Double Derangement Permutations

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Received 28 September 2015; accepted 9 April 2016; published 12 April 2016

Abstract

Let $n$ be a positive integer. A permutation $a$ of the symmetric group $S_n$ of permutations of $\{1, 2, \ldots, n\}$ is called a derangement if $a(i) \neq i$ for each $i \in [n]$. Suppose that $x$ and $y$ are two arbitrary permutations of $S_n$. We say that a permutation $a$ is a double derangement with respect to $x$ and $y$ if $a(i) \neq x(i)$ and $a(i) \neq y(i)$ for each $i \in [n]$. In this paper, we give an explicit formula for $nD_{xy}$, the number of double derangements with respect to $x$ and $y$. Let $0 \leq k \leq n$ and let $\{i_1, \ldots, i_k\}$ and $\{a_1, \ldots, a_k\}$ be two subsets of $[n]$ with $i_j \neq a_j$ and $\ell = |i_1, \ldots, i_k| \cap |a_1, \ldots, a_k|$. Suppose that $\Delta(n, k, \ell)$ denotes the number of derangements $x$ such that $x(i_j) = a_j$. As the main result, we show that if $0 \leq m \leq n$ and $z$ is a permutation such that $z(i) \neq i$ for $i > m$, then $D_n(e, z) = \sum_{k=0}^{m} \sum_{\ell = |i_1, \ldots, i_k|} (-1)^\ell \Delta(n, k, \ell(i_1, \ldots, i_k))$, where $\ell(i_1, \ldots, i_k) = |i_1, \ldots, i_k| \cap |z(i_1), \ldots, z(i_k)|$.

Keywords
Symmetric Group of Permutations, Derangement, Double Derangement

1. Introduction

Let $n$ be a positive integer. A derangement is a permutation of the symmetric group $S_n$ of permutations of $[n] = \{1, 2, \ldots, n\}$ such that none of the elements appear in their original position. The number of derangements of $S_n$ is denoted by $D_n$ or $\nu$. A simple recursive argument shows that
The number of derangements also satisfies the relation \( D_n = nD_{n-1} + (-1)^n \). It can be proved by the inclusion-exclusion principle that \( D_n \) is explicitly determined by \( n! \sum_{i=0}^{n} (-1)^i \frac{1}{i!} \). This implies that \( \lim_{n \to \infty} \frac{D_n}{n!} = \frac{1}{e} \).

These facts and some other results concerning derangements can be found in [1]. There are also some generalizations of this notion. The problème des rencontres asks how many permutations of the set \( \{1, 2, \ldots, n\} \) have exactly \( k \) fixed points. The number of such permutations is denoted by \( D_{n,k} \) and is given by \( D_{n,k} = \frac{n!}{k!} D_{n-k} \).

Thus, we can say that \( \lim_{n \to \infty} \frac{D_{n,k}}{n!} = \frac{1}{k!e} \). Some probabilistic aspects of this concept and the related notions concerning the permutations of \( S_n \) is discussed in [2] and [3].

Let \( e \) be the identity element of the symmetric group \( S_n \), which is defined by \( e(i) = i \) for each \( i \in [n] \). We can say that a permutation \( a \) of \( [n] \) is a derangement if \( a(i) \neq e(i) \) for each \( i \in [n] \). We denote this by \( a \perp e \). Thus, \( D_n \) is the number of \( a \) with \( a \perp e \). If \( e \) is any fixed element of \( S_n \), then the number of \( a \in S_n \) with \( a \perp x \) is also \( D_n \), since \( a \perp x \) if and only if \( ax^{-1} \perp e \). In the present paper, we extend the concept of a derangement to a double derangement with respect to two fixed elements \( x \) and \( y \) of \( S_n \).

### 2. The Result

In the following, we assume that \( n \) is a positive integer and the identity permutation of the symmetric group \( S_n \) of permutations of \( [n] \) is denoted by \( e \). Moreover, for two permutations \( a \) and \( b \) of \( S_n \), the notation \( a \perp b \) means that \( a(i) \neq b(i) \) for each \( i \in [n] \). We also denote the number of elements of a set \( A \) by \( |A| \).

**Definition 1.** Suppose that \( x \) and \( y \) are two arbitrary permutations of \( S_n \). We say that a permutation \( a \) is a double derangement with respect to \( x \) and \( y \) if \( a \perp x \) and \( a \perp y \). The number of double derangements with respect to \( x \) and \( y \) is denoted by \( D_n(x,y) \).

