x \in V(G)$, either the set of colors used on edges incident to $x$ or the set of colors not used on edges incident to $x$ forms an interval of integers. In this paper, we provide a new proof of the result on the colors in cyclically interval edge colorings of simple cycles which was first proved by Rafayel R. Kamalian in the paper "On a Number of Colors in Cyclically Interval Edge Colorings of Simple Cycles, Open Journal of Discrete Mathematics, 2013, 43-48".

Keywords

Edge Coloring, Interval Edge Coloring, Cyclically Interval Edge Coloring

1. Introduction

All graphs considered in this paper are finite undirected simple graphs. For a graph $G$, let $V(G)$ and $E(G)$ denote the sets of vertices and edges of $G$, respectively. For a vertex $x \in V(G)$, let $J_G(x)$ and $d_G(x)$ denote the subset of $E(G)$ incident with the vertex $x$, and the degree of the vertex $x$ in $G$, respectively. We denote $\Delta(G)$ the maximum degree of vertices of $G$. A simple path with $n \geq 1$ edges is denoted by $P_n$. A simple cycle with $n \geq 3$ edges is denoted by $C_n$.

For an arbitrary finite set $A$, we denote by $|A|$ the number of elements of $A$. The set of positive integers is denoted by $\mathbb{N}$. An arbitrary nonempty subset of consecutive integers is called an interval. An interval with the
minimum element \( p \) and the maximum element \( q \) is denoted by \([p,q]\). We denote \( \alpha([a,b]) \) and \( \omega([a,b]) \) the sets of even and odd integers in \([a,b]\), respectively. An interval \( D \) is called a \( h \)-interval if \(|D| = h\).

A function \( \alpha : E(G) \to [1,t] \) is called a proper edge \( t \)-coloring of a graph \( G \), if all colors are used, and no two adjacent edges receive the same color. The minimum value of \( t \) for which there exists a proper edge \( t \)-coloring of a graph \( G \) is denoted by \( \chi'(G) \). If \( E_0 \subseteq E(G) \), and \( \alpha \) is a proper edge \( t \)-coloring of a graph \( G \), then let \( \alpha|_{E_0} = \{ \alpha(e) | e \in E_0 \} \). A proper edge \( t \)-coloring \( \alpha \) of a graph \( G \) is called an interval \( t \)-coloring of \( G \) if for any \( x \in V(G) \), the set \( \alpha|_{\langle x \rangle} \) is a \( d_G(x) \)-interval. A graph \( G \) is interval colorable if it has an interval \( t \)-coloring for some positive integer \( t \). The concept of interval edge coloring of graphs was introduced by Asratian and Kamalian [1]. In [1] [2], the authors showed that if \( G \) is interval colorable, then \( \chi'(G) = \Delta(G) \).

For any \( t \in \mathbb{N} \), we denote by \( \mathcal{M} \) the set of graphs for which there exists an interval \( t \)-coloring. Let \( \mathcal{M} = \bigcup_{t \in \mathbb{N}} \mathcal{M}_t \). For any graph \( G \in \mathcal{M} \), the minimum and the maximum values of \( t \) for which \( G \) has an interval \( t \)-coloring are denoted by \( w(G) \) and \( W(G) \), respectively. For a graph \( G \in \mathcal{M} \), let \( \theta(G) = \{ t | G \in \mathcal{M}_t \} \).

A proper edge \( t \)-coloring \( \alpha \) of a graph \( G \) is called an interval cyclic \( t \)-coloring of \( G \), if for any \( x \in V(G) \), at least one of the following two conditions holds:

1) \( \alpha|_{\langle x \rangle} \) is a \( d_G(x) \)-interval,

2) \([1,t] \setminus \alpha|_{\langle x \rangle} \) is a \((t-d_G(x))\)-interval.

A graph \( G \) is interval cyclically colorable if it has a interval cyclic \( t \)-coloring for some positive integer \( t \). This type of edge coloring under the name of “\( \pi \)-coloring” was first considered by Kotzig [3], and the concept of cyclically interval edge coloring of graphs was explicitly introduced by de Werra and Solot [4].

For any \( t \in \mathbb{N} \), we denote by \( \mathcal{M}_t \) the set of graphs for which there exists an interval cyclic \( t \)-coloring. Let \( \mathcal{M}_t = \bigcup_{t \in \mathbb{N}} \mathcal{M}_t \). For any graph \( G \in \mathcal{M}_t \), the minimum and the maximum values of \( t \) for which \( G \) has a cyclic interval \( t \)-coloring are denoted by \( w_t(G) \) and \( W_t(G) \) respectively. For a graph \( G \in \mathcal{M}_t \), let \( \Theta(G) = \{ t | G \in \mathcal{M}_t \} \).

