A Lemma on Almost Regular Graphs and an Alternative Proof for Bounds on $\gamma_t(P_k \Box P_m)$

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Received July 16, 2013; revised August 10, 2013; accepted September 3, 2013

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ABSTRACT

Gravier et al. established bounds on the size of a minimal totally dominant subset for graphs $P_k \Box P_m$. This paper offers an alternative calculation, based on the following lemma: Let $k, r \in \mathbb{N}$ so $k \geq 3$ and $r \geq 2$. Let $H$ be an $r$-regular finite graph, and put $G = P_r \Box H$. 1) If a perfect totally dominant subset exists for $G$, then it is minimal; 2) If $r > 2$ and a perfect totally dominant subset exists for $G$, then every minimal totally dominant subset of $G$ must be perfect. Perfect dominant subsets exist for $P_r \Box C_n$ when $k$ and $n$ satisfy specific modular conditions. Bounds for $\gamma_t(P_k \Box P_m)$, for all $k, m$ follow easily from this lemma. Note: The analogue to this result, in which we replace “totally dominant” by simply “dominant”, is also true.

Keywords: Domination; Total Domination; Matrix; Linear Algebra

1. Introduction

Let $G = (V(G), E(G))$ be a graph. In this paper, each edge of a graph must have two different endpoints; also, two vertices may be linked by at most one edge. A subset $Z$ of vertices is said to totally dominate $G$ if every vertex of $G$ has a neighbor in $Z$. We say $Z$ perfectly totally dominates if every vertex has exactly one neighbor in $Z$. Next, suppose that $G$ is finite. In this case, we say a totally dominant subset $Z$ is minimal if $|Z|$ is the smallest size possible among all dominant subsets. This minimal size is denoted by $\gamma_t(G)$.

For $r \in \mathbb{N}$, we say that a graph $G$ is $r$-regular if every vertex is the endpoint of exactly $r$ edges. Suppose $G$ is regular. A subset $Z$ which perfectly totally dominates is clearly minimal. If a perfect dominant set does not exist, we can search for minimality among dominant subsets $Z$ by counting “overlaps”. That is, for each $v \in V(G)$, let $o_l(v, G, Z)$ be the number of neighbors of $v$ which lie in $Z$, minus 1. If $Z_1$ and $Z_2$ are two totally dominant subsets, then $|Z_1| < |Z_2|$ happens if and only if the sum of $Z_1$-overlaps is strictly less than the sum of $Z_2$-overlaps.

These elementary links between minimality, perfection and overlaps may fail if $G$ is not regular. For arbitrary graphs, all sorts of behavior is possible. For graph theorists, a challenge is to specific assertions that apply to a broad family of graphs.

The following conventions will be used here.

(1a) For $k \in \mathbb{N}$, $k \geq 2$, let $P_k$, the $k$-path be the graph whose vertices are the numbers $1, 2, \cdots, k$, and whose edges are links from $i$ to $i + 1$ for each $1 \leq i < k$. There is an infinite member of this family: Interpret $Z$ as a graph in which edges consist of links from $i$ to $i + 1$ for all $i$.

(1b) Let $k > 2$. The graph consisting of $P_k$ plus an edge between 1 and $k$ called the $k$-cycle. It is denoted by $C_k$.

(1c) For $G$ and $H$, the product graph $G \Box H$ is defined as follows. The set of vertices $V(G \Box H) = V(G) \times V(H)$. Two vertices $(x_1, y_1)$ and $(x_2, y_2)$ are linked by an edge if and only if
- either $x_1 = x_2$ and $y_1y_2$ is an edge of $H$, or
- $x_1x_2$ is an edge of $G$ and $y_1 = y_2$.

For example, for $k, n \in \mathbb{N}$, $P_k \Box P_n$ is the familiar $k \times n$ grid map. A product of a list of paths and circuits by $\Box$ is called a grid graph.

A product of $n$ copies of $Z$ corresponds to the set $Z^*$ with the “Manhattan metric” notion of the edge: two tupels $(x_1, \cdots, x_n)$ and $(y_1, \cdots, y_n)$ are linked if and only if there is an index $i$ such that $|x_i - y_i| = 1$ and
\(x_j = y_j\) for all \(j \neq i\).

Tiling is the route that Gravier [1] takes in computing \(\gamma_v\) for grid graphs. The program begins with the work by herself, Molland and Payan [2] on the tiling question. The solution generates perfectly dominant subsets on \(\mathbb{Z}^n\). Now, finite grid graphs can be interpreted as rectangular subsets, or (for products with \(C_n\) factors) as such subsets with some “opposed” sides identified. Domination becomes a problem of refining the patterns at the edges.

