Some Results on Prime Labeling*

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Abstract

In the present work we investigate some classes of graphs and disjoint union of some classes of graphs which admit prime labeling. We also investigate prime labeling of a graph obtained by identifying two vertices of two graphs. We also investigate prime labeling of a graph obtained by identifying two edges of two graphs. Prime labeling of a prism graph is also discussed. We show that a wheel graph of odd order is switching invariant. A necessary and sufficient condition for the complement of $W_n$ to be a prime graph is investigated.

Keywords

Graph Labeling, Prime Labeling, Switching of a Vertex, Switching Invariance

1. Introduction

We begin with simple, finite, undirected and non-trivial graph $G = (V, E)$ with the vertex set $V$ and the edge set $E$. The number of elements of $V$, denoted as $|V|$ is called the order of the graph $G$ while the number of elements of $E$, denoted as $|E|$ is called the size of the graph $G$. In the present work $C_n$ denotes the cycle with $n$ vertices and $P_n$ denotes the path of $n$ vertices. In the wheel $W_n = C_n + K_1$ the vertex corresponding to $K_1$ is called the apex vertex and the vertices corresponding to $C_n$ are called the rim vertices. For various graph theoretic notations and terminology we follow Gross and Yellen [1] whereas for number theory we follow D. M. Burton [2]. We will give brief summary of definitions and other information which are useful for the present investigations.

Definition 1.1: If the vertices of the graph are assigned values subject to certain conditions then it is known as graph labeling.

For latest survey on graph labeling we refer to J. A. Gallian [3]. Vast amount of literature is available on different types of graph labeling and more than 1000 research papers have been published so far in last four decades. For any graph labeling problem following three features are really noteworthy:

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• a set of numbers from which vertex labels are chosen;
• a rule that assigns a value to each edge;
• a condition that these values must satisfy.

The present work is aimed to discuss one such labeling known as prime labeling.

**Definition 1.2:** A prime labeling of a graph $G$ of order $n$ is an injective function $f: V \rightarrow \{1, 2, \ldots, n\}$ such that for every pair of adjacent vertices $u$ and $v$, $gcd(f(u), f(v)) = 1$. The graph which admits prime labeling is called a prime graph.

The notion of prime labeling was originated by Entringer and was discussed in A.Tout [4]. Many researchers have studied prime graphs. It has been proved by H. L. Fu and C. K. Huang [5] that $P_n$ is a prime graph. It has been proved by S. M. Lee [6] that wheel graph $W_n$ is a prime graph if and only if $n$ is even. T. Deretsky [7] has proved that cycle $C_n$ is a prime graph.

**Definition 1.3:** A vertex switching $G_v$ of a graph $G$ is the graph obtained by taking a vertex $v$ of $G$, removing all the edges incident to $v$ and adding edges joining to every other vertex which is not adjacent to $v$ in $G$.

**Definition 1.4:** A prime graph is said to be switching invariant if for every vertex $v$ of $G$, the graph $G_v$ obtained by switching the vertex $v$ in $G$ is also a prime graph.

**Definition 1.5:** For two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ their cartesian product $G_1 \times G_2$ is defined as the graph whose vertex set is $V_1 \times V_2$ and two vertices $(u, v_1)$ and $(u, v_2)$ in $G_1 \times G_2$ are adjacent if $u = u_2$ and $v_1$ is adjacent to $v_2$ or $u_1$ is adjacent to $u_2$ and $v_1 = v_2$.

**Definition 1.6:** $C_n \times P_2$ is called a prism graph.

**Bertrand’s Postulate 1.7:** For every positive integer $n > 1$ there is a prime $p$ such that $n < p < 2n$.

### 2. Some Results on Prime Labeling

**Theorem 2.1:** If $G_1$ is a prime graph with order $n$, where $n$ is even and $G_2$ is a graph with order 3 then disjoint union of $G_1$ and $G_2$ is a prime graph.

**Proof:** Let $u_1, u_2, u_3, \ldots, u_n$ be the vertices of $G_1$ and $v_1, v_2, v_3$ be the vertices of $G_2$. Let $G$ be a disjoint union of $G_1$ and $G_2$. Now $G_1$ is a prime graph, so there is an injective function $f: \{u_1, u_2, u_3, \ldots, u_n\} \rightarrow \{1, 2, \ldots, n\}$ such that for any arbitrary edge $e = u_i u_j$, we have $gcd(f(u_i), f(u_j)) = 1$.

