Edge-Vertex Dominating Sets and Edge-Vertex Domination Polynomials of Cycles

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Abstract

Let $G = (V, E)$ be a simple graph. A set $S \subseteq E(G)$ is an edge-vertex dominating set of $G$ (or simply an ev-dominating set), if for all vertices $v \in V(G)$; there exists an edge $e \in S$ such that $e$ dominates $v$. Let $D_{ev}(C_n, i)$ denote the family of all ev-dominating sets of $C_n$ with cardinality $i$. Let $d_{ev}(C_n, i) = |D_{ev}(C_n, i)|$. In this paper, we obtain a recursive formula for $d_{ev}(C_n, i)$. Using this recursive formula, we construct the polynomial, $D_{ev}(C_n, x) = \sum_{i=1}^{n} d_{ev}(C_n, i) x^i$, which we call edge-vertex domination polynomial of $C_n$ (or simply an ev-domination polynomial of $C_n$) and obtain some properties of this polynomial.

Keywords

ev-Domination Set, ev-Domination Number, ev-Domination Polynomials

1. Introduction

Let $G = (V, E)$ be a simple graph of order $|V| = n$. A set $S \subseteq V(G)$ is a dominating set of $G$, if every vertex $v \in S$ is adjacent to at least one vertex in $S$. For any vertex $v \in V$, the open neighbourhood of $v$ is the set $N(v) = \{u \in V \mid uv \in E\}$ and the closed neighbourhood of $v$ is the set $\overline{N}(v) = N(v) \cup \{v\}$. For a set $S \subseteq V$, the open neighbourhood of $S$ is $N(S) = \bigcup_{v \in S} N(v)$ and the closed neighbourhood of $S$ is $\overline{N}(S) = N(S) \cup S$. The domination number of a graph $G$ is defined as the minimum size of a dominating set in $G$ and it is denoted as $\gamma(G)$. A cycle is defined as a closed path, and is denoted by $C_n$.
Definition 1.1
For a graph $G = (V, E)$, an edge $e = uv \in E(G)$, $ev$-dominates a vertex $w \in V(G)$ if
1) $u = w$ or $v = w$ ($w$ is incident to $e$) or
2) $uv$ or $vu$ is an edge in $G$ ($w$ is adjacent to $u$ or $v$).

Definition 1.2 [1]
A set $S \subseteq E(G)$ is an edge-vertex dominating set of $G$ (or simply an $ev$-dominating set), if for all vertices $v \in V(G)$, there exists an edge $e \in S$ such that $e$ dominates $v$. The $ev$-domination number of a graph $G$ is defined as the minimum size of an $ev$-dominating set of edges in $G$ and it is denoted as $\gamma_{ev}(G)$.

Definition 1.3
Let $D_{ev}(C_n, i)$ be the family of $ev$-dominating sets of a graph $C_n$ with cardinality $i$ and let $d_{ev}(C_n, i) = |D_{ev}(C_n, i)|$. We call the polynomial $D_{ev}(C_n, x) = \sum_{i=1}^{n} d_{ev}(C_n, i) x^{i}$ the $ev$-domination polynomial of the graph $C_n$.

In the next section, we construct the families of the $ev$-dominating sets of cycles by recursive method. As usual we use $\lfloor x \rfloor$ for the largest integer less than or equal to $x$ and $\lceil x \rceil$ for the smallest integer greater than or equal to $x$. Also, we denote the set $\{e_1, e_2, \ldots, e_s\}$ by $[e_s]$ and the set $\{1, 2, \ldots, n\}$ by $[n]$, throughout this paper.

2. Edge-Vertex Dominating Sets of Cycles

Let $D_{ev}(C_n, i)$ be the family of $ev$-dominating sets of $C_n$ with cardinality $i$. We investigate the $ev$-dominating sets of $C_n$. We need the following lemma to prove our main results in this section.

Lemma 2.1: [2] $\gamma_{ev}(C_n) = \left\lceil \frac{n}{4} \right\rceil$.

By Lemma 2.1 and the definition of $ev$-domination number, one has the following Lemma:

Lemma 2.2: $D_{ev}(C_n, i) = \Phi$ if and only if $i > n$ or $i < \left\lceil \frac{n}{4} \right\rceil$.

Lemma 2.3: If a graph $G$ contains a simple path of length $4k - 1$, then every $ev$-dominating set of $G$ must contain at least $k$ vertices of the path.

Proof: The path has $4k$ vertices. As every edge dominates at most 4 vertices, the $4k$ vertices are covered by at least $k$ edges.

Lemma 2.4. If $Y \subseteq D_{ev}(C_{4i-3}, i-1)$, and there exists $x \in [e_3]$ such that $Y \cup \{x\} \subseteq D_{ev}(C_n, i)$ then $Y \subseteq D_{ev}(C_{4i-3}, i-1)$.

Proof: Suppose that $Y \not\subseteq D_{ev}(C_{4i-3}, i-1)$. Since $Y \subseteq D_{ev}(C_{4i-3}, i-1)$, $Y$ contains at least one edge labelled $e_{4i-3}, e_{4i-6}$ or $e_{4i-7}$.

1) If $e_{4i-3} \in Y$, then $Y \subseteq D_{ev}(C_{4i-3}, i-1)$, a contradiction. Hence $e_{4i-3}$ or $e_{4i-7} \in Y$, but then in this case $Y \cup \{x\} \in D_{ev}(C_n, i)$ for any $x \in [e_3]$, also a contradiction.

Lemma 2.5. [3]

1) If $D_{ev}(C_{4i-3}, i-1) = D_{ev}(C_{4i-3}, i-1) = \Phi$ then $D_{ev}(C_{4i-3}, i-1) = D_{ev}(C_{4i-3}, i-1) = \Phi$.
2) If $D_{ev}(C_{4i-3}, i-1) \not\subseteq \Phi$, and $D_{ev}(C_{4i-3}, i-1) \not\subseteq \Phi$ then $D_{ev}(C_{4i-3}, i-1) \not\subseteq \Phi$ and $D_{ev}(C_{4i-3}, i-1) \not\subseteq \Phi$.
3) If $D_{ev}(C_{4i-3}, i-1) = \Phi$, $D_{ev}(C_{4i-3}, i-1) = \Phi$, $D_{ev}(C_{4i-3}, i-1) = \Phi$, and $D_{ev}(C_{4i-3}, i-1) = \Phi$, then $D_{ev}(C_{4i-3}, i-1) = \Phi$.

Proof: 1) Since $D_{ev}(C_{4i-3}, i-1) = D_{ev}(C_{4i-3}, i-1) = \Phi$, by Lemma 2.2, $i-1 > n-1$ or $i-1 < \left\lfloor \frac{n-4}{4} \right\rfloor$. In either case, we have $D_{ev}(C_{4i-3}, i-1) = \Phi$ and $D_{ev}(C_{4i-3}, i-1) = \Phi$.

2) Since $D_{ev}(C_{4i-3}, i-1) \not\subseteq \Phi$ and $D_{ev}(C_{4i-3}, i-1) \not\subseteq \Phi$, by Lemma 2.2, we have $\left\lfloor \frac{n-1}{4} \right\rfloor \leq i-1 \leq n-1$ and $\left\lfloor \frac{n-3}{4} \right\rfloor \leq i-1 \leq n-3$. Therefore,
Lemma 2.6. [4] If \( D_{\nu}(C_n,i) \neq \Phi \), then

1) \( D_{\nu}(C_{n-1},i-1) = D_{\nu}(C_{n-2},i-1) = D_{\nu}(C_{n-3},i-1) = \Phi \) and \( D_{\nu}(C_{n-4},i-1) = \Phi \) if and only if \( n = 4k \) and \( i = k \) for some \( k \in \mathbb{N} \).