Our current work exploits the abundance of perfect dominations on graphs \(G = P_i \boxtimes C_n\). A calculation with matrices leads to a lower bound on \(\gamma_v(G)\) that can only be attained by a perfectly totally dominant subset. Once we classify which indices \(k, n\) admit perfect dominations, an elementary trick provides upper and lower bounds for all graphs \(P_i \boxtimes C_n\). The bounds here do not improve on the earlier work, but are almost as narrow.

Suppose \(H\) is a finite \(r\)-regular graph for some natural number \(r\), and put \(G = P_i \boxtimes H\) for \(k \geq 3\). Then the majority of vertices of \(G\) have a degree \(r + 2\). The vertices of the degree \(r + 1\) form two connected subgraphs. A crude bound for a minimal totally dominant subset of \(G\) is \(k|H|/(r + 2)\). However, this bound is too low by a positive number times \(|H|\).

We find a subtler minimal bound using matrices. The computation also shows that

1. \(P_i \boxtimes C_n\) is even. In this case, \(P_i \boxtimes C_n\) is bipartite. Identify \(C_n\) with \(\mathbb{Z}/n\mathbb{Z}\) in the standard way. We can “color” the vertices: we say \((i, j)\) (where \(j\) is read \(\text{mod}(n)\)) is black if \(i + j\) is even and white if \(i + j\) is odd. Then every edge links a black vertex with a white one. If \(Z\) dominates \(P_i \boxtimes C_n\), then the set of black members of \(Z\) dominates all white vertices, and the white vertices of \(Z\) dominate all the black. Consequently, a minimal dominant subset is a disjoint union of two minimal “color” dominant subsets; each a subset of one color vertices that dominates all vertices of the other color. Furthermore, the “shift by 1” automorphism of \(P_i \boxtimes C_n\) identifies the sets of different colored vertices.

Figure 1 shows a pattern of vertices of one color. Provided that \(k\) is odd, this pattern will totally dominate all vertices of the opposite color.

If \(k\) is even, this pattern does not work. Instead, as illustrated in Figure 2 for \(k = 8\), one can build a pattern by taking triangular wedges of the first pattern, and pairing them with a skew reflection. The latter pattern can be repeated throughout \(P_i \boxtimes C_n\) provided that \(2(k+1)\) divides \(n\).

The contribution of this paper is an alternate construction of a lower bound. The bound is met for three perfect subsets. Next, using these subsets, one can establish a general upper bound for \(P_i \boxtimes P_m\) for all \(m\).

1.2. A Tie with Perfection

Gravier [1] proves that the set \(Z\) consisting of the middle row of \(P_i \boxtimes P_m\), for any \(n\), is a minimal totally dominant subset. Obviously, this choice of minimal

\[\text{Figure 1. One color dominance, } k \text{ odd.}\]

\[\text{Figure 2. One color dominance, } k = 8.\]
subset produces many overlaps. By rotating $3 \times 3$ blocks, we can produce other minimal dominant sets with fewer overlaps, as in Figure 3. Furthermore, if $n$ is a multiple of 4, there is a variation which is a perfect total domination of $P_2 \Box C_n$, as in Figure 4. The flexibility in the number of vertices which are dominated by more than one member of $Z$ reflects the presence of vertices of two degrees, namely 3 and 4.

In this example, the size of a minimal, imperfect totally dominant subset “ties” the size of a perfect totally dominant set. Can a minimal subset be smaller than a perfect one? We prove that a tie is rare, and that beating is impossible.

1.3. Weights

We have two sets of theorems based on series.

**Definition 1** Let $r$ be a real number. Let $\Xi[r]$ be the set of infinite sequences of real numbers $\{a_i\}_{i=0}^{\infty}$ such that

$$\forall i > 1, a_i = ra_{i-1} - a_{i-2}.$$  

Clearly, $\Xi[r]$ is a real vector space, and the function $\{a_i\}_{i=1}^{\infty} \mapsto (a_0, a_1)$ is a linear isomorphism from it onto $\mathbb{R}^2$.

For $r$ real, let $i \mapsto \lambda(r,i)$ be the unique member of $\Xi[r]$ such that $\lambda(r,0)=0$ and $\lambda(r,1)=1$. Observe that $\lambda(r,2) = r$.