Define a function $g: \{u_1, u_2, u_3, \ldots, u_n, v_1, v_2, v_3\} \rightarrow \{1, 2, \ldots, n, n+1, n+2, n+3\}$ as follows:

$$g(u) = \begin{cases} f(u_i) & \text{for } u = u_i, i = 1, 2, \ldots, n; \\ n+i & \text{for } u = v_i, i = 1, 2, 3. \end{cases}$$

Clearly $g$ is an injective function.

If $e = u_i u_j$ is any edge of $G$ then $gcd(g(u_i), g(u_j)) = gcd(f(u_i), f(u_j)) = 1$. If $e = v_i v_j$ then $gcd(g(v_i), g(v_j)) = gcd(n+1, n+2) = 1$. If $e = v_i v_j$ then $gcd(g(v_i), g(v_j)) = gcd(n+2, n+3) = 1$. If $e = v_i v_j$ then $gcd(g(v_i), g(v_j)) = gcd(n+1, n+3) = 1$ as $n$ is even.

Thus $G$ admits a prime labeling. So $G$ is a prime graph.

**Theorem 2.2:** If $G_1$ is a prime graph with order $n$, where $n$ is divisible by 6 and $G_2$ is a prime graph with order 5 then disjoint union of $G_1$ and $G_2$ is a prime graph.

**Proof:** Let $u_1, u_2, u_3, \ldots, u_n$ be the vertices of $G_1$ and $v_1, v_2, v_3, v_4, v_5$ be the vertices of $G_2$. Let $G$ be the disjoint union of $G_1$ and $G_2$. Now $G_1$ is a prime graph, so there exists an injective function $f: \{u_1, u_2, u_3, \ldots, u_n\} \rightarrow \{1, 2, \ldots, n\}$ such that for any arbitrary edge $e = u_i u_j$, we have $gcd(f(u_i), f(u_j)) = 1$.

Also $G_2$ is a prime graph, so there is an injective function $g: \{v_1, v_2, v_3, v_4, v_5\} \rightarrow \{1, 2, 3, 4, 5\}$ such that for any arbitrary edge $e = v_i v_j$, we have $gcd(g(v_i), g(v_j)) = 1$.

Define a function $h: \{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_5\} \rightarrow \{1, 2, \ldots, n+4, n+5\}$ as follows:

$$h(u) = \begin{cases} f(u_i) & \text{for } u = u_i, i = 1, 2, \ldots, n; \\ n+g(v_i) & \text{for } u = v_i, i = 1, 2, 3, 4, 5. \end{cases}$$

Clearly $h$ is an injective function. To prove $h$ is a prime labeling of $G$ we have the following cases:
Case 1: If \( e = u_j u_j \) is any edge of \( G_i \) then \( \gcd(h(u_j), h(u_j)) = \gcd(f(u_j), f(u_j)) = 1 \).

Case 2: Suppose \( e = v_j v_j \) is any edge of \( G_2 \) and \( \gcd(h(v_j), h(v_j)) = \gcd(n + g(v_j), n + g(v_j)) = d \). Here \( d \) is an odd natural number as \( n \) is even and at least one of \( g(v_j) \) and \( g(v_j) \) is odd. As \( d = (n + g(v_j)) \) and \( d\{n + g(v_j)\} \) so \( d\{n + g(v_j)\} \). But possible values of \( \gcd(v_j, g(v_j)) \) are 1, 2, 3 and 4, and \( d \) is odd. So \( d = 1 \) or \( d = 3 \). If \( d = 3 \) then \( 3\{n + g(v_j)\} \) and \( 3\{n + g(v_j)\} \). Also \( 3\{n\} \), therefore \( 3\{g(v_j)\} \) and \( 3\{g(v_j)\} \), which is not possible as \( \gcd(g(v_j), g(v_j)) = 1 \). Thus \( d = 1 \), hence \( \gcd(h(v_j), h(v_j)) = 1 \).

Thus \( G \) admits prime labeling. So \( G \) is a prime graph.

S. K. Vaidya and U. M. Prajapati [8] has proved that if \( n_i \g 4 \) is an even integer and \( n_2 \) is a natural number, then the graph obtained by identifying any of the rim vertices of a wheel \( W_n \) with an end vertex of a path \( P_n \) is a prime graph. But in the following theorem we have proved that if \( n_2 \) is odd then also it is prime.