2) \( D_{\nu}(C_{n-1},i-1) = D_{\nu}(C_{n-3},i-1) = D_{\nu}(C_{n-4},i-1) = \Phi \) and \( D_{\nu}(C_{n-1},i-1) = \Phi \) if and only if \( i = n \).

3) \( D_{\nu}(C_{n-1},i-1) = \Phi \), \( D_{\nu}(C_{n-2},i-1) = \Phi \), \( D_{\nu}(C_{n-3},i-1) = \Phi \), \( D_{\nu}(C_{n-4},i-1) = \Phi \), if and only if \( n = 4k + 2 \) and \( i = \left[ \frac{4k + 2}{4} \right] \) for some \( k \in \mathbb{N} \).

4) \( D_{\nu}(C_{n-1},i-1) = \Phi \), \( D_{\nu}(C_{n-2},i-1) = \Phi \), \( D_{\nu}(C_{n-3},i-1) = \Phi \) and \( D_{\nu}(C_{n-4},i-1) = \Phi \) if and only if \( i = n - 2 \).

5) \( D_{\nu}(C_{n-1},i-1) = \Phi \), \( D_{\nu}(C_{n-2},i-1) = \Phi \), \( D_{\nu}(C_{n-3},i-1) = \Phi \) and \( D_{\nu}(C_{n-4},i-1) = \Phi \) if and only if \( i = n - 1 \).

6) \( D_{\nu}(C_{n-1},i-1) = \Phi \), \( D_{\nu}(C_{n-2},i-1) = \Phi \), \( D_{\nu}(C_{n-3},i-1) = \Phi \) and \( D_{\nu}(C_{n-4},i-1) = \Phi \) if and only if \( \frac{n - 3}{4} + 1 \leq i \leq n - 3 \).

Proof: 1) \( \Rightarrow \) Since \( D_{\nu}(C_{n-1},i-1) = D_{\nu}(C_{n-2},i-1) = D_{\nu}(C_{n-3},i-1) = \Phi \), by Lemma 2.2, \( i - 1 > n - 1 \) or \( i - 1 < \left[ \frac{n - 3}{4} \right] \). If \( i - 1 > n - 1 \), then \( i > n \) and by Lemma 2.2, \( D_{\nu}(C_n,i) = \Phi \), a contradiction.

So

\[ i - 1 < \left[ \frac{n - 3}{4} \right] \]  \hspace{1cm} (2.1)

and since \( D_{\nu}(C_{n-4},i-1) \neq \Phi \), we have

\[ \left[ \frac{n - 4}{4} \right] \leq i - 1 \leq n - 4 \]  \hspace{1cm} (2.2)

From (2.1) and (2.2),

\[ \left[ \frac{n - 4}{4} \right] \leq i - 1 < \frac{n - 3}{4} \]  \hspace{1cm} (2.3)

When \( n \) is a multiple of 4, \( \frac{n - 4}{4} = \frac{n}{4} - 1 \) and \( \frac{n - 3}{4} = \frac{n}{4} \). Therefore, \( \frac{n}{4} - 1 \leq i - 1 < \frac{n}{4} \). Therefore, \( i - 1 = \frac{n}{4} - 1 \), we get \( i = \frac{n}{4} \). Thus, when \( n = 4k \), (2.3) holds good and \( i = \frac{n}{4} = k \). When \( n \neq 4k \),

\[ \frac{n - 4}{4} = \frac{n}{4} - 1 \text{ and } \frac{n - 3}{4} = \frac{n}{4} - 1. \]

Therefore, \( \frac{n}{4} - 1 \leq i - 1 < \frac{n}{4} - 1 \), which is not possible.

Hence \( n = 4k \) and \( i = k \)

\( \Leftarrow \) If \( n = 4k \) and \( i = k \) for some \( k \in \mathbb{N} \), then by Lemma 2.2, \( \frac{n - 1}{4} = \frac{4k - 1}{4} = k \). Hence, \( i = k > i - 1 \). Therefore, \( i - 1 < \left[ \frac{n - 1}{4} \right] \), which implies \( D_{\nu}(C_{n-4},i-1) = \Phi \). Similarly, \( D_{\nu}(C_{n-2},i-1) = \Phi \) and \( D_{\nu}(C_{n-3},i-1) = \Phi \).
Now \[
\left\lceil \frac{n-4}{4} \right\rceil = \left\lfloor \frac{4k-4}{4} \right\rfloor = k-1 = i-1. \text{ Therefore, } \left\lceil \frac{n-4}{4} \right\rceil \leq i-1, \text{ which implies } D_{e\nu}(C_{n-4}, i-1) \neq \Phi. 
\]

2) \((\Rightarrow)\) Since \( D_{e\nu}(C_{n-2}, i-1) = D_{e\nu}(C_{n-3}, i-1) = D_{e\nu}(C_{n-4}, i-1) = \Phi \), by Lemma 2.2, \( i-1 > n-2 \) or \( i-1 < \left\lceil \frac{n-4}{4} \right\rceil \). If \( i-1 < \left\lceil \frac{n-4}{4} \right\rceil \) then by Lemma 2.2, \( D_{e\nu}(C_{n-1}, i-1) = \Phi \), a contradiction.

So
\[
i-1 > n-2 \quad \quad \quad (2.4)
\]

Since,
\[
D_{e\nu}(C_{n-1}, i-1) \neq \Phi, \quad \left\lceil \frac{n-1}{4} \right\rceil \leq i-1 \leq n-1 \quad \quad \quad (2.5)
\]

From (2.4) and (2.5), we have \( n-1 \geq i-1 > n-2 \). Therefore, \( i-1 = n-1 \). Therefore, \( i = n \)

\((\Leftarrow)\) If \( i = n \), then by Lemma 2.2, \( \left\lceil \frac{n}{4} \right\rceil \leq i \leq n \). Therefore, \( \left\lceil \frac{n-1}{4} \right\rceil \leq i-1 \leq n-1 \)

Therefore, \( D_{e\nu}(C_{n-1}, i-1) \neq \Phi \)

3) \((\Rightarrow)\) Since \( D_{e\nu}(C_{n-1}, i-1) = \Phi \), by Lemma 2.2, \( i-1 > n-1 \) or \( i-1 < \left\lceil \frac{n-1}{4} \right\rceil \). If \( i-1 > n-1 \), then \( i-1 > n-2 > n-3 > n-4 \), by Lemma 2.2, \( D_{e\nu}(C_{n-2}, i-1) = D_{e\nu}(C_{n-3}, i-1) = D_{e\nu}(C_{n-4}, i-1) = \Phi \), a contradiction.

Therefore, \( i-1 < \left\lceil \frac{n-1}{4} \right\rceil \), which implies,
\[
i < \left\lceil \frac{n-1}{4} \right\rceil + 1 \quad \quad \quad (2.6)
\]

Since, \( D_{e\nu}(C_{n-2}, i-1) \neq \Phi, \quad \left\lceil \frac{n-2}{4} \right\rceil \leq i-1 \leq n-2 \).