**Proposition 1.** Let \( 0 \leq k \leq n \) and let \( \{i_1, \ldots, i_k\} \) and \( \{a_1, \ldots, a_k\} \) be two subsets of \( [n] \) with \( i_j \neq a_j \) and \( \ell = \left| \{i_1, \ldots, i_k\} \cap \{a_1, \ldots, a_k\} \right| \). Then \( \Delta(n,k,\ell) \), the number of derangements \( x \) such that \( x(i_j) = a_j \), is determined by

\[
\Delta(n,k,\ell) = \begin{cases} \sum_{i=0}^{k-\ell-1} \frac{(k-\ell-1)!}{i!} D_{(n+1)-(k+i)} & \text{if } k \neq \ell \text{ and } 2k-\ell \leq n \\ D_{n-k} & \text{if } k = \ell \\ 0 & \text{otherwise} \end{cases}
\]

**Proof.** Let \( a_j \in \{i_1, \ldots, i_k\} \cap \{a_1, \ldots, a_k\} \). Thus \( a_j = a_s \) for some \( s \neq r \). Now there are two cases:

**Case 1.** \( a_j \in \{i_1, \ldots, i_k\} \). Let \( a_j = i_k \). In this case a derangement \( x \) satisfies the condition \( x(i_j) = a_j \) if and only if the derangement \( x' \) of the set \( [n] \setminus \{i_k\} \) satisfies the condition \( x'(i_{j}) = a_{s} \) for all \( j \neq i_k \), where \( a_{j} = a_{j} \) for \( j \neq s \) and \( a_{s} = a_{i_k} \). This provides a one to one correspondence between the derangements \( x \) of \( [n] \) with \( x(i_j) = a_j \) and the given sets \( \{i_1, \ldots, i_k\} \) and \( \{a_1, \ldots, a_k\} \) with \( \ell \) elements in their intersections, and the derangements \( x' \) of \( [n] \setminus \{i_k\} \) with \( x_{j} = a_{j} \) for the given sets \( \{i_1, \ldots, i_{k-1}\} \setminus \{i_k\} \) and \( \{a_1, \ldots, a_{k-1}\} \setminus \{a_{s}\} \) with \( \ell - 1 \) elements in their intersections.

**Case 2.** \( a_j \notin \{i_1, \ldots, i_k\} \). In this case a derangement \( x \) satisfies the condition \( x(i_j) = a_j \) if and only if the derangement \( x' \) of the set \( [n] \setminus \{a_j\} \) satisfies the condition \( x'(i_{j}) = a_{s} \) for all \( j \neq s \). This provides a one to one correspondence between the derangements \( x \) of \( [n] \) with \( x(i_j) = a_j \) and the given sets \( \{i_1, \ldots, i_k\} \) and \( \{a_1, \ldots, a_k\} \) with \( \ell \) elements in their intersections, and the derangements \( x' \) of \( [n] \setminus \{a_j\} \) with \( x'(i_{j}) = a_{j} \) for the given sets \( \{i_1, \ldots, i_k\} \setminus \{a_j\} \) and \( \{a_1, \ldots, a_j\} \setminus \{a_{s}\} \) with \( \ell - 1 \) elements in their intersections.

These considerations show that \( \Delta(n,k,\ell) = \Delta(n-1,k-1,\ell-1) \). Iterating this argument, we have

\[
\Delta(n,k,\ell) = \Delta(n-1,k-1,\ell-1) = \Delta(n-2,k-2,\ell-2) = \cdots = \Delta(n-\ell,k-\ell,0).
\]

We can therefore assume that \( \ell = 0 \). We thus evaluate \( \Delta(n,k,0) \), where \( 2k \leq n \). For \( k = 0 \), we obviously have \( \Delta(n,0,0) = D_n \). For \( k \geq 1 \), we claim that
\[\Delta(n,k,0) = \Delta(n-1,k-1,0) + \Delta(n-2,k-1,0).\]

For a derangement \(x\) satisfying \(x(i) = a_i\) there are two cases: \(x(a_i) = i\) or \(x(a_i) \neq i\).

If the first case occurs then we have to evaluate the number of derangements of the set \([n]\) \(\setminus\{a_i\}\) for the given sets \(\{i_2,\ldots,i_k\}\) and \(\{a_2,\ldots,a_k\}\) with 0 elements in their intersections. The number is equal to \(\Delta(n-2,k-1,0)\).

If the second case occurs then we have to evaluate the number of derangements of the set \([n]\) \(\setminus\{a_i\}\) for the given sets \(\{i_2,\ldots,i_k\}\) and \(\{a_2,\ldots,a_k\}\) with 0 elements in their intersections. The number is equal to \(\Delta(n-1,k-1,0)\).