Theorem 2.3: If \( n_i + n_2 = p \), where \( p \) is prime then the graph obtained by identifying one of the rim vertices of \( W_n \) with an end vertex of \( P_n \) is prime.

Proof: Let \( u_j \) be an apex vertex of \( W_n \) and \( u_i u_i, u_i, u_i, \ldots, u_i \) be consecutive rim vertices of \( W_n \) and \( v_i, v_i, v_i, \ldots, v_i \) be consecutive vertices of \( P_n \). Without loss of generality we can assume that \( G(V, E) \) be the graph obtained by identifying a rim vertex \( u_i \) of \( W_n \) with an end vertex \( v_i \) of \( P_n \). Define \( f : V \rightarrow \{1, 2, \ldots, |V|\} \) as follows:

\[
f(u) = \begin{cases} 
  p & \text{for } u = u_j; \\
  i & \text{for } u = u_i, i = 1, 2, \ldots, n_2; \\
  n_i + j - 1 & \text{for } u = v_i, j = 2, 3, \ldots, n_2.
\end{cases}
\]

Clearly \( f \) is an injective function. Let \( e \) be an arbitrary edge of \( G \). To prove \( f \) is a prime labeling of \( G \) we have the following cases:

Case 1: If \( e = u_j u_i \) then \( \gcd(f(u_j), f(u_i)) = \gcd(p, i) = 1, \forall i = 1, 2, \ldots, n_2 \).

Case 2: If \( e = u_j u_{i+1} \) then \( \gcd(f(u_j), f(u_{i+1})) = \gcd(i, i+1) = 1, \forall i = 1, 2, \ldots, n_2 - 1 \).

Case 3: If \( e = v_i v_{i+1} \) then \( \gcd(f(v_i), f(v_{i+1})) = \gcd(n_i + j - 1, n_i + j) = 1, \forall j = 2, 3, \ldots, n_2 - 1 \).

Case 4: If \( e = u_j v_i \) then \( \gcd(f(u_j), f(v_i)) = \gcd(1, n_i + 1) = 1 \).

Case 5: If \( e = u_i u_j \) then \( \gcd(f(u_i), f(u_j)) = \gcd(n_i, 1) = 1 \).

Thus \( G \) admits a prime labeling. So \( G \) is a prime graph.

Theorem 2.4: A path \( P_n \) and \( m \) copies of cycle \( C_n \) are given, then the graph obtained by identifying each edge of \( P_n \) with an edge of a corresponding copy of the cycle \( C_n \) is prime.

Proof: Let \( v_i v_i, v_i, v_i, \ldots, v_i \) be the vertex of \( P_n \) and \( u_i, u_{i+1}, u_{i+2}, \ldots, u_{i+m} \) be the vertices of \( i^{th} \) copy of cycle \( C_n \) where \( i = 1, 2, \ldots, m \). Let \( G \) be a graph obtained by identifying an edge \( u_i u_{i+m} \) of \( i^{th} \) copy of cycle \( C_n \) with an edge \( v_i v_{i+m} \) of path \( P_n \), where \( i = 1, 2, \ldots, m \). Let \( V \) be the set of vertices of \( G \) then \( |V| = m(n - 1) + 1 \). Define a function \( f : V \rightarrow \{1, 2, \ldots, |V|\} \) as follows:

\[
f(u) = \begin{cases} 
  (1 + (i-1)(n-1)) & \text{for } u = v_i, i = 1, 2, \ldots, m; \\
  (i-1)(n-1) + j & \text{for } u = u_{j}, i = 1, 2, \ldots, m, j = 2, 3, \ldots, n-1.
\end{cases}
\]

Clearly \( f \) is an injective function. Let \( e \) be an arbitrary edge of \( G \). To prove \( f \) is a prime labeling of \( G \) we have the following cases:

Case 1: If \( e = v_i v_{i+1} \) then \( \gcd(f(v_i), f(v_{i+1})) = \gcd(1 + (i-1)(n-1), 1 + i(n-1)) = 1 \), for \( i = 1, 2, \ldots, m \).

Case 2: If \( e = u_i u_{i+m} \) then \( \gcd(f(u_i), f(u_{i+m})) = \gcd(i - 1)(n - 1) + j, (i-1)(n-1) + j + 1) = 1 \), for \( j = 2, 3, \ldots, n-2 \) and \( i = 1, 2, \ldots, m \).

