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Note on Cyclically Interval Edge Colorings of Simple Cycles

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Abstract

A proper edge t-coloring of a graph G is a coloring of its edges with colors $1,2,\cdots,t$ such that all colors are used, and no two adjacent edges receive the same color. A cyclically interval t-coloring of a graph G is a proper edge t-coloring of G such that for each vertex $x \in V(G)$, either the set of colors used on edges incident to x or the set of colors not used on edges incident to x forms an interval of integers. In this paper, we provide a new proof of the result on the colors in cyclically interval edge colorings of simple cycles which was first proved by Rafayel R. Kamalian in the paper "On a Number of Colors in Cyclically Interval Edge Colorings of Simple Cycles, *Open Journal of Discrete Mathematics*, 2013, 43-48".

Keywords

Edge Coloring, Interval Edge Coloring, Cyclically Interval Edge Coloring

1. Introduction

All graphs considered in this paper are finite undirected simple graphs. For a graph G, let $V(G)$ and $E(G)$ denote the sets of vertices and edges of G, respectively. For a vertex $x \in V(G)$, let $J_G(x)$ and $d_G(x)$ denote the subset of $E(G)$ incident with the vertex x, and the degree of the vertex x in G, respectively. We denote $\Delta(G)$ the maximum degree of vertices of G. A simple path with $n \geq 1$ edges is denoted by P_n. A simple cycle with $n \geq 3$ edges is denoted by C_n.

For an arbitrary finite set A, we denote by $|A|$ the number of elements of A. The set of positive integers is denoted by $\mathbb{N}$. An arbitrary nonempty subset of consecutive integers is called an interval. An interval with the

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minimum element p and the maximum element q is denoted by $[p,q]$. We denote $\Diamond[a,b]$ and $\circ[a,b]$ the sets of even and odd integers in $[a,b]$, respectively. An interval D is called a h-interval if $|D|=h$.

A function $\alpha: E(G)\to[1,t]$ is called a proper edge t-coloring of a graph G, if all colors are used, and no two adjacent edges receive the same color. The minimum value of t for which there exists a proper edge t-coloring of a graph G is denoted by $\chi'(G)$. If $E_0\subseteq E(G)$, and α is a proper edge t-coloring of a graph G, then let $\alpha|_{E_0}=\{\alpha(e)\,|\,e\in E_0\}$. A proper edge t-coloring α of a graph G is called an interval t-coloring of G if for any $x\in V(G)$, the set $\alpha|_{J_D(x)}$ is a $d_G(x)$-interval. A graph G is interval colorable if it has an interval t-coloring for some positive integer t. The concept of interval edge coloring of graphs was introduced by Asratian and Kamalian [1]. In [1] [2], the authors showed that if G is interval colorable, then $\chi'(G)=\Delta(G)$.

For any $t\in\mathbb{N}$, we denote by $\mathfrak{N}_t$ the set of graphs for which there exists an interval t-coloring. Let $\mathfrak{N}=\bigcup_{t\ge1}\mathfrak{N}_t$. For any graph $G\in\mathfrak{N}$, the minimum and the maximum values of t for which G has an interval t-coloring are denoted by $w(G)$ and $W(G)$, respectively. For a graph $G\in\mathfrak{N}$, let $\theta(G)=\{t\,|\,G\in\mathfrak{N}_t\}$.

A proper edge t-coloring α of a graph G is called a interval cyclic t-coloring of G, if for any $x\in V(G)$, at least one of the following two conditions holds:

1) $\alpha|_{J_G(x)}$ is a $d_G(x)$-interval,

2) $[1,t]\setminus\alpha|_{J_G(x)}$ is a $(t-d_G(x))$-interval.

A graph G is interval cyclically colorable if it has a cyclically interval t-coloring for some positive integer t. This type of edge coloring under the name of "π-coloring" was first considered by Kotzig [3], and the concept of cyclically interval edge coloring of graphs was explicitly introduced by de Werra and Solot [4].

For any $t\in\mathbb{N}$, we denote by $\mathfrak{M}_t$ the set of graphs for which there exists a interval cyclic t-coloring. Let $\mathfrak{M}=\bigcup_{t\ge1}\mathfrak{M}_t$. For any graph $G\in\mathfrak{M}$, the minimum and the maximum values of t for which G has a cyclically interval t-coloring are denoted by $w_c(G)$ and $W_c(G)$ respectively. For a graph $G\in\mathfrak{M}$, let $\Theta(G)=\{t\,|\,G\in\mathfrak{M}_t\}$.