We now use induction on \(k\) to show that

\[
\Delta(n,k,0) = \sum_{i=0}^{k-1} \binom{k-1}{i} D_{\Delta(n-1-i,k-i)} n^{-k+i-1}. \\
2 \leq 2k \leq n.
\]

For \(k = 1\) we have

\[
\Delta(n,1,0) = \Delta(n-1,0,0) + \Delta(n-2,0,0) = D_{n-1} + D_{n-2} = \frac{D_n}{n-1}.
\]

Now let the result be true for \(k - 1\). We can write

\[
\Delta(n,k,0) = \Delta(n-1,k-1,0) + \Delta(n-2,k-1,0)
\]

\[
= \sum_{i=0}^{k-1} \binom{k-1}{i} D_{\Delta(n-1-i,k-i)} (n-1-2k+i) + \sum_{i=0}^{k-1} \binom{k-1}{i} D_{\Delta(n-1-i,k-i)} (n-2-k+i)
\]

\[
= \sum_{i=0}^{k-1} \binom{k-2}{i} D_{\Delta(n-1-i,k-i)} (n-k+1-i) + \sum_{i=0}^{k-1} \binom{k-2}{i} D_{\Delta(n-1-i,k-i)} (n-k+2+i)
\]

\[
= \frac{D_{\Delta(n-1-k+i)}}{n-k} + \sum_{i=1}^{k-1} \binom{k-2}{i} D_{\Delta(n-1-k+i)} (n-k+i) + \sum_{i=1}^{k-1} \binom{k-2}{i} D_{\Delta(n-1-k+i)} (n-k+i)
\]

\[
= \frac{D_{\Delta(n-1-k)+1}}{n-k} + \sum_{i=1}^{k-1} \binom{k-2}{i} D_{\Delta(n-1-k+i)} (n-k+i) + \sum_{i=1}^{k-1} \binom{k-2}{i} D_{\Delta(n-1-k+i)} (n-k+i)
\]

\[
= \sum_{i=0}^{k-1} \binom{k-1}{i} D_{\Delta(n-1-k+i)} n^{-k+i-1}
\]

**Corollary 1.** Let \(k\) be a positive integer. Then

\[
\sum_{i=0}^{k-1} \binom{k-1}{i} D_{\Delta(n-1-k+i)} = k!.
\]

**Proof.** Let \(n = 2k\), \(i_j = j\) and \(a_j = k+j\) for \(j = 1,\ldots,k\). Then a derangement \(x\) satisfies the condition \(x(i) = a_j\) if and only if \(x'\) defined by \(x'(i) = x(i) + i\) for \(i \in [k]\) is a permutation of \([k]\). The number of such permutations \(x'\) is \(k!\).

The following Table 1 gives some small values of \(\Delta(n,k,0)\).

The following lemma can be easily proved.

**Lemma 1.** Let \(x\) and \(y\) be two arbitrary permutations and \(m \geq 0\) be the number of \(i\)'s for which \(x(i) \neq y(i)\). Then there is a permutation \(z\) such that \(z(i) \neq i\) for \(i \leq m\) and \(z(i) = i\) for \(i > m\) and \(D_x(x,y) = D_z(x,z)\).

**Theorem 2.** Let \(0 \leq m \leq n\) and let \(z\) be a permutation such that \(z(i) \neq i\) for \(i \leq m\) and \(z(i) = i\) for \(i > m\). Then

\[
D_x(e,z) = \sum_{k=0}^{m} \sum_{1 \leq i_1 < \ldots < i_k \leq m} (-1)^k D_{\Delta(n,k,\ell(i_1,\ldots,i_k))},
\]
Table 1. Values of $\Delta(n,k,0)$ for $1 \leq n \leq 10$ and $1 \leq 2k \leq n$.

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where $\ell(i_1,\ldots,i_k) = |[i_1,\ldots,i_k] \cap [z(i_1),\ldots,z(i_k)]|$.

Proof. Let $E_i$ be the set of all derangements $x$ for which $x(i) = z(i)$, where $1 \leq i \leq m$. Then $D_n(e,z) = D_n - \bigcup_{i=1}^m E_i$. We use the inclusion-exclusion principle to determine $\bigcup_{i=1}^m E_i$. For each $0 \leq k \leq m$ and $1 \leq i_1 < \cdots < i_k \leq m$ we have

$$|E_{i_1} \cap \cdots \cap E_{i_k}| = \Delta(n,k,\ell(i_1,\ldots,i_k)),$$

where $\ell(i_1,\ldots,i_k) = |[i_1,\ldots,i_k] \cap [z(i_1),\ldots,z(i_k)]|$. This implies the result.