Hence,
\[
\left\lceil \frac{n-2}{4} \right\rceil + 1 \leq i \leq n-1 \quad \quad \quad (2.7)
\]

Similarly,
\[
\left\lceil \frac{n-3}{4} \right\rceil + 1 \leq i \leq n-2 \quad \quad \quad (2.8)
\]

and
\[
\left\lceil \frac{n-4}{4} \right\rceil + 1 \leq i \leq n-3 \quad \quad \quad (2.9)
\]

From (2.6), (2.7), (2.8) and (2.9),
\[
\left\lceil \frac{n-2}{4} \right\rceil + 1 \leq i < \left\lceil \frac{n-1}{4} \right\rceil + 1 \quad \quad \quad (2.10)
\]

Therefore, (2.10) hold when \( k = \frac{n-2}{4} \) or \( n = 4k+2 \) and \( i = k+1 = \left\lceil \frac{4k+2}{4} \right\rceil \), for some \( k \in N \). Suppose \( n = 4k+2 \), then \( \left\lceil \frac{n-2}{4} \right\rceil + 1 = k+1 \) and \( \left\lceil \frac{n-1}{4} \right\rceil + 1 = k+2 \). Therefore, from (2.10), we have, \( k+1 \leq i < k+2 \), which implies \( i = k+1 \). Suppose \( n \neq 4k+2 \) i.e., \( n = 4k, 4k+1, 4k+3 \).
Case 1) When \( n = 4k \).
From (2.10), we get \( \left\lfloor \frac{4k - 2}{4} \right\rfloor + 1 = k + 1 \) and \( \left\lfloor \frac{4k - 1}{4} \right\rfloor + 1 = k + 1 \). Therefore, \( k + 1 \leq i < k + 1 \), which is not possible.

Case 2) When \( n = 4k + 1 \). From (2.10), we get \( \left\lfloor \frac{4k + 1 - 2}{4} \right\rfloor + 1 = k + 1 \) and \( \left\lfloor \frac{4k + 1 - 1}{4} \right\rfloor + 1 = k + 1 \). Therefore, \( k + 1 \leq i < k + 1 \), which is not possible.

Case 3) When \( n = 4k + 3 \). From (2.10), we get \( \left\lfloor \frac{4k + 3 - 2}{4} \right\rfloor + 1 = k + 2 \) and \( \left\lfloor \frac{4k + 3 - 1}{4} \right\rfloor + 1 = k + 2 \). Therefore, \( k + 2 \leq i < k + 2 \), which is not possible. Therefore, \( n = 4k + 2 \)

(\( \Leftarrow \)) If \( n = 4k + 2 \) and \( i = \left\lfloor \frac{4k + 2}{4} \right\rfloor \) for some \( k \in \mathbb{N} \), and \( D_{\alpha}(C_{n}, i) \neq \Phi \), then by Lemma 2.2, \( \left\lfloor \frac{n}{4} \right\rfloor \leq i \leq n \), \( \left\lfloor \frac{n}{4} \right\rfloor = \left\lfloor \frac{4k + 2}{4} \right\rfloor = i > i - 1 \). Therefore, \( i - 1 < \left\lfloor \frac{n - 1}{4} \right\rfloor \). Therefore, \( D_{\alpha}(C_{n - 1}, i - 1) = \Phi \).

Also, \( \left\lfloor \frac{n - 2}{4} \right\rfloor = k \). Therefore, \( \left\lfloor \frac{n - 2}{4} \right\rfloor \leq i - 1 \leq n - 2 \) and \( \left\lfloor \frac{n - 3}{4} \right\rfloor \leq i - 1 \leq n - 3 \) and \( \left\lfloor \frac{n - 4}{4} \right\rfloor \leq i - 1 \leq n - 4 \). Hence \( D_{\alpha}(C_{n - 2}, i - 1) \neq \Phi \), \( D_{\alpha}(C_{n - 3}, i - 1) \neq \Phi \), \( D_{\alpha}(C_{n - 4}, i - 1) \neq \Phi \).

4) (\( \Rightarrow \)) Since \( D_{\alpha}(C_{n}, i) \neq \Phi \), by Lemma 2.2,

\[
i - 1 > n - 4 \text{ or } i - 1 < \left\lfloor \frac{n - 4}{4} \right\rfloor
\]
(2.11)

Since \( D_{\alpha}(C_{n - 3}, i - 1) \neq \Phi \), by Lemma 2.2,

\[
\left\lfloor \frac{n - 3}{4} \right\rfloor \leq i - 1 \leq n - 3
\]
(2.12)

Similarly, \( D_{\alpha}(C_{n - 2}, i - 1) \neq \Phi \) and \( D_{\alpha}(C_{n - 1}, i - 1) \neq \Phi \), by Lemma 2.2

\[
\left\lfloor \frac{n - 2}{4} \right\rfloor \leq i - 1 \leq n - 2
\]
(2.13)

and

\[
\left\lfloor \frac{n - 1}{4} \right\rfloor \leq i - 1 \leq n - 1
\]
(2.14)

From (2.11), we get \( i - 1 < \left\lfloor \frac{n - 4}{4} \right\rfloor \) which is not possible.

Therefore, \( i - 1 > n - 4 \Rightarrow i > n - 3 \Rightarrow i \geq n - 2 \) \( \quad \text{(2.15)} \)

From (2.12), \( i - 1 \leq n - 3 \Rightarrow i \leq n - 2 \) \( \quad \text{(2.16)} \)

From (2.15) and (2.16), \( i = n - 2 \)

(\( \Leftarrow \)) If \( i = n - 2 \), \( i - 1 = n - 3 \) then by Lemma 2.2, \( i - 1 > n - 4 \) or \( i - 1 < \left\lfloor \frac{n - 4}{4} \right\rfloor \). Therefore, \( D_{\alpha}(C_{n - 4}, i - 1) = \Phi \). Also \( \left\lfloor \frac{n - 1}{4} \right\rfloor \leq i - 1 \leq n - 1 \), therefore, \( D_{\alpha}(C_{n - 1}, i - 1) \neq \Phi \); \( \left\lfloor \frac{n - 2}{4} \right\rfloor \leq i - 1 \leq n - 2 \), therefore, \( D_{\alpha}(C_{n - 2}, i - 1) \neq \Phi \); and \( \left\lfloor \frac{n - 3}{4} \right\rfloor \leq i - 1 \leq n - 3 \), therefore, \( D_{\alpha}(C_{n - 3}, i - 1) \neq \Phi \).

5) (\( \Rightarrow \)) Since \( D_{\alpha}(C_{n}, i - 1) = \Phi \) by Lemma 2.2,
i−1 > n−4 or i−1 < n−4
\[ (2.17) \]

Since \( D_{ev}(C_{n-3}, i-1) = \Phi \) by Lemma 2.2,

\[ i−1 > n−3 \text{ or } i−1 < n−3 \]
\[ (2.18) \]

Since \( D_{ev}(C_{n-2}, i-1) \neq \Phi \) and \( D_{ev}(C_{n-1}, i-1) \neq \Phi \), by Lemma 2.2,

\[ \left[ \frac{n−2}{4} \right] \leq i−1 \leq n−2 \]
\[ (2.19) \]

and

\[ \left[ \frac{n−1}{4} \right] \leq i−1 \leq n−1 \]
\[ (2.20) \]

From (2.19) and (2.20), we have \( \left[ \frac{n−1}{4} \right] \leq i−1 \leq n−2 \). From (2.18), we have \( i−1 > n−3 \). Therefore, \( i−1 \geq n−2 \). But \( i−1 \leq n−2 \). Therefore, \( i = n−1 \).