It is clear that for any \( t \in \mathbb{N} \), \( \mathcal{M}_t \subseteq \mathcal{M}_s \) and \( \mathcal{M}_s \subseteq \mathcal{M}_t \). Note that for an arbitrary graph \( G \), \( \Theta(G) \subseteq \Theta(G) \). It is also clear that for any \( G \in \mathcal{M}_t \), the following inequality is true:

\[ \Delta(G) \leq \chi'(G) \leq w_t(G) \leq w(G) \leq W(G) \leq W_t(G) \leq |E(G)|. \]

Let \( T \) be a tree. Kamalian [5] [6] showed that \( T \in \mathcal{M} \), \( \theta(T) \) was an interval, and provided the exact values of the parameters \( w(T) \) and \( W(T) \). Kamalian [7] [8] also proved that \( \Theta(T) = \Theta(T) \). Some interesting results on cyclically interval \( t \)-colorings and related topics were obtained in [3] [4] [9]-[14]. For any integer \( n \geq 3 \), Kamalian [13] proved that \( C_n \in \mathcal{M}_n \), determined the set \( \Theta(C_n) \), and provided the following theorem.

**Theorem 1** (R. R. Kamalian [13]) For any integers \( n \geq 3 \) and \( t \in [2, n] \), \( C_n \in \mathcal{M}_t \) if and only if

\[
t \in \begin{cases} 
[c, n], & \text{if } n \text{ is odd;} \\
\left[\frac{n}{2} + 2, n\right] \cup \left[\frac{n}{2} + 1\right], & \text{if } n \text{ is even.}
\end{cases}
\]

In this paper, we provide a new proof of the theorem. The terms and concepts that we do not define can be found in [15].

**2. Main Result**

**Proof of Theorem 1.** Suppose that, in clockwise order along the cycle \( C_n \), the vertices of \( C_n \) are \( v_1, v_2, \ldots, v_n \) and the edges of \( C_n \) are \( e_1, e_2, \ldots, e_n \), where \( e_i = v_i v_{i+1} \) for \( i = 1, 2, \ldots, n \), and \( v_{n+1} = v_1 \). Since \(|E(C_n)| = n\) and

\[
\chi'(C_n) = \begin{cases} 
2, & \text{if } n \text{ is even;} \\
3, & \text{if } n \text{ is odd.}
\end{cases}
\]

We know that if \( t > n \) or

\[
t < \begin{cases} 
2, & \text{if } n \text{ is even;} \\
3, & \text{if } n \text{ is odd.}
\end{cases}
\]
then \( C_n \notin \mathcal{M} \).

First we prove that if \( n \geq 3 \) and

\[
t \in \begin{cases} 
  [3, n], & \text{if } n \text{ is odd;} \\
  \emptyset \left[ \frac{n}{2} + 2, n \right] \cup \left[ \frac{n}{2} + 1, n \right], & \text{if } n \text{ is even,}
\end{cases}
\]

then \( C_n \notin \mathcal{M} \).

Case 1. \( n \) is odd. For any \( t \in \mathbb{N} \setminus \mathcal{M} \), let

\[
\alpha(e_i) = \begin{cases} 
  i, & i \in [1, t] \\
  1, & i \in \emptyset [t + 1, n] \\
  2, & i \in \emptyset [t + 1, n].
\end{cases}
\]

It is easy to check that \( \alpha \) is a cyclically interval-\( t \)-coloring of \( C_n \).

Case 2. \( n \) is even. For any \( t \in \emptyset [2, n] \), let

\[
\alpha(e_i) = \begin{cases} 
  i, & i \in [1, t] \\
  1, & i \in \emptyset [t + 1, n] \\
  2, & i \in \emptyset [t + 1, n].
\end{cases}
\]

If \( t = \frac{n}{2} + 1 \) is odd, then let

\[
\alpha(e_i) = \begin{cases} 
  i, & i \in [1, t] \\
  n - i + 2, & i \in [t + 1, n].
\end{cases}
\]

For any \( t \in \mathbb{N} \setminus \mathcal{M} \), let

\[
\alpha(e_i) = \begin{cases} 
  i, & i \in [1, t] \\
  i - t, & i \in [t + 1, 2t] \\
  1, & i \in \emptyset [2t + 1, n] \\
  2, & i \in \emptyset [2t + 1, n].
\end{cases}
\]

It is easy to check that, in each case, \( \alpha \) is a cyclically interval-\( t \)-coloring of \( C_n \).

Now let us prove that if \( n \geq 3 \), \( t \in [2, n] \) and \( C_n \in \mathcal{M} \), then

\[
t \in \begin{cases} 
  [3, n], & \text{if } n \text{ is odd;} \\
  \emptyset \left[ \frac{n}{2} + 2, n \right] \cup \left[ \frac{n}{2} + 1, n \right], & \text{if } n \text{ is even.}
\end{cases}
\]

By contradiction. Suppose that there are \( n_0 \in \mathbb{N} \), \( n_0 \geq 3 \), \( t_0 \in [2, n_0] \) and

\[
t_0 \not\in \begin{cases} 
  [3, n], & \text{if } n_0 \text{ is odd;} \\
  \emptyset \left[ \frac{n}{2} + 2, n \right] \cup \left[ \frac{n}{2} + 1, n \right], & \text{if } n_0 \text{ is even,}
\end{cases}
\]

such that \( C_{n_0} \) has a cyclically interval-\( t_0 \)-coloring \( \alpha \).