In the opening section, we defined the overlap function $ol_i(v,G,Z)$ for totally dominant subsets $Z$ of a graph $G$. In addition, for $G$ a graph and $Z$ a dominant (but possibly not totally) subset, and $v \in V(G)$, let $ol_i(v,G,Z)$ be $ol_i(v,G,Z)$ if $v \not\in Z$ and $ol_i(v,G,Z)+1$ if $v \in Z$. For $k \geq 3$, $G = P_2 \Box H$ for some graph $H$ and $v \in V(G)$, define row(v) the row of v to be the first coordinate of v.

**Lemma 2** Let $r,k \in \mathbb{N}$ such that $r \geq 2$ and $k \geq 3$.

For each integer $1 \leq j \leq k$, put

$$\omega_j = \lambda(r,k+1)+(-1)^{j+1}\lambda(r,k+1-j)+(-1)^{j+1}\lambda(r,j).$$

For each $1 \leq j \leq k$,

(4a) $\omega_j \geq 0$, and

(4b) $\omega_j = 0$ if and only if $r = 2$, $k$ is odd and $j$ is even.

We refer to $\omega_1, \ldots, \omega_k$ as the weight system for parameters $r,k$.

**Definition 3** Let $r,k \in \mathbb{N}$ such that $r \geq 2$ and $k \geq 3$. Let $\omega_1, \ldots, \omega_k$ be the weight system for $r,k$. Also, let $v_1, \ldots, v_k$ be the weight system for parameters $r+1,k$. Define

$$\mu(r,k) = \frac{(rk+2k+2)\lambda(r,k+1)+2\lambda(r,k)+(-1)^{j+1}2}{(r+2)^2 \lambda(r,k+1)}.$$

Suppose $H$ is an $r$-regular graph, and put $n = |H|$ and $G = P_2 \Box H$. Define two functions on $Z \subseteq V(G)$:

$$\text{score}(Z) = \sum_{v \in V(G)} ol_i(v,G,Z) \cdot \omega_{\text{row}(v)}$$

$$\text{score}(Z) = \sum_{v \in V(G)} ol_i(v,G,Z) \cdot v_{\text{row}(v)}$$

**Theorem 4** Assume the hypothesis and construction of Lemma 2 and Definition 3. Let $H$ be a finite graph, and put $n = |H|$ and $G = P_2 \Box H$.

(A) If $Z \subseteq V(G)$ is totally dominant, then

$$|Z| = n\mu(r,k) + \frac{\text{score}(Z)}{(r+2)\lambda(r,k+1)}.$$  

(B) If $Z \subseteq V(G)$ is dominant, then

$$|Z| = n\mu(r+1,k) + \frac{\text{score}(Z)}{(r+3)\lambda(r+1,k+1)}.$$  

A trivial consequence of this theorem and the preceding lemma is:

**Corollary 5** Assume the hypothesis of Theorem 4.

(A) Suppose $r \geq 3$. If $Z_1,Z_2$ are totally dominant subsets of $G$, then

$$|Z_1| < |Z_2| \iff \text{score}(Z_1) < \text{score}(Z_2).$$

(B) If $Z_1,Z_2$ are dominant subsets of $G$, then

$$|Z_1| < |Z_2| \iff \text{score}(Z_1) < \text{score}(Z_2).$$

2. Modeled with Matrices

Our results are based on a simple linear algebra model. For convenience,

(5) For $k \in \mathbb{N}$, let $\text{Ind}(k) = \{1, \ldots, k\}$.

**Notation 6** Let $k \in \mathbb{N}$. We identify the real vector space $\mathbb{R}^k$ with length $k$ column vectors. We use trans-
pose notation to write these horizontally:

\[(z_1, \cdots, z_k)^T \text{ for } \begin{bmatrix} z_1 \\ \vdots \\ z_k \end{bmatrix} \]

For each \(1 \leq i \leq k\), let \(\pi_i\) be the projection function from each vector \((z_1, \cdots, z_k)\) to its \(i\)-coordinate \(z_i\). Also define a linear functional \(\mathbb{R}^k \to \mathbb{R}\)

\[\text{sum}(z) = \sum_{i=1}^{k} \pi_i(z).\]

We denote the zero vector by \(\hat{0}\).

In what follows, let \(k, r \in \mathbb{N}\), and let \(H\) be a finite, \(r\)-regular graph. Put \(G = P_r \Box H\).

For \(Z \subseteq V(G)\), define the row count vector \(z\) for \(Z\) to be \((z_1, \cdots, z_k)^T\) in which \(z_i\) is the number of members of \(Z\) in the \(i\)-th row. Obviously, \(\text{sum}(z) = |Z|\).

Now suppose \(Z \subseteq V(G)\) totally dominates, and let \(z = (z_1, \cdots, z_k)^T\) be its row count vector. Let \(1 \leq i \leq k\).

The sum of \(\text{ol}_i(v, Z, G)\) over all \(v\) in the \(i\)-th row, plus \(|H|\), equals

\[rz_1 + z_2, \quad \text{for } i = 1,\]

\[z_i + rz_{i+1} + z_{i+1}, \quad \text{for } 1 \leq i \leq k-1,\text{ and}\]

\[z_{k-1} + rz_k, \quad \text{for } i = k.\]

In particular,

**(7a)** If \(Z\) totally dominates, then each of these expressions must be \(\geq |H|\), and

**(7b)** If \(Z\) perfectly totally dominates, then each of these expressions must equal \(|H|\).

If we replace \(\text{totally domination}\) with \(\text{simple domination}\), the analogous assertions hold after the \(r\) terms in (6) are changed to \(r+1\).

These remarks motivate our next definition.

**Definition 7** Let \(r\) be a real number and let \(k\) be a natural number \(> 1\). Define \(L[r, k]\) to be the \(k \times k\) matrix such that

\[\forall i, j \in \text{Ind}(k), L[r, k]_{ij} = \begin{cases} r & \text{if } i = j, \\ 1 & \text{if } i - j \text{ is } 1 \text{ or } -1, \\ 0 & \text{otherwise} \end{cases}\]

Note that \(L[r, k]\) is symmetric.

Also, for these parameters, define \(M[r, k]\) to be the \(k \times k\) matrix such that

\[M[r, k]_{ij} = \begin{cases} \lambda(-r, i)\lambda(-r, k+1-j) & \text{if } i \leq j, \\ \lambda(-r, j)\lambda(-r, k+1-i) & \text{if } j < i. \end{cases}\]

Note that the case \(i = j\) is covered in both parts of this conditional definition.

As we shall see, the matrix \(M[r, k]\) is essentially \(L[r, k]^{1/3}\).

### 3. Relevant Sequences

There is a discrete analogy to \(\text{convexity}\) for functions of a single real variable. We recall some basics.

**Definition 8** Let \(\{a_i\}_{i=0}^{\infty}\) be a sequence of real numbers, starting at index 0. We say that the sequence is convex if

\[\forall i \in \mathbb{N}, \quad a_{i+1} - a_i \geq a_i - a_{i-1}.\]

We say the sequence is strictly convex if

\[a_{i+1} - a_i > a_i - a_{i-1}, \quad \text{for each } i.\]

**Lemma 9** Let \(\{a_i\}_{i=0}^{\infty}\) be a convex sequence. For \(u, v \in \mathbb{N}\),

\[a_{u+v} \geq a_u + a_v - a_0.\]

Moreover, \(a_{u+v} = a_u + a_v - a_0\) if and only if there is a number \(t\) such that

\[\forall i \in \text{Ind}(u+v), \quad a_i = t + a_{i-1}.\]

**Proof.** We may interchange \(u\) and \(v\) without loss of generality. Hence, assume \(u \geq v\). For each \(i \in \mathbb{N}\), put \(b_i = a_i - a_{i-1}\). Then \(\{b_i\}_{i=0}^{\infty}\) is a weakly increasing sequence. Then

\[a_{u+v} - a_0 - (a_u - a_0) = \left(\sum_{i=1}^{u+v} b_i\right) - \left(\sum_{i=1}^{u} b_i\right) - \left(\sum_{i=1}^{v} b_i\right) - (u-v)\]

\[\Leftrightarrow a_{u+v} - a_u - a_v + a_0 = \left(\sum_{i=1}^{u+v} b_i\right) - \left(\sum_{i=1}^{u} b_i\right) - (u-v)\]

\[\Leftrightarrow a_{u+v} - a_u - a_v + a_0 = \sum_{i=1}^{u+v-1} b_i - b_v\]

Observe that

\[u+v+1-i \geq i \iff (u-v) + 2(v-i) + 1 \geq 0.\]

For each index \(i\) in the last sum, the term has the format \(b_p - b_q\) where \(p > q\). Therefore

\[a_{u+v} - a_u - a_v + a_0 \geq 0.\]

Now suppose \(a_{u+v} - a_u - a_v + a_0 = 0\). Then every term in the final sum of (8) must be 0. When \(i = 1\), we get \(b_1 = b_0 = 0\). Since \(b_1\) is an increasing sequence, it follows that \(b_i = b_1\) for every index \(i \leq u+v\). □

We focus on the sequences \(\lambda(r, i)\) of Definition 1. The first remark is that the sign can be separated from the magnitude.

**Lemma 10** Let \(r\) be a real number. Then

\[\forall i \in \mathbb{N}, \quad (-1)^{i+1}\lambda(r, i) = \lambda(-r, i).\]

**Proof.** Trivial. □
Many of the positive sequences $\lambda(r,i)$ are convex.

**Lemma 11** Each member of $\Xi[2]$ is a linear sequence.

**Proof.** Trivial. □

**Lemma 12** Let $r > 2$, and let $\{b_i\} \in \Xi[r]$ such that $b_i \geq b_0 \geq 0$. If $b_i > 0$, then $\{b_i\}$ is increasing and strictly convex. Furthermore, $b_i = b_{i-1}$ can occur only if $i = 1$.

**Proof.** For $i \geq 2$, we can rewrite the relation $b_i = rb_{i-1} - b_{i-2}$ as

(9a) $b_i = (r - 2)b_{i-2} + b_{i-1} + (b_{i-1} - b_{i-2})$, and

(9b) $(b_i - b_{i-1}) = (r - 2)b_{i-1} + (b_{i-1} - b_{i-2})$.

Use the two identities to induct on the double hypothesis that both $b_i > b_{i-1} > 0$ and $(b_i - b_{i-1}) > (b_{i-1} - b_{i-2}) > 0$.

**Corollary 13** Let $r \in \mathbb{R}$ and $k \in \mathbb{N}$ such that $|r| \geq 2$. Then $\lambda(r,k) \neq 0$.

**Proof.** This is an easy consequence of this lemma and Lemma 10.

The next two propositions play roles in our analysis.

**Lemma 14** Let $r$ be a real number other than 2. For $k \geq 1$,

$$\sum_{i=1}^{k} \lambda(r,i) = \frac{\lambda(r,k+1) - \lambda(r,k)-1}{r-2}. \quad (10)$$

**Proof.** In what follows, a sum from any integer $m$ to $m-1$ is defined to be 0. For this proof, we abbreviate $\lambda(k)$ for $\lambda(r,k)$.

For each $k \in \mathbb{N} \cup \{0\}$, define

$$s_k = \sum_{i=0}^{k} \lambda(r,i).$$

Then for $k \geq 2$,

$$s_k = \lambda(0) + \lambda(1) + \sum_{i=2}^{k} [\lambda(r(i-1)) - \lambda(i-2)]$$

$$= 1 + r \left[ \sum_{j=1}^{k-1} \lambda(j) \right] - \sum_{j=0}^{k-2} \lambda(j)$$

$$= 1 + r \cdot s_{k-1} - s_{k-2}.$$ 

Define a new sequence by $t_i = s_i + \frac{1}{r-2}$. Replace $s_i = t_i - \frac{1}{r-2}$ into the previous relation to get

$$\forall k \geq 2, \quad t_k = r \cdot t_{k-1} - t_k.$$

Hence, $\{t_i\}$ belongs to $\Xi[r]$.

Now

$$t_0 = s_0 + \frac{1}{r-2}, \quad t_i = s_i + \frac{1}{r-2} = \frac{1}{r-2}.$$

In the vector space $\mathbb{R}^2$,

$$\left( \frac{1}{r-2}, \frac{1}{r-2} \right) = \frac{1}{r-2} \cdot (r, r) - \frac{1}{r-2} \cdot (0, 1).$$

The sequences $t_i$ and

$$i \mapsto \frac{1}{r-2} \lambda(i+1) - \frac{1}{r-2} \lambda(i)$$

both belong to $\Xi[r]$, and agree on the first two indices. Hence, they are the same sequence. This gives the equality of (10). □

**Lemma 15** Let $r$ be a real number, and let $j, k \in \mathbb{N}$ such that $k \geq j$. Then

$$\lambda(r, k+1) = \lambda(r, j) \lambda(r, k+2 - j) - \lambda(r, j-1) \lambda(r, k+1 - j).$$

**Proof.** We write $\lambda(i)$ for $\lambda(r, i)$ in this argument.

If $k = j$, then $\lambda(k+2 - j) = \lambda(2) = r$,

$$\lambda(k+1 - j) = 1,$$

and the result follows from the recursive definition.

The remaining cases follow from a proof is by induction on $j$. The inductive hypothesis is

$$\forall k > j, \lambda(k+1) = \lambda(j) \lambda(k+2 - j) - \lambda(j-1) \lambda(k+1 - j).$$

For $j = 1$, this follows from the fact that $\lambda(1) = 1$ and $\lambda(0) = 0$.

Assume $j \in \mathbb{N