Case 3: If \( e = v_i v_{i+m} \) then \( \gcd(f(v_i), f(u_{i+m})) = \gcd(1 + (i-1)(n-1), 2 + (i-1)(n-1)) = 1 \).
Case 4: If \( e = u_{(n-1),j}v_{i+1} \) then \( \gcd \left( f(u_{(n-1),j}), f(v_{i+1}) \right) = \gcd \left( i(n-1), 1+i(n-1) \right) = 1. \)

Thus \( G \) is a prime graph. So \( G \) is a prime graph.

**Theorem 2.5:** A cycle \( C_n \) and \( m \) copies of a cycle \( C_n \) are given, then the graph obtained by identifying each edge of \( C_n \) with an edge of corresponding copy of the cycle \( C_n \) is prime.

**Proof:** Let \( v_1,v_2,v_3,\cdots,v_m \) be the vertices of \( C_m \) and \( u_{j1},u_{j2},u_{j3},\cdots,u_{jm} \) be the vertices of \( i^{\text{th}} \) copy of cycle \( C_n \) where \( i=1,2,\cdots,m \). Let \( G \) be a graph obtained by identifying an edge \( u_{ij},u_{im} \) of \( i^{\text{th}} \) copy of cycle \( C_n \) with an edge \( v_iv_j \) of cycle \( C_m \), where \( i=1,2,\cdots,m-1 \) and an edge \( u_{im}u_{jm} \) of \( m^{\text{th}} \) copy of cycle \( C_n \) with an edge \( v_iv_j \) of cycle \( C_m \). Let \( V \) be the vertex set of \( G \) then \( |V'| = m(n-1) \). Define a function \( f : V \rightarrow \{1,2,\cdots,|V'|\} \) as follows:

\[
f(u) = \begin{cases} 
1+\left( \frac{i-1}{n-1} \right)(n-1) & \text{for } u = v_i, i = 1,2,\cdots,m; \\
\left( \frac{i-1}{n-1} \right)(n-1) + j & \text{for } u = u_{ij}, j = 1,2,\cdots,m-1, i = 2,3,\cdots,m. 
\end{cases}
\]

Clearly \( f \) is an injective function. Let \( e \) be an arbitrary edge of \( G \). To prove \( f \) is a prime labeling of \( G \) we have the following cases:

**Case 1:** If \( e = v_1v_{i+1} \) then \( \gcd \left( f(v_1), f(v_{i+1}) \right) = \gcd \left( 1+(i-1)(n-1), 1+i(n-1) \right) = 1, \) for \( i = 1,2,\cdots,m-1. \)

**Case 2:** If \( e = v_kv_j \) then \( \gcd \left( f(v_k), f(v_j) \right) = \gcd \left( 1+(m-1)(n-1), 1 \right) = 1. \)

**Case 3:** If \( e = u_{ij}u_{ij} \) then \( \gcd \left( f(u_{ij}), f(u_{ij}) \right) = \gcd \left( (i-1)(n-1) + j, (i-1)(n-1) + j+1 \right) = 1, \) for \( j = 2,3,\cdots,n-2 \) and \( i = 1,2,\cdots,m. \)

**Case 4:** If \( e = u_{(n-1),j}v_{i+1} \) then \( \gcd \left( f(u_{(n-1),j}), f(v_{i+1}) \right) = \gcd \left( (i-1)(n-1), 1+i(n-1) \right) = 1, \) for \( i = 1,2,\cdots,m-1. \)

**Case 5:** If \( e = u_{(n-1),j}v_i \) then \( \gcd \left( f(u_{(n-1),j}), f(v_i) \right) = \gcd \left( 1+(i-1)(n-1), 2+i(n-1) \right) = 1, \) for \( i = 1,2,\cdots,m. \).

Thus \( G \) admits a prime labeling. So \( G \) is a prime graph.

S. K. Vaidya and U. M. Prajapati [9] have proved that switching the apex vertex in \( W_n \) is a prime graph and switching a rim vertex in \( W_n \) is a prime graph if \( n+1 \) is prime. But in the following theorem we have proved that \( W_n \) is switching invariant if \( n \) is even.