Case 1. \( n_0 \) is odd. Clearly, \( t_0 \in \emptyset [2, n_0 - 1] \). Let \( e_s \) and \( e_t \) be two edges of \( C_{n_0} \) such that \( \alpha(e_s) = 1 \) and \( \alpha(e_t) = t_0 \).

Without loss of generality, we may assume \( s < t \). Let \( L_1 \) be the subgraph induced by \( \{ e_i | s \leq i \leq t \} \), and \( L_2 \) be the subgraph induced by \( \{ e_j | j \leq s \text{ or } j \geq t \} \), respectively. Since \( t_0 \) is even and \( \alpha \) is a cyclically interval
Let \( C_{n_0} \) be the graph removing from the graph \( C_n \) the edges with the colors except 1 and \( t_0 \), and \( H_0 \) the graph removing from the graph \( H \) all its isolated vertices.

Case 2.1. \( H_0 \) is connected.

Let \( F \) be the subgraph of \( C_{n_0} \) induced by \( E(C_{n_0}) \setminus E(H_0) \cup \{e', e''\} \), where \( e' \) and \( e'' \) are the two pendant edges of \( H_0 \).

Clearly, \( t_0 \in \left[ \frac{n_0}{2} + 2, n_0 - 1 \right] \). If \( |E(H_0)| \) is odd, then \( \alpha(e') = \alpha(e'') \). Since \( \alpha \) is a cyclically interval \( t_0 \)-coloring of \( C_{n_0} \), then \( \alpha|_{\{e', e''\}} \) is a interval \((t_0 - 1)\)-coloring with \( \alpha(e') = \alpha(e'') \). So we have \( n_0 > |E(F)| \geq 2t_0 - 3 \geq n_0 + 1 \), a contradiction.

If \( |E(H_0)| \) is even, then \( \alpha(e') \neq \alpha(e'') \). Since \( \alpha \) is a cyclically interval \( t_0 \)-coloring of \( C_{n_0} \), then \( \alpha|_{\{e', e''\}} \) is a interval \( t_0 \)-coloring. So we know that \( |E(F)| \) is odd, and then \( n_0 = |E(H_0)| + |E(F)| - 2 \) is odd, a contradiction.

Case 2.2. \( H_0 \) is a graph with \( m \) connected components, \( m \geq 2 \).

Suppose that, in clockwise order along the cycle \( C_n \), the \( m \) connected components of \( H_0 \) are \( H_1, H_2, \ldots, H_m \). Without loss of generality, we may also assume that \( v_1, v_2 \in V(H_1) \) and \( v_n \notin V(H_1) \).

Clearly, \( t_0 \in \left[ \frac{n_0}{2} + 2, n_0 - 1 \right] \) and \( \min \{i \mid e_i \in E(H_1)\} = 1 \). Let \( r_1 = \max \{i \mid e_i \in E(H_1)\} \), \( r_2 = \min \{i \mid e_i \in E(H_2)\} \) and \( r_3 = \max \{i \mid e_i \in E(H_m)\} \). Let \( L_1 \) be the subgraph induced by \( \{e_1 \mid r_1 \leq i \leq r_2\} \), and \( L_2 \) be the subgraph induced by \( \{e_j \mid j = 1 \text{ or } j \geq r_3\} \), respectively. Let \( \alpha(e_1) = \alpha(e_2) \) or \( \alpha(e_1) = \alpha(e_3) \), say \( \alpha(e_1) = \alpha(e_2) \). Since \( \alpha \) is a cyclically interval \( t_0 \)-coloring of \( C_{n_0} \), then \( \alpha|_{\{e_1, e_2\}} \) is a interval \((t_0 - 1)\)-coloring with \( \alpha(e_1) = \alpha(e_2) \). So we have \( n_0 > |E(L_1)| \geq 2t_0 - 3 \geq n_0 + 1 \), a contradiction.

Now let \( \alpha(e_1) \neq \alpha(e_2) \) and \( \alpha(e_1) \neq \alpha(e_3) \). Since \( \alpha \) is a cyclically interval \( t_0 \)-coloring of \( C_{n_0} \), then \( \alpha|_{\{e_1, e_2\}} \) and \( \alpha|_{\{e_1, e_4\}} \) are all interval \( t_0 \)-coloring. So we have \( n_0 > |E(L_1)| + |E(L_4)| - 2 \geq 2t_0 - 2 \geq n_0 + 2 \), a contradiction.

\[ \square \]

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References


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