(\( \Leftarrow \)) If \( i = n−1 \), \( i−1 = n−2 \) then by Lemma 2.2, \( D_{ev}(C_{n-3}, i-1) = \Phi \) and \( i−1 > n−4 \) therefore, \( D_{ev}(C_{n-4}, i-1) = \Phi \) and \( \left[ \frac{n−1}{4} \right] \leq i−1 \leq n−1 \), therefore, \( D_{ev}(C_{n-1}, i-1) \neq \Phi \) and \( \left[ \frac{n−2}{4} \right] \leq i−2 \leq n−2 \), therefore, \( D_{ev}(C_{n-2}, i-1) \neq \Phi \).

6) (\( \Rightarrow \)) Since \( D_{ev}(C_{n-1}, i-1) \neq \Phi \), \( D_{ev}(C_{n-2}, i-1) \neq \Phi \), \( D_{ev}(C_{n-3}, i-1) \neq \Phi \), and \( D_{ev}(C_{n-4}, i-1) \neq \Phi \), by Lemma 2.2, \( \left[ \frac{n−1}{4} \right] \leq i−1 \leq n−1 \), \( \left[ \frac{n−2}{4} \right] \leq i−1 \leq n−2 \), \( \left[ \frac{n−3}{4} \right] \leq i−1 \leq n−3 \), and \( \left[ \frac{n−4}{4} \right] \leq i−1 \leq n−4 \).

So \( \left[ \frac{n−1}{4} \right] \leq i−1 \leq n−4 \) and hence \( \left[ \frac{n−1}{4} \right] +1 \leq i \leq n−3 \).

(\( \Leftarrow \)) If \( \left[ \frac{n−1}{4} \right] +1 \leq i \leq n−3 \), then by Lemma 2.2 we have, \( \left[ \frac{n−1}{4} \right] \leq i−1 \leq n−1 \), \( \left[ \frac{n−2}{4} \right] \leq i−1 \leq n−2 \), \( \left[ \frac{n−3}{4} \right] \leq i−1 \leq n−3 \), and \( \left[ \frac{n−4}{4} \right] \leq i−1 \leq n−4 \).

Therefore, \( D_{ev}(C_{n-1}, i-1) \neq \Phi \), \( D_{ev}(C_{n-2}, i-1) \neq \Phi \), \( D_{ev}(C_{n-3}, i-1) \neq \Phi \), and \( D_{ev}(C_{n-4}, i-1) \neq \Phi \).

Theorem 2.7 [5]

For every \( n \geq 5 \) and \( i \geq \left[ \frac{n}{4} \right] \),

1) If \( D_{ev}(C_{n-1}, i-1) = D_{ev}(C_{n-2}, i-1) = D_{ev}(C_{n-3}, i-1) = \Phi \) and \( D_{ev}(C_{n-4}, i-1) \neq \Phi \) then

\[ D_{ev}(C_{n}, i) = \{e_1, e_5, \ldots, e_{n-3} \}, \{e_2, e_6, \ldots, e_{n-2} \}, \{e_3, e_7, \ldots, e_{n-1} \}, \{e_4, e_8, \ldots, e_n \} \].

2) If \( D_{ev}(C_{n-2}, i-1) = D_{ev}(C_{n-1}, i-1) = \Phi \) and \( D_{ev}(C_{n-3}, i-1) \neq \Phi \) then

\[ D_{ev}(C_{n}, i) = \{e_n \} \].

3) If \( D_{ev}(C_{n-1}, i-1) = \Phi, D_{ev}(C_{n-2}, i-1) = \Phi, D_{ev}(C_{n-3}, i-1) \neq \Phi \) and \( D_{ev}(C_{n-4}, i-1) \neq \Phi \) then

\[ D_{ev}(C_{n}, i) = Y_1 \cup Y_2 \cup Y_3 \], where

\[ Y_1 = \{e_1, e_5, \ldots, e_{n-3}, e_1 \}, \{e_2, e_6, \ldots, e_{n-2}, e_2 \}, \{e_3, e_7, \ldots, e_{n-1}, e_3 \}, \{e_4, e_8, \ldots, e_n, e_4 \} \}

\[ Y_2 = \begin{cases} X_2 \cup \{e_n \}, & \text{if } e_1 \in X_2, X_2 \in D_{ev}(C_{n-3}, i-1) \\ \{e_n \}, & \text{otherwise} \end{cases} \]

\[ Y_3 = \begin{cases} X_2 \cup \{e_n \}, & \text{if } e_2 \in X_2, X_2 \in D_{ev}(C_{n-3}, i-1) \\ \{e_n \}, & \text{otherwise} \end{cases} \]
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\[ Y_2 = \begin{cases} \quad \{e_i\}, & \text{if } e_i \in X_i / X_2 \in D_{e_{n,i}}(C_{n,i-1}) \\ \{e_{i+1}\}, & \text{if } e_i \in X_2 / X_2 \in D_{e_{n,i}}(C_{n,i-1}) \\ \{e_{i+2}\}, & \text{otherwise} \end{cases} \]

4) If \( D_{e_{n,i}}(C_{n-1,i-1}) = \Phi \), \( D_{e_{n,i}}(C_{n-2,i-1}) \neq \Phi \), and \( D_{e_{n,i}}(C_{n-4,i-1}) \neq \Phi \) then
\[ D_{e_{n,i}}(C_{n,i}) = \{e_n\} \]

5) If \( D_{e_{n,i}}(C_{n-1,i-1}) \neq \Phi \), \( D_{e_{n,i}}(C_{n-2,i-1}) \neq \Phi \), \( D_{e_{n,i}}(C_{n-3,i-1}) \neq \Phi \) and \( D_{e_{n,i}}(C_{n-4,i-1}) \neq \Phi \) then
\[ D_{e_{n,i}}(C_{n,i}) = \left\{ \{X_1 \cup \{e_i\} / X_1 \in D_{e_{n,i}}(C_{n-1,i-1})\} \cup \{X_1 \cup \{e_{n-1}\} / X_2 \in D_{e_{n,i}}(C_{n-2,i-1})\} \cup \{X_1 \cup \{e_{n-2}\} / X_3 \in D_{e_{n,i}}(C_{n-3,i-1})\} \cup \{X_1 \cup \{e_{n-3}\} / X_4 \in D_{e_{n,i}}(C_{n-4,i-1})\} \right\} \]

**Proof:**

1) Since, \( D_{e_{n,i}}(C_{n-1,i-1}) = D_{e_{n,i}}(C_{n-2,i-1}) = D_{e_{n,i}}(C_{n-3,i-1}) = \Phi \) and \( D_{e_{n,i}}(C_{n-4,i-1}) \neq \Phi \), by Lemma 2.6 (i)
\[ n = 4k \] and \( i = k \) for some \( k \in N \). The sets \( \{e_1,e_2,\ldots,e_{n-3}\}, \{e_2,e_3,\ldots,e_{n-2}\}, \{e_3,e_4,\ldots,e_{n-1}\}, \{e_4,e_5,\ldots,e_n\} \)
have \( \frac{n}{4} \) elements and each one covers all vertices. Also, no other sets of cardinality \( \frac{n}{4} \) covers all vertices.

Therefore, the collection of \( ev \)-dominating sets of cardinality \( \frac{n}{4} \) is
\[ \{\{e_1,e_2,\ldots,e_{n-3}\}, \{e_2,e_3,\ldots,e_{n-2}\}, \{e_3,e_4,\ldots,e_{n-1}\}, \{e_4,e_5,\ldots,e_n\}\} \]

Hence, \( D_{e_{n,i}}(C_{n,i}) = \{e_i\} \).