**Theorem 2.6:** \( W_{2n} \) is switching invariant.

**Proof:** Let \( v_1,v_2,v_3,\cdots,v_{2n} \) be rim vertices and \( v_0 \) be an apex vertex of \( W_{2n} \). According to the degree of vertices of \( W_{2n} \) we can take the following cases.

**Case 1:** Let \( G \) be a graph obtained by switching a rim vertex \( v_{2n} \). Let \( p \) be a largest prime less than \( 2n \).

Define a function \( f : V(G) \rightarrow \{1,2,\cdots,|V(G)|\} \) as follows:

\[
f(v_i) = \begin{cases} 
1 & \text{for } i = 0; \\
p + i & \text{for } i = 1,2,\cdots,2n-p+1; \\
-i-2n+p & \text{for } i = 2n-p+2,2n-p+3,\cdots,2n-1; \\
p & \text{for } i = 2n. 
\end{cases}
\]

Clearly \( f \) is an injective function. Let \( e \) be an arbitrary edge of \( G \). To prove \( f \) is a prime labeling of \( G \) we have the following cases:

- If \( e = v_0v_i, i \neq 2n \) then \( \gcd \left( f(v_0), f(v_i) \right) = \gcd \left( 1, f(v_i) \right) = 1. \)
- If \( e = v_iv_{i+1}, i = 1,2,\cdots,2n-p \) then \( \gcd \left( f(v_i), f(v_{i+1}) \right) = \gcd \left( p+i, p+i+1 \right) = 1. \)
- If \( e = v_{2n-p+1}v_{2n-p+2} \) then \( \gcd \left( f(v_{2n-p+1}), f(v_{2n-p+2}) \right) = \gcd \left( 2n+1, 2 \right) = 1. \)
- If \( e = v_{(2n-p)+1}v_{(2n-p)+2} \) then \( \gcd \left( f(v_i), f(v_{i+1}) \right) = \gcd \left( i-2n+p, i-2n+p+1 \right) = 1. \)
- If \( e = v_{2n-1}v_{2n}, i = 2,3,\cdots,2n-2 \) then \( \gcd \left( f(v_i), f(v_{2n}) \right) = \gcd \left( p+i, p \right) = \gcd \left( i, p \right) = 1, \) as \( p \) is the largest prime less than \( 2n. \)
Case 2: Let $G$ be a graph obtained by switching an apex vertex $v_0$. Define a function $f : V(G) \rightarrow \{1, 2, \ldots, |V'(G)|\}$ as follows:

$$f(v_i) = \begin{cases} 
  i & \text{for } i = 1, 2, \ldots, 2n; \\
  2n+1 & \text{for } i = 0. 
\end{cases}$$

Clearly $f$ is an injective function. It can be easily verified that $f$ is a prime labeling. Thus from both the cases it follows that $G$ is a prime graph.

Theorem 2.7: The complement of $W_n$ is prime if and only if $3 \leq n \leq 6$.

Proof: We can easily see that $\overline{W_n}$ is prime for $n = 3, 4, 5$ and 6 from Figure 1. Now if $n \geq 7$ then $(n-3) \geq 4$ and every rim vertex of $\overline{W_n}$ is adjacent to other $(n-3)$ rim vertices. We have total \( \left\lceil \frac{n+1}{2} \right\rceil \) even numbers to assign $n+1$ vertices. If one of the rim vertices is labeled as even number then other $n-3$ vertices can not be labeled as even number. Also remaining two rim vertices are adjacent, so only one of them can be labeled as even number. The apex vertex can also be labeled as even number. Thus maximum three vertices can be labeled as even number. But if $n \geq 7$ then we have 4 or more even numbers to label. So it is not possible. Thus $W_n$ is not prime for $n \geq 7$.

Theorem 2.8: Let $p \geq 3$ be a prime number and take $p-2$ copies of $C_{p^2}$, then the graph obtained by identifying one vertex of each copy of $C_{p^2}$ with corresponding pendant vertex of $K_{1,p^2}$ is prime.

Proof: Let $u_0$ be an apex vertex and $u_1, u_2, u_3, \ldots, u_{p-2}$ be pendant vertices of $K_{1,p^2}$. Also let $v_{i,1}, v_{i,2}, v_{i,3}, \ldots, v_{i,p^2}$ be the vertices of $i$th copy of $C_{p^2}$. Now let $G$ be the graph obtained by identifying a pendant vertex $u_i$ of $K_{1,p^2}$ with a vertex $v_{i,p^1}$ of $f^i$ copy of $C_{p^1}$, where $i = 1, 2, \ldots, p-2$.