Our ultimate goal is to find an explicit formula for evaluating $D_n(e,c)$ for an arbitrary cycle $c$. Prior to that we need to state two elementary enumerative problems concerning subsets $A$ of the set $[n]$ with $k$ elements and exactly $\ell$ consecutive members.

Lemma 2. Let $S(n,k,\ell)$ be the number of subsets $A = \{a_1,\ldots,a_k\}$ of $[n]$ for which the equation $r = s + 1$ has exactly $\ell$ solutions for $r$ and $s$ in $A$. If $0 \leq \ell < k \leq n$ then

$$S(n,k,\ell) = \binom{n-k+1}{k-\ell} \binom{k-1}{\ell}.$$

Moreover, $S(n,0,0) = 1$ and $S(n,k,\ell) = 0$ for other values of $n,k,\ell$.

Proof. We can restate the problem as follows: We want to put $k$ ones and $n-k$ zeros in a row in such a way that there are exactly $\ell$ appearance of one-one. To do this we put $n-k$ zeros and choose $k-\ell$ places of the $n-k+1$ possible places for putting $k-\ell$ blocks of ones in $\binom{n-k+1}{k-\ell}$ ways. Let the number of ones in the $i$-th block be $r_i \geq 1$. We then must have $r_1 + \cdots + r_{\ell-1} = k$. The number of solutions for the latter equation is $\binom{k-1}{\ell}$.

Now suppose that we write $1,2,\ldots,n$ around a circle. We thus assume that 1 is after $n$ and so $n,1$ is also assumed to be consecutive. Under this assumption we have the following result.

Lemma 3. Let $C(n,k,\ell)$ be the number of subsets $A = \{a_1,\ldots,a_k\}$ of $[n]$ for which the equation $r = s + 1 \pmod{n}$ has exactly $\ell$ solutions for $r$ and $s$ in $A$. If $0 \leq \ell < k < n$ then

$$C(n,k,\ell) = n \binom{n-k-1}{k} \binom{k}{\ell}.$$

Moreover, $C(n,0,0) = C(n,n,n) = 1$ and $C(n,k,\ell) = 0$ for other values of $n,k,\ell$. 

Proof. Similar to the above argument, we want to put \( k \) ones and \( n-k \) zeros around a circle in such a way that there are exactly \( \ell \) appearances of one-one. At first, we put them in a row. There are four cases:

Case 1. There is no block of ones before the first zero and after the last zero. In this case we put \( n-k \) zeros and choose \( k-\ell \) places of the \( n-k-1 \) possible places for putting \( k-\ell \) blocks of ones in \( \binom{n-k-1}{k-\ell} \) ways. Let the number of ones in the \( i \)-th block be \( r_i \geq 1 \). We then must have \( r_1 + \cdots + r_{n-k} = k \). The number of solutions for the latter equation is \( \binom{k-1}{\ell} \).

Case 2. There is no block of ones before the first zero but there is a block after the last zero. In this case we put \( n-k \) zeros and choose \( 1 \) place of the \( n-k \) possible places for putting \( 1 \) block of ones in \( \binom{n-k-1}{1} \) ways. Let the number of ones in the \( i \)-th block be \( r_i \geq 1 \). We then must have \( r_1 = k \). The number of solutions for the latter equation is \( \binom{k-1}{\ell} \).

Case 3. There is a block of ones before the first zero but there is no block after the last zero. This is similar to the above case.

Case 4. There is a block of ones before the first zero and a block of ones after the last zero. In this case we must have \( \ell - 1 \) appearance of one-one in the row format, since we want to achieve \( \ell \) appearance of one-one in the circular format. Thus we put \( n-k \) zeros and choose \( k-(\ell-1)-2 \) places of the \( n-k-1 \) possible places for putting \( k-(\ell-1)-2 \) blocks of ones in \( \binom{n-k-1}{k-(\ell-1)-1} \) ways. Let the number of ones in the \( i \)-th block be \( r_i \geq 1 \). We then must have \( r_1 + \cdots + r_{k-(\ell-1)} = k \). The number of solutions for the latter equation is \( \binom{k-1}{\ell-1} \).