It is clear that for any $t\in\mathbb{N}$, $\mathfrak{N}_t\subseteq\mathfrak{M}_t$ and $\mathfrak{N}\subseteq\mathfrak{M}$. Note that for an arbitrary graph G, $\theta(G)\subseteq\Theta(G)$. It is also clear that for any $G\in\mathfrak{N}$, the following inequality is true:

$$\Delta(G)\le\chi'(G)\le w_c(G)\le w(G)\le W(G)\le W_c(G)\le|E(G)|.$$

Let T be a tree. Kamalian [5] [6] showed that $T\in\mathfrak{N}$, $\theta(T)$ was an interval, and provided the exact values of the parameters $w(T)$ and $W(T)$. Kamalian [7] [8] also proved that $\Theta(T)=\theta(T)$. Some interesting results on cyclically interval t-colorings and related topics were obtained in [3] [4] [9]-[14]. For any integer $n\ge3$, Kamalian [13] proved that $C_n\in\mathfrak{M}$, determined the set $\Theta(C_n)$, and provided the following theorem.

Theorem 1 (R. R. Kamalian [13]) *For any integers* $n\ge3$ *and* $t\in[2,n]$, $C_n\in\mathfrak{M}_t$ *if and only if*

$$t\in\begin{cases}\circ[3,n], & \text{if } n \text{ is odd;}\\ \Diamond\left[\frac{n}{2}+2,n\right]\cup\left[2,\frac{n}{2}+1\right], & \text{if } n \text{ is even.}\end{cases}$$

In this paper, we provide a new proof of the theorem. The terms and concepts that we do not define can be found in [15].

2. Main Result

Proof of Theorem 1. Suppose that, in clockwise order along the cycle C_n, the vertices of C_n are $v_1,v_2,\cdots,v_n$ and the edges of C_n are $e_1,e_2,\cdots,e_n$, where $e_i=v_iv_{i+1}$ for $i=1,2,\cdots,n$, and $v_{n+1}=v_1$. Since $|E(C_n)|=n$ and

$$\chi'(C_n)=\begin{cases}2, & \text{if } n \text{ is even;}\\ 3, & \text{if } n \text{ is odd.}\end{cases}$$

We know that if $t>n$ or

$$t<\begin{cases}2, & \text{if } n \text{ is even;}\\ 3, & \text{if } n \text{ is odd,}\end{cases}$$

then $C_n \notin \mathfrak{M}_t$.

First we prove that if $n \geq 3$ and

$$t \in \begin{cases} \circ[3,n], & \text{if } n \text{ is odd;} \\ \lozenge\left[\frac{n}{2}+2,n\right] \cup \left[2,\frac{n}{2}+1\right], & \text{if } n \text{ is even,} \end{cases}$$

then $C_n \in \mathfrak{M}_t$.

Case 1. n is odd.

For any $t \in \circ[3,n]$, let

$$\alpha(e_i) = \begin{cases} i, & i \in [1,t]; \\ 1, & i \in \lozenge[t+1,n]; \\ 2, & i \in \circ[t+1,n]. \end{cases}$$

It is easy to check that α is a cyclically interval t-coloring of C_n.

Case 2. n is even.

For any $t \in \lozenge[2,n]$, let

$$\alpha(e_i) = \begin{cases} i, & i \in [1,t], \\ 1, & i \in \circ[t+1,n]; \\ 2, & i \in \lozenge[t+1,n]. \end{cases}$$

If $t = \frac{n}{2}+1$ is odd, then let

$$\alpha(e_i) = \begin{cases} i, & i \in [1,t]; \\ n-i+2, & i \in [t+1,n]. \end{cases}$$

For any $t \in \circ\left[3,\frac{n}{2}\right]$, let

$$\alpha(e_i) = \begin{cases} i, & i \in [1,t]; \\ i-t, & i \in [t+1,2t]; \\ 1, & i \in \circ[2t+1,n]; \\ 2, & i \in \lozenge[2t+1,n]. \end{cases}$$

It is easy to check that, in each case, α is a cyclically interval t-coloring of C_n.