Define a function $f : V(G) \rightarrow \{1, 2, \ldots, |V'|\}$, where $|V'| = (p-2)(p+1)+1$ as follows:

$$f(u) = \begin{cases} 
  1 & \text{for } u = u_0; \\
  i(p+1)+1 & \text{for } u = u_i = v_{i,p^1}, i = 1, 2, \ldots, p-2; \\
  (i-1)(p+1)+j+1 & \text{for } u = v_{i,j}, i = 1, 2, \ldots, p-2, j = 1, 2, \ldots, p. 
\end{cases}$$

Clearly $f$ is an injective function. Let $e$ be an arbitrary edge of $G$. To prove $f$ is a prime labeling of $G$ we have the following cases:

Case 1: If $e = u_0u_i = u_0v_{i,p^1}$, $\gcd(f(u_0), f(u_i)) = \gcd(1, i(p+1)+1) = 1$, for $i = 1, 2, \ldots, p-2$.

Case 2: If $e = v_{i,j}v_{i+1,j}$, $\gcd(f(v_{i,j}), f(v_{i+1,j})) = \gcd((i-1)(p+1)+j+1, (i-1)(p+1)+j+2) = 1$ for $i = 1, 2, \ldots, p-2$ and $j = 1, 2, \ldots, p$.

Case 3: If $e = v_{i,1}v_{i,p^1}$, then for $i = 1, 2, \ldots, p-2$ and $j = 1, 2, \ldots, p$,

$\gcd(f(v_{i,1}), f(v_{i,p^1})) = \gcd((i-1)(p+1)+2, i(p+1)+1)$

$= \gcd((i-1)(p+1)+2, i(p+1)+1-2)$

$= \gcd((i-1)(p+1)+2, p)$

$= \gcd((i-1)(p+1)+2-1, p)$

$= \gcd(i+1, p)$

$= 1$ (as $i < p-1$ so $i+1 < p$ and $p$ is a prime.)

Figure 1. Prime labeling of $\overline{W_3}$, $\overline{W_4}$, $\overline{W_5}$ and $\overline{W_6}$. 
Thus $G$ admits a prime labeling. So $G$ is a prime graph.

**Theorem 2.9:** If $n \geq 3$ is an odd integer then the prism graph $C_n \times P_2$ is not prime.

**Proof:** In the prism graph $C_n \times P_2$ there are two cycles $C_n$. So total number of vertices are $2n$. So we have to use 1 to $2n$ natural numbers to label these vertices, and from 1 to $2n$ there are $n$ even integers. If $n$ is odd then we can use at the most $\frac{n-1}{2}$ even integers to label the vertices of a cycle $C_n$. We have such two cycles, so we can use at the most $\frac{n-1}{2} + \frac{n-1}{2} = n-1$ even integers to label the vertices of $C_n \times P_2$. But from 1 to $2n$ there are $n$ even integers. So such prime labeling is not possible.

Thus $C_n \times P_2$ is not prime if $n \geq 3$ is an odd integer.

**Theorem 2.10:** If $p \geq 3$ is a prime number then the prism graph $C_{p-1} \times P_2$ is prime.

**Proof:** In the prism graph $C_{p-1} \times P_2$, let $v_{1,1}, v_{1,2}, v_{1,3}, \ldots, v_{p-1,1}$ be the vertices of one cycle and $v_{2,1}, v_{2,2}, v_{2,3}, \ldots, v_{2,p-1}$ be the vertices of the other cycle and a vertex $v_{2,i}$ is joined with $v_{2,j}$ by an edge for $i = 1, 2, \ldots, p-1$. Define a function $f : V(G) \rightarrow \{1, 2, \ldots, 2p-2\}$ as follows:

$$f(v_{i,j}) = \begin{cases} 
  j & \text{for } i = 1, j = 1, 2, \ldots, p-1; \\
  p + j & \text{for } i = 2, j = 1, 2, \ldots, p-2; \\
  p & \text{for } i = 2, j = p-1.
\end{cases}$$

Clearly $f$ is an injective function. Let $e$ be an arbitrary edge of $G$. To prove $f$ is a prime labeling of $G$ we have the following cases:

**Case 1:** If $e = v_{i,j}v_{i+1,j}$ then $gcd(f(v_{i,j}), f(v_{i+1,j})) = gcd(j, j+1) = 1$, for $j = 1, 2, \ldots, p-2$.