These considerations prove that

\[
C(n,k,\ell) = \binom{n-k-1}{k-\ell}\binom{k-1}{\ell} + 2\binom{n-k-1}{k-\ell-1}\binom{k-1}{\ell} + \binom{n-k-1}{k-\ell-1}\binom{k-1}{\ell-1}.
\]

A straightforward computation gives the result.

The following Table 2 gives some small values of \( C(10,k,\ell) \).

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Theorem 3. Let $c$ be a cycle of length $m \leq n$. Then
\[
D_n(e,c) = \sum_{0\leq i_1 \leq \cdots \leq i_m \leq n} (-1)^i \frac{C(m,k,\ell)}{C(n,k,\ell)} \Delta(n,k,\ell).
\]

Proof. Let $c_n$ be the cycle defined by $c_n(j) = j + 1$ for $1 \leq j \leq m - 1$, $c_n(m) = 1$ and $c_n(i) = i$ for $m + 1 \leq i \leq n$. Then $D_n(e,c) = D_n(e,c_n)$.

Using the notations of Theorem 2, $\ell(i_1,\ldots,i_k) = \ell$ if and only if the subset $A = \{i_1,\ldots,i_k\}$ of $[m]$ has exactly $\ell$ solutions for the equation $r = s + 1 \pmod{n}$ for $r,s \in A$. Thus the number of $\{i_1,\ldots,i_k\}$ with the property $\ell(i_1,\ldots,i_k) = \ell$ is $C(m,k,\ell)$. Applying Theorem 2, we have the result.

Example 1. We evaluate $D_5(e,c_5)$ and $D_5(e,c_3)$. Applying Theorem 3 with $m = 5$ we have
\[
D_5(e,c_5) = C(5,0,0)\Delta(5,0,0) - C(5,1,0)\Delta(5,1,0) + C(5,2,0)\Delta(5,2,0)
+ C(5,2,1)\Delta(5,2,1) - C(5,3,1)\Delta(5,3,1) - C(5,3,2)\Delta(5,3,2)
+ C(5,4,3)\Delta(5,4,3) - C(5,5,3)\Delta(5,5,3)
= C(5,0,0)\Delta(5,0,0) - C(5,1,0)\Delta(5,1,0) + C(5,2,0)\Delta(5,2,0)
+ C(5,2,1)\Delta(4,1,0) - C(5,3,1)\Delta(4,2,0) - C(5,3,2)\Delta(3,1,0)
+ C(5,4,3)\Delta(2,1,0) - C(5,5,3)\Delta(0,0,0)
\]
\[
= 1 \times 44 - 5 \times 11 + 5 \times 4 + 5 \times 3 - 5 \times 2 - 5 \times 1 + 5 \times 1 - 1 \times 1 = 13,
\]
and $\{(x(1),x(2),x(3),x(4),x(5))\}$ for the 13 double derangements $x$ with respect to $e$ and $c_5$ are
\[
(3,1,5,2,4),(3,4,5,1,2),(3,5,1,2,4),(4,3,5,2,4),(4,1,5,2,3),
(4,1,5,3,2),(4,5,1,2,3),(4,5,1,3,2),(4,5,2,1,3),(5,1,2,3,4),
(5,4,1,2,3),(5,4,1,3,2),(5,4,2,1,3).
\]

Applying Theorem 3 with $m = 3$ we have
\[
D_5(e,c_3) = C(3,0,0)\Delta(5,0,0) - C(3,1,0)\Delta(5,1,0)
+ C(3,2,1)\Delta(5,2,1) - C(3,3,1)\Delta(5,3,3)
= 1 \times 44 - 3 \times 11 + 3 \times 3 - 1 \times 1 = 19,
\]
and $\{(x(1),x(2),x(3),x(4),x(5))\}$ for the 19 double derangements with respect to $e$ and $c_3$ are
\[
(3,4,5,1,2),(3,5,4,1,2),(3,4,5,2,1),(3,5,4,2,1),(4,5,2,1,3),
(5,4,2,1,3),(4,5,2,3,1),(5,4,2,3,1),(4,1,5,2,3),(5,1,4,2,3),
(4,1,5,3,2),(5,1,4,3,2),(5,2,1,4),(3,4,2,5,1),(3,1,5,2,4),
(3,1,4,5,2),(5,1,2,3,4),(4,1,2,5,3),(3,1,2,5,4).
\]

The above example shows that how can we evaluate $D_n(e,c)$ for a cycle $c$. Moreover, Theorem 2 gives a formula for evaluating $D_n(e,z)$ for any permutation $z$. Applying Lemma 1, we can compute $D_n(x,y)$ for any permutations $x$ and $y$.

References