2) We have \( D_{e_{n,i}}(C_{n-2,i-1}) = D_{e_{n,i}}(C_{n-3,i-1}) = \Phi \) and \( D_{e_{n,i}}(C_{n-4,i-1}) \neq \Phi \). By Lemma 2.6 (2), we have \( i = n \). So, \( D_{e_{n,i}}(C_{n,i}) = \{e_n\} \).

3) We have \( D_{e_{n,i}}(C_{n-1,i-1}) = \Phi \), \( D_{e_{n,i}}(C_{n-2,i-1}) \neq \Phi \), \( D_{e_{n,i}}(C_{n-3,i-1}) \neq \Phi \) and \( D_{e_{n,i}}(C_{n-4,i-1}) \neq \Phi \), by Lemma 2.6 (3), \( n = 4k + 2 \) and \( i = \left\lfloor \frac{4k+2}{4} \right\rfloor = k+1 \), for some \( k \in N \).

Let \( Y_1 = \{\{e_1,e_2,\ldots,e_{4k+1}\}, \{e_2,e_3,\ldots,e_{4k+2}\}, \{e_3,e_4,\ldots,e_{4k+3}\}, \{e_4,e_5,\ldots,e_{4k+4}\}\} \)
\[ Y_2 = \left\{ \begin{aligned} \{e_1\}, & \text{if } e_1 \in X_2 / X_2 \in D_{e_{4k+1}}(C_{4k+1}) \\ \{e_{i+2}\}, & \text{otherwise} \end{aligned} \right\} \]
\[ Y_3 = \left\{ \begin{aligned} \{e_{i+1}\}, & \text{if } e_i \in X_2 / X_2 \in D_{e_{4k+2}}(C_{4k+2}) \\ \{e_{i+2}\}, & \text{otherwise} \end{aligned} \right\} \]
\[ Y_4 = \left\{ \begin{aligned} \{e_{i+1}\}, & \text{if } e_i \in X_3 / X_3 \in D_{e_{4k+2}}(C_{4k+2}) \\ \{e_{i+2}\}, & \text{otherwise} \end{aligned} \right\} \]

We shall prove that \( D_{e_{n,i}}(C_{4k+2,k+1}) = Y_1 \cup Y_2 \cup Y_3 \). It is clear that \( Y_1 \subseteq D_{e_{n,i}}(C_{4k+2,k+1}) \),\( Y_2 \subseteq D_{e_{n,i}}(C_{4k+2,k+1}) \), and \( Y_3 \subseteq D_{e_{n,i}}(C_{4k+2,k+1}) \). Therefore, \( Y_1 \cup Y_2 \cup Y_3 \subseteq D_{e_{n,i}}(C_{4k+2,k+1}) \).

Conversely, let \( Y \in D_{e_{n,i}}(C_{4k+2,k+1}) \). Suppose, \( Y \) is of the form \( \{e_1,e_2,\ldots,e_{4k+1}\} \), \( \{e_2,e_3,\ldots,e_{4k+2}\} \), \( \{e_3,e_4,\ldots,e_{4k+3}\} \), \( \{e_4,e_5,\ldots,e_{4k+4}\} \) then \( Y \in Y_1 \subseteq D_{e_{n,i}}(C_{4k+2,k+1}) \). Now suppose,
Y \not\in Y_i$ and $Y$ is of the form \[ X_2 \cup \begin{cases} \{e_4\}, & \text{if } e_4 \in X_2 / X_2 \in D_{ev} (C_{4k+1}, k) \\ \{e_{4k+1}\}, & \text{if } e_{4k+1} \in X_2 / X_2 \in D_{ev} (C_{4k+1}, k) \\ \{e_{4k+2}\}, & \text{otherwise} \end{cases} \] then $Y \in Y_2 \subseteq D_{ev} (C_{4k+1}, k)$.

Now suppose, $Y \not\in Y_1$, $Y \not\in Y_2$ and $Y \in D_{ev} (C_{4k+2}, k+1)$. We split $D_{ev} (C_{4k+2}, k)$ as four parts. If $X_1 \in D_{ev} (C_{4k+2}, k)$ with $e_1 \in X_1$ then $X_1 \cup \{e_{4k+1}\} \in Y_1$ and $X_1 \cup \{e_{4k+2}\} \not\subseteq Y_2$ and $\not\subseteq Y_i$. If $X_2 \in D_{ev} (C_{4k+2}, k)$ with $e_2 \in X_2$ then $X_2 \cup \{e_4\} \in Y_2$ and $X_2 \cup \{e_{4k+1}\} \not\subseteq Y_2$ and $\not\subseteq Y_i$. If $X_3 \in D_{ev} (C_{4k+2}, k)$ with $e_3 \in X_3$ then $X_3 \cup \{e_{4k+1}\} \in Y_1$ and $X_3 \cup \{e_{4k+2}\} \not\subseteq Y_2$ and $\not\subseteq Y_i$. If $X_4 \in D_{ev} (C_{4k+2}, k)$ with $e_4 \in X_4$ then $X_4 \cup \{e_{4k+1}\} \subseteq Y_1$ and $X_4 \cup \{e_{4k+2}\} \not\subseteq Y_2$ and $\not\subseteq Y_i$. In this case $Y$ is of the form $X_4 \cup \begin{cases} \{e_{4k+1}\}, & \text{if } e_{4k+1} \in X_4 / X_4 \in D_{ev} (C_{4k+2}, k) \\ \{e_{4k+2}\}, & \text{if } e_{4k+2} \in X_4 / X_4 \in D_{ev} (C_{4k+2}, k) \\ \{e_4\}, & \text{otherwise} \end{cases}$ then $Y \in Y_3 \in D_{ev} (C_{4k+2}, k)$. Therefore, $D_{ev} (C_{4k+2}, k+1) \subseteq Y_1 \cup Y_2 \cup Y_3$. Thus, we have proved that

$$D_{ev} (C_{n,i}) = Y_1 \cup Y_2 \cup Y_3$$

where $Y_i = \{\{e_1, e_2, \ldots, e_{n-5}, e_{n+1}\}, \{e_2, e_3, \ldots, e_{n-4}, e_n\}, \{e_3, e_4, \ldots, e_{n-3}, e_{n-1}\}, \{e_4, e_5, \ldots, e_{n-2}, e_n\}\}$

4) If $D_{ev} (C_{n-3}, i-1) = \Phi$, $D_{ev} (C_{n-2}, i-1) \neq \Phi$, and $D_{ev} (C_{n-1}, i-1) \neq \Phi$, by Lemma 3.6 (iv) $i = n-1$.

Therefore $D_{ev} (C_{n,i}) = \{\{e_{n-3}\} / x \in [e_{n-1}]\}$. 

5) $D_{ev} (C_{n-3}, i-1) = \Phi$, $D_{ev} (C_{n-2}, i-1) \neq \Phi$, $D_{ev} (C_{n-1}, i-1) \neq \Phi$ and $D_{ev} (C_{n-4}, i-1) \neq \Phi$.

Clearly, $\{\{X_1 \cup \{e_1\} / x \in D_{ev} (C_{n-1}, i-1) \} \cup \{X_2 \cup \{e_{n-1}\} / X_2 \in D_{ev} (C_{n-2}, i-1)\} \} \subseteq \{D_{ev} (C_{n,i})\}$.