Now let us prove that if $n \geq 3$, $t \in [2,n]$ and $C_n \in \mathfrak{M}_t$, then

$$t \in \begin{cases} \circ[3,n], & \text{if } n \text{ is odd;} \\ \lozenge\left[\frac{n}{2}+2,n\right] \cup \left[2,\frac{n}{2}+1\right], & \text{if } n \text{ is even.} \end{cases}$$

By contradiction. Suppose that there are $n_0 \in \mathbb{N}$, $n_0 \geq 3$, $t_0 \in [2,n_0]$ and

$$t_0 \notin \begin{cases} \circ[3,n], & \text{if } n_0 \text{ is odd;} \\ \lozenge\left[\frac{n}{2}+2,n\right] \cup \left[2,\frac{n}{2}+1\right], & \text{if } n_0 \text{ is even,} \end{cases}$$

such that C_{n_0} has a cyclically interval t_0-coloring α.

Case 1. n_0 is odd.

Clearly, $t_0 \in \lozenge[2,n_0-1]$. Let e_s and e_t be two edges of C_{n_0} such that $\alpha(e_s)=1$ and $\alpha(e_t)=t_0$. Without loss of generality, we may assume $s<t$. Let L_1 be the subgraph induced by $\{e_i \mid s \leq i \leq t\}$, and L_2 be the subgraph induced by $\{e_j \mid j \leq s \text{ or } j \geq t\}$, respectively. Since t_0 is even and α is a cyclically interval

t_0-coloring of C_{n_0}, then $|E(L_1)|$ and $|E(L_2)|$ are all even. So we have that n_0 is even, a contradiction.

Case 2. n_0 is even.

Let H be the graph removing from the graph C_{n_0} the edges with the colors except 1 and t_0, and H_0 the graph removing from the graph H all its isolated vertices.

Case 2.1. H_0 is connected.

Let F be the subgraph of C_{n_0} induced by $E(C_{n_0})\setminus E(H_0)\cup\{e',e''\}$, where e' and e'' are the two pendant edges of H_0.

Clearly, $t_0 \in \circ\left[\frac{n_0}{2}+2, n_0-1\right]$. If $|E(H_0)|$ is odd, then $\alpha(e')=\alpha(e'')$. Since α is a cyclically interval t_0-coloring of C_{n_0}, then $\alpha|_{E(F)}$ is a interval (t_0-1)-coloring with $\alpha(e')=\alpha(e'')$. So we have $n_0 > |E(F)| \ge 2t_0-3 \ge n_0+1$, a contradiction.

If $|E(H_0)|$ is even, then $\alpha(e')\ne\alpha(e'')$. Since α is a cyclically interval t_0-coloring of C_{n_0}, then $\alpha|_{E(F)}$ is a interval t_0-coloring. So we know that $|E(F)|$ is odd, and then $n_0=|E(H_0)|+|E(F)|-2$ is odd, a contradiction.

Case 2.2. H_0 is a graph with m connected components, $m\ge 2$.

Suppose that, in clockwise order along the cycle C_n, the m connected components of H_0 are $H_1, H_2, \cdots, H_m$. Without loss of generality, we may also assume that $v_1, v_2 \in V(H_1)$ and $v_{n_0}\notin V(H_1)$.

Clearly, $t_0 \in \circ\left[\frac{n_0}{2}+2, n_0-1\right]$ and $\min\{i \mid e_i \in E(H_1)\}=1$. Let $r_1=\max\{i \mid e_i\in E(H_1)\}$, $r_2=\min\{i \mid e_i \in E(H_2)\}$ and $r_3=\max\{i \mid e_i\in E(H_m)\}$. Let L_3 be the subgraph induced by $\{e_i \mid r_1\le i\le r_2\}$, and L_4 be the subgraph induced by $\{e_j \mid j=1 \text{ or } j\ge r_3\}$, respectively. Let $\alpha(e_{r_1})=\alpha(e_{r_2})$ or $\alpha(e_{r_3})=\alpha(e_1)$, say $\alpha(e_{r_1})=\alpha(e_{r_2})$. Since α is a cyclically interval t_0-coloring of C_{n_0}, then $\alpha|_{E(L_3)}$ is a interval (t_0-1)-coloring with $\alpha(e_{r_1})=\alpha(e_{r_2})$. So we have $n_0>|E(L_3)|\ge 2t_0-3\ge n_0+1$, a contradiction.

Now let $\alpha(e_{r_1})\ne\alpha(e_{r_2})$ and $\alpha(e_{r_3})\ne\alpha(e_1)$. Since α is a cyclically interval t_0-coloring of C_{n_0}, then $\alpha|_{E(L_3)}$ and $\alpha|_{E(L_4)}$ are all interval t_0-coloring. So we have $n_0>|E(L_3)|+|E(L_4)|-2\ge 2t_0-2\ge n_0+2$, a contradiction. □

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