**Case 2:** If $e = v_{i,j}v_{i,j+1}$ then $gcd(f(v_{i,j}), f(v_{i,j+1})) = gcd(j+1, j+2) = 1$, for $j = 1, 2, \ldots, p-3$.

**Case 3:** If $e = v_{i,j}v_{i+1,j}$ then $gcd(f(v_{i,j}), f(v_{i+1,j})) = gcd(j, j+1) = 1$.

**Case 4:** If $e = v_{2,i}v_{2,i+1}$ then $gcd(f(v_{2,i}), f(v_{2,i+1})) = gcd(2p, 2p+1) = 1$.

**Case 5:** If $e = v_{2,i}v_{2,i+1}$ then $gcd(f(v_{2,i}), f(v_{2,i+1})) = gcd(p+1, p) = 1$.

**Case 6:** If $e = v_{i,j}v_{i+1,j}$ then $gcd(f(v_{i,j}), f(v_{i+1,j})) = gcd(j, j+1) = 1$, for $j = 1, 2, \ldots, p-2$.

**Case 7:** If $e = v_{i,j}v_{i+1,j}$ then $gcd(f(v_{i,j}), f(v_{i+1,j})) = gcd(j, j+1) = 1$, for $j = 1, 2, \ldots, p-1$.

Thus $G$ admits a prime labeling. So $G$ is a prime graph.

**Theorem 2.11:** A graph obtained by joining every rim vertex of a wheel graph $W_{p-1}$ with corresponding vertex of a cycle $C_{p-1}$ is a prime graph, where $p$ is a prime number not less than 3.

**Proof:** Let $v_0$ be an apex vertex and $v_1, v_2, v_3, \ldots, v_{p-1}$ be rim vertices of $W_{p-1}$. Also $u_1, u_2, u_3, \ldots, u_{p-3}$ are the vertices of $C_{p-1}$. Let $G$ be the graph obtained by joining a vertex $v_i$ of $W_{p-1}$ with a vertex $u_j$ of $C_{p-1}$ by an edge, where $i = 1, 2, \ldots, p-1$. Define a function $f : V(G) \rightarrow \{1, 2, \ldots, 2p-1\}$ as follows:

$$f(u) = \begin{cases} 
  i & \text{for } u = v_i, i = 1, 2, \ldots, p-1; \\
  p & \text{for } u = v_0; \\
  p+i & \text{for } u = u_i, i = 1, 2, \ldots, p-1.
\end{cases}$$

Clearly $f$ is an injective function. Let $e$ be an arbitrary edge of $G$. To prove $f$ is a prime labeling of $G$ we have the following cases:

**Case 1:** If $e = v_{i}v_{i+1}$ then $gcd(f(v_i), f(v_{i+1})) = gcd(i, i+1) = 1$, for $i = 1, 2, \ldots, p-2$.

**Case 2:** If $e = v_{i}v_{i+1}$ then $gcd(f(v_i), f(v_{i+1})) = gcd(1, p-1) = 1$.

**Case 3:** If $e = v_{i}v_{i+1}$ then $gcd(f(v_i), f(v_{i+1})) = gcd(p, i) = 1$, for $i = 1, 2, \ldots, p-1$.

**Case 4:** If $e = v_{i}v_{i+1}$ then $gcd(f(v_i), f(v_{i+1})) = gcd(p+i, p+i+1) = 1$, for $i = 1, 2, \ldots, p-2$. 

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Case 5: If $e = u_iu_{p-i}$ then $\gcd\left(f(u_i), f(u_{p-i})\right) = \gcd(p+1, 2p-1) = 1$.

Case 6: If $e = v_iu_i$ then $\gcd\left(f(v_i), f(u_i)\right) = \gcd(i, p+i) = 1$, for $i = 1, 2, \ldots, p-1$.

Thus $G$ admits a prime labeling. So $G$ is a prime graph.

3. Concluding Remarks

Study of relatively prime numbers is very interesting in the theory of numbers and it is challenging to investigate prime labeling of some families of graphs. Here we investigate several results of some classes of graphs about prime labeling. Extending the study to other graph families is an open area of research.

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References


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