Conversely, let $Y \in D_{ev} (C_{n,i})$. Then $e_1$ or $e_{n-2}$ or $e_{n-3}$ $\in Y$. If $e_1 \in Y$, then we can write $Y = X_1 \cup \{e_1\}$, for some $X_1 \in D_{ev} (C_{n-1}, i-1)$. If $e_{n-2} \in Y$ and $e_{n-3} \in Y$, then we can write $Y = X_3 \cup \{e_{n-2}\}$, for some $X_3 \in D_{ev} (C_{n-2}, i-1)$. If $e_{n-3} \in Y$, then $e_{n-2} \not\subseteq Y$, $e_{n-3} \not\subseteq Y$, $e_{n-3} \not\subseteq Y$, then we can write $Y = X_4 \cup \{e_{n-3}\}$, for some $X_4 \in D_{ev} (C_{n-4}, i-1)$.

Therefore we proved that

$$D_{ev} (C_{n,i}) = \{\{X_1 \cup \{e_1\} / x \in D_{ev} (C_{n-1}, i-1)\} \cup \{X_2 \cup \{e_{n-1}\} / X_2 \in D_{ev} (C_{n-2}, i-1)\} \} \cup \{X_3 \cup \{e_{n-2}\} / X_3 \in D_{ev} (C_{n-3}, i-1)\} \cup \{X_4 \cup \{e_{n-3}\} / X_4 \in D_{ev} (C_{n-4}, i-1)\}.$$
Hence,
\[
D_v(C_n,i) = \left\{\left[\left\{X_1 \cup \{e_n\} / X_1 \in D_v(C_{n-1},i-1)\right\}\cup \left\{X_2 \cup \{e_n\} / X_2 \in D_v(C_{n-2},i-1)\right\}\cup \left\{X_3 \cup \{e_n\} / X_3 \in D_v(C_{n-3},i-1)\right\}\cup \left\{X_4 \cup \{e_n\} / X_4 \in D_v(C_{n-4},i-1)\right\}\right\}
\]

3. Edge-Vertex Domination Polynomials of Cycles

Let \( D_v(C_n,x) = \sum_{i=0}^{n} d_v(C_n,i)x^i \) be the ev-domination polynomial of a cycle \( C_n \). In this section, we derive the expression for \( D_v(C_n,x) \).

**Theorem 3.1** [6]

1) If \( D_v(C_n,i) \) is the family of ev-dominating sets with cardinality \( i \) of \( C_n \), then
\[
d_v(C_n,i) = d_v(C_{n-1},i-1) + d_v(C_{n-2},i-1) + d_v(C_{n-3},i-1) + d_v(C_{n-4},i-1)
\]
where \( d_v(C_n,i) = |D_v(C_n,i)| \).

2) For every \( n \geq 5 \),
\[
D_v(C_n,x) = x\left[D_v(C_{n-1},x) + D_v(C_{n-2},x) + D_v(C_{n-3},x) + D_v(C_{n-4},x)\right]
\]

with the initial values
\[
D_v(C_1,x) = x,
D_v(C_2,x) = 2x^2 + x,
D_v(C_3,x) = 3x^3 + 3x^2 + x,
D_v(C_4,x) = 4x^4 + 6x^3 + 4x^2 + x.
\]

**Proof:**

1) Using (1), (2), (3), (4) and (5) of Theorem 2.7, we prove (1) part.
Suppose, \( D_v(C_{n-1},i) = D_v(C_{n-2},i-1) = D_v(C_{n-3},i-1) = \Phi \) and \( D_v(C_{n-4},i-1) = \Phi \) then,
\[
D_v(C_n,i) = \left\{\left\{e_1, e_2, \ldots, e_{n-3}\right\}, \left\{e_2, e_3, \ldots, e_{n-2}\right\}, \left\{e_3, e_4, \ldots, e_{n-1}\right\}, \left\{e_4, e_5, \ldots, e_n\right\}\right\}.
\]
Therefore, \( |D_v(C_n,i)| = |\left\{e_1, e_2, \ldots, e_{n-3}\right\}, \left\{e_2, e_3, \ldots, e_{n-2}\right\}, \left\{e_3, e_4, \ldots, e_{n-1}\right\}, \left\{e_4, e_5, \ldots, e_n\right\}| = 4 \). In this case
\[
|D_v(C_{n-4},i-1)| = |\left\{e_1, e_2, \ldots, e_{n-3}\right\}, \left\{e_2, e_3, \ldots, e_{n-2}\right\}, \left\{e_3, e_4, \ldots, e_{n-1}\right\}, \left\{e_4, e_5, \ldots, e_n\right\}| = 4 \] and
\[
|D_v(C_{n-1},i-1)| = |D_v(C_{n-2},i-1)| = |D_v(C_{n-3},i-1)| = 0. Therefore, in this case the theorem holds.

Suppose, \( D_v(C_{n-2},i-1) = D_v(C_{n-3},i-1) = D_v(C_{n-4},i-1) = \Phi \) and \( D_v(C_{n-1},i-1) = \Phi \), then
\[
D_v(C_n,i) = \left\{\left\{e_1\right\}\right\}.
\]
Therefore, \( |D_v(C_n,i)| = |\left\{e_1\right\}| = 1 \). In this case \( D_v(C_{n-1},i-1) = \left\{\left\{e_1\right\}\right\} \). Therefore,
\[
|D_v(C_{n-1},i-1)| = 1 \text{ and } |D_v(C_{n-2},i-1)| = |D_v(C_{n-3},i-1)| = |D_v(C_{n-4},i-1)| = 0. Therefore, in this case the theorem holds.

Suppose, \( D_v(C_{n-1},i-1) = \Phi \), \( D_v(C_{n-2},i-1) = \Phi \), \( D_v(C_{n-3},i-1) = \Phi \) and \( D_v(C_{n-4},i-1) = \Phi \). In this case,
\[
D_v(C_n,i) = Y_1 \cup Y_2 \cup Y_3
\]
where
\[
Y_1 = \left\{\left\{e_1, e_2, \ldots, e_{n-5}, e_{n-1}\right\}, \left\{e_2, e_3, \ldots, e_{n-4}, e_n\right\}, \left\{e_3, e_4, \ldots, e_{n-3}, e_1\right\}, \left\{e_4, e_5, \ldots, e_{n-2}, e_n\right\}\right\}
\]
\[
Y_2 = \left\{\left\{e_{n-2}\right\}, \text{ if } 1 \in X_2 \cup X_2 \in D_v(C_{n-2},i-1)\right\}\cup \left\{e_{n-1}\right\}, \text{ if } 2 \in X_2 \cup X_2 \in D_v(C_{n-2},i-1)\right\}\cup \left\{e_n\right\}, \text{ otherwise}
\]
$Y_3 = \begin{cases} 
X_3 \cup \{ e_{n-3} \}, & \text{if } 1 \in X_3 / X_3 \in D_{e_v}(C_{n-3},i-1) \\
\{ e_{n-2} \}, & \text{if } 2 \in X_3 / X_3 \in D_{e_v}(C_{n-3},i-1) \\
\{ e_{n-1} \}, & \text{if } 3 \in X_3 / X_3 \in D_{e_v}(C_{n-3},i-1) \\
\{ e_n \}, & \text{otherwise} 
\end{cases}$

Therefore, $|D_{e_v}(C_{n},i)| = 4 + |D_{e_v}(C_{n-3},i-1)| + |D_{e_v}(C_{n-4},i-1)|$. Also, $|D_{e_v}(C_{n-1},i-1)| = 0$ and $|D_{e_v}(C_{n-2},i-1)| = 4$. Therefore, $|D_{e_v}(C_{n},i)| = |D_{e_v}(C_{n-1},i-1)| + |D_{e_v}(C_{n-3},i-1)| + |D_{e_v}(C_{n-4},i-1)|$ and $|D_{e_v}(C_{n-1},i-1)| = 0$. Therefore, in this case the theorem holds.

Suppose,

$D_{e_v}(C_{n-1},i-1) \neq \Phi$, $D_{e_v}(C_{n-2},i-1) \neq \Phi$ and $D_{e_v}(C_{n-3},i-1) = \Phi$.

Then we have $D_{e_v}(C_{n},i) = \left[ \{ e_r \} - \{ x^i \} / x \in [e_r] \right]$.

Therefore, $|D_{e_v}(C_{n},i)| = n$. In this case, $|D_{e_v}(C_{n-1},i-1)| = n-1$, $|D_{e_v}(C_{n-2},i-1)| = 1$ and $|D_{e_v}(C_{n-3},i-1)| = 0$. Therefore,

$|D_{e_v}(C_{n},i)| = |D_{e_v}(C_{n-1},i-1)| + |D_{e_v}(C_{n-2},i-1)| + |D_{e_v}(C_{n-3},i-1)| = n-1+1+0 = n$.

Therefore, in this case the theorem holds.

Suppose, $D_{e_v}(C_{n-1},i-1) \neq \Phi$, $D_{e_v}(C_{n-2},i-1) \neq \Phi$, $D(C_{n-3},i-1) \neq \Phi$ and $D_{e_v}(C_{n-4},i-1) \neq \Phi$. In this case, we have

$D_{e_v}(C_{n},i) = \left[ \{ X_1 \} \cup \{ e_{n-3} \} / X_1 \in D_{e_v}(C_{n-3},i-1) \right] \cup \left[ \{ X_2 \} \cup \{ e_{n-4} \} / X_2 \in D_{e_v}(C_{n-4},i-1) \right] \cup \left[ \{ X_3 \} \cup \{ e_{n-2} \} / X_3 \in D_{e_v}(C_{n-2},i-1) \right] \cup \left[ \{ X_4 \} \cup \{ e_{n-3} \} / X_4 \in D_{e_v}(C_{n-4},i-1) \right]$.

Therefore,

$|D_{e_v}(C_{n},i)| = |D_{e_v}(C_{n-1},i-1)| + |D_{e_v}(C_{n-2},i-1)| + |D_{e_v}(C_{n-3},i-1)| + |D_{e_v}(C_{n-4},i-1)|$.

Hence,

$d_{e_v}(C_{n},i) = d_{e_v}(C_{n-1},i-1) + d_{e_v}(C_{n-2},i-1) + d_{e_v}(C_{n-3},i-1) + d_{e_v}(C_{n-4},i-1)$.

$d_{e_v}(C_{n},i)x^i = d_{e_v}(C_{n-1},i-1)x^i + d_{e_v}(C_{n-2},i-1)x^i + d_{e_v}(C_{n-3},i-1)x^i + d_{e_v}(C_{n-4},i-1)x^i$.

$\Sigma d_{e_v}(C_{n},i)x^i = \Sigma d_{e_v}(C_{n-1},i-1)x^i + \Sigma d_{e_v}(C_{n-2},i-1)x^i + \Sigma d_{e_v}(C_{n-3},i-1)x^i + \Sigma d_{e_v}(C_{n-4},i-1)x^i$.

$\Sigma d_{e_v}(C_{n},i)x^i = x \Sigma d_{e_v}(C_{n-1},i-1)x^{i-1} + x \Sigma d_{e_v}(C_{n-2},i-1)x^{i-1} + x \Sigma d_{e_v}(C_{n-3},i-1)x^{i-1} + x \Sigma d_{e_v}(C_{n-4},i-1)x^{i-1}$.

$\Sigma d_{e_v}(C_{n},i)x^i = x\left[ \Sigma d_{e_v}(C_{n-1},i-1)x^{i-1} + \Sigma d_{e_v}(C_{n-2},i-1)x^{i-1} + \Sigma d_{e_v}(C_{n-3},i-1)x^{i-1} + \Sigma d_{e_v}(C_{n-4},i-1)x^{i-1} \right]$.

$D_{e_v}(C_{n},x) = x \left[ D_{e_v}(C_{n-1},x) + D_{e_v}(C_{n-2},x) + D_{e_v}(C_{n-3},x) + D_{e_v}(C_{n-4},x) \right]$.

with the initial values

$D_{e_v}(C_{0},x) = x$

$D_{e_v}(C_{1},x) = 2x^2 + x$

$D_{e_v}(C_{2},x) = 3x^3 + 3x^2 + x$

$D_{e_v}(C_{3},x) = 4x^4 + 6x^3 + 4x^2 + x$
We obtain \( d_{e^v} (C_n, i) \) for \( 1 \leq n \leq 16 \) as shown in Table 1.

In the following Theorem, we obtain some properties of \( d_{e^v} (C_n, i) \).

**Theorem 3.2**

The following properties hold for the coefficients of \( D_{e^v} (C_n, x) \):

1) \( d_{e^v} (C_{4n}, n) = 4 \), for every \( n \in \mathbb{N} \).

2) \( d_{e^v} (C_n, n) = 1 \), for every \( n \in \mathbb{N} \).

3) \( d_{e^v} (C_n, n - 1) = n \), for every \( n \geq 2 \).

4) \( d_{e^v} (C_n, n - 2) = nC_2 = \frac{n(n-1)}{2} \), for every \( n \geq 3 \).

5) \( d_{e^v} (C_n, n - 3) = nC_3 = \frac{n(n-1)(n-2)}{6} \), for every \( n \geq 4 \).

6) \( d_{e^v} (C_n, n - 4) = nC_4 - n = \frac{n(n-1)(n-2)(n-3)}{4!} - n \), for every \( n \geq 5 \).

7) \( d_{e^v} (C_{4n-1}, n) = 4n-1 \), for every \( n \in \mathbb{N} \).

**Proof:**

1) Since \( D_{e^v} (C_{4n}, n) = \{\{e_1, e_5, \ldots, e_{2n-3}\}, \{e_2, e_6, \ldots, e_{2n-2}\}, \{e_3, e_7, \ldots, e_{2n-1}\}, \{e_4, e_8, \ldots, e_{2n}\}\} \), we have \( d_{e^v} (C_{4n}, n) = 4 \).

2) Since \( D_{e^v} (C_n, n) = \{\{e_n\}\} \), we have the result \( d_{e^v} (C_n, n) = 1 \) for every \( n \in \mathbb{N} \).

3) Since \( D_{e^v} (C_n, n - 1) = \{\{e_n\} - \{x\} / x \in [e_n]\} \), we have \( d_{e^v} (C_n, n - 1) = n \) for \( n \geq 2 \).

4) By induction on \( n \). The result is true for \( n = 3 \). L.H.S. = \( d_{e^v} (C_3, 1) = 3 \) (from Table 1) R.H.S. = \( \frac{3 \times 2}{2} = 3 \).

Therefore, the result is true for \( n = 3 \). Now suppose that the result is true for all numbers less than ‘\( n \)’ and we prove it for \( n \). By Theorem 3.1

**Table 1.** \( d_{e^v} (C_n, i) \), the number of \( ev \)-dominating set of \( C_n \) with cardinality \( i \).

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\[ d_v(C_n, n-2) = d_v(C_{n-1}, n-3) + d_v(C_{n-2}, n-3) + d_v(C_{n-3}, n-3) + d_v(C_{n-4}, n-3) \]
\[ = \frac{(n-1)(n-2)}{2} + (n-1) = \frac{1}{2}[(n-1)(n-2) + 2(n-1)] = \frac{1}{2}[(n-1)(n-2 + 2)] = \frac{1}{2}((n-1)n) \]

5) By induction on \( n \), the result is true for \( n = 4 \). L.H.S. = \( d_v(C_4, 1) = 4 \) (from Table 1).
R.H.S. = \( \frac{4 \cdot 3 \cdot 2}{6} = 4 \). Therefore, the result is true for \( n = 4 \). Now suppose the result is true for all natural numbers less than \( n \). By Theorem 3.1,
\[ d_v(C_n, n-3) = d_v(C_{n-1}, n-4) + d_v(C_{n-2}, n-4) + d_v(C_{n-3}, n-4) + d_v(C_{n-4}, n-4) \]
\[ = \frac{(n-1)(n-2)(n-3)}{6} + \frac{(n-2)(n-3)}{2} + (n-2) \]
\[ = \frac{1}{6}[(n-1)(n-2)(n-3) + 3(n-2)(n-3) + 6(n-2)] \]
\[ = \frac{(n-2)}{6}[(n-1)(n-3) + 3(n-3) + 6] \]
\[ = \frac{(n-2)}{6}[(n-1)(n-3) + 3(n-1)] \]
\[ = \frac{(n-2)(n-1)}{6}[(n-3 + 3)] = \frac{(n-2)(n-1)n}{6} \]

6) By induction on \( n \), the result is true for \( n = 5 \). L.H.S. = \( d_v(C_5, 1) = 0 \) (from Table 1)
R.H.S. = \( \frac{(n-1)(n-2)(n-3)}{1 \times 2 \times 3 \times 4} = \frac{5 \times 4 \times 3 \times 2}{1 \times 2 \times 3 \times 4} - 5 = 0 \)

Therefore the result is true for \( n = 5 \).
Now suppose that the result is true for all natural numbers less than \( n \) and we prove it for \( n \). By Theorem 3.1,
\[ d_v(C_n, n-4) = d_v(C_{n-1}, n-5) + d_v(C_{n-2}, n-5) + d_v(C_{n-3}, n-5) + d_v(C_{n-4}, n-5) \]
\[ = \frac{(n-1)(n-2)(n-3)(n-4)}{24} + (n-1) + \frac{(n-2)(n-3)(n-4)}{6} + \frac{(n-3)(n-4)}{2} + (n-4) \]
\[ = \frac{1}{24}[(n-1)(n-2)(n-3)(n-4) + 4(n-2)(n-3)(n-4) + 12(n-3)(n-4) + 24(n-4)] - n \]
\[ = \frac{1}{24}[(n-1)(n-2)(n-3)(n-4) + 4(n-2)(n-3)(n-4) + 12(n-3)(n-4) + 24(n-4) + 24] - n \]
\[ = \frac{1}{24}[(n-1)(n-2)(n-3)(n-4) + 4(n-2)(n-3)(n-4) + 12(n-3)(n-4) + 24(n-3)] - n \]
\[ = \frac{(n-3)}{24}[(n-1)(n-2)(n-4) + 4(n-2)(n-4) + 12(n-4) + 24] - n \]
\[ = \frac{(n-3)(n-2)}{24}[(n-1)(n-4) + 4(n-4) + 12] - n \]
\[ = \frac{(n-3)(n-2)}{24}[(n-1)(n-4) + 4(n-4) + 3] - n \]
\[ = \frac{(n-1)(n-2)(n-3)}{24} - n \]

7) From the table it is true.
Theorem 3.3

1) \[ \sum_{i=n}^{4n} d_{ev}(C_i) = 4 \sum_{i=n}^{4n-4} d_{ev}(C_i, n-1), \quad n \geq 2. \]

2) For every \( j \geq \left\lceil \frac{n}{4} \right\rceil \), \( d_{ev}(C_{n+1}, j+1) - d_{ev}(C_n, j+1) = d_{ev}(C_n, j) - d_{ev}(C_{n-4}, j). \)

3) If \( S_n = \sum_{i=n}^{n+1} d_{ev}(C_n, i) \), then for every \( n \geq 5 \), \( S_n = S_{n-1} + S_{n-2} + S_{n-3} + S_{n-4} \) with initial values \( S_1 = 1 \), \( S_2 = 3 \), \( S_3 = 7 \), and \( S_4 = 15 \).

Proof:

1) We prove by induction on \( n \).

First suppose that \( n = 2 \) then,

\[ \sum_{i=2}^{8} d_{ev}(C_i, 2) = 4 \sum_{i=2}^{4} d_{ev}(C_i, 1) = 40. \]

\[ \sum_{i=2}^{4k} d_{ev}(C_i, k) = \sum_{i=2}^{4k} d_{ev}(C_{i-1}, k-1) + \sum_{i=2}^{4k} d_{ev}(C_{i-2}, k-1) + \sum_{i=2}^{4k} d_{ev}(C_{i-3}, k-1) + \sum_{i=2}^{4k} d_{ev}(C_{i-4}, k-1) = 4 \sum_{i=2}^{4(k-1)} d_{ev}(C_{i-1}, k-2) + 4 \sum_{i=2}^{4(k-1)} d_{ev}(C_{i-2}, k-2) + 4 \sum_{i=2}^{4(k-1)} d_{ev}(C_{i-3}, k-2) + 4 \sum_{i=2}^{4(k-1)} d_{ev}(C_{i-4}, k-2). \]

We have the result.

2) By Theorem 3.1, we have

\[ d_{ev}(C_{n+1}, j+1) - d_{ev}(C_n, j+1) = d_{ev}(C_n, j) + d_{ev}(C_{n-1}, j) + d_{ev}(C_{n-2}, j) + d_{ev}(C_{n-3}, j) - d_{ev}(C_{n-1}, j) - d_{ev}(C_{n-2}, j) - d_{ev}(C_{n-3}, j) - d_{ev}(C_{n-4}, j) \]

Therefore, we have the result.

3) By Theorem 3.1, we have

\[ S_n = \sum_{i=n}^{n+1} d_{ev}(C_n, i) \]

\[ S_n = \sum_{i=n}^{n+1} \left[ d_{ev}(C_{n-1}, i-1) + d_{ev}(C_{n-2}, i-1) + d_{ev}(C_{n-3}, i-1) + d_{ev}(C_{n-4}, i-1) \right] \]

\[ = \sum_{i=n}^{n-1} d_{ev}(C_{n-1}, i) + \sum_{i=n}^{n-1} d_{ev}(C_{n-2}, i) + \sum_{i=n}^{n-1} d_{ev}(C_{n-3}, i) + \sum_{i=n}^{n-1} d_{ev}(C_{n-4}, i) \]

\[ S_n = S_{n-1} + S_{n-2} + S_{n-3} + S_{n-4}. \]

4. Concluding Remarks

In [7], the domination polynomial of cycle was studied and obtained the very important property,

\[ d(C_n, i) = d(C_{n-1}, i-1) + d(C_{n-2}, i-1) + d(C_{n-3}, i-1). \]

It is interesting that we have derived an analogous relation for the edge-vertex domination of cycles of the form,

\[ d_{ev}(C_n, i) = d_{ev}(C_{n-1}, i-1) + d_{ev}(C_{n-2}, i-1) + d_{ev}(C_{n-3}, i-1) + d_{ev}(C_{n-4}, i-1). \]

One can characterise the roots of the polynomial \( D_{ev}(C_n, x) \) and identify whether they are real or complex. Another interesting character to be investigated is whether \( D_{ev}(C_n, x) \) is log-concave or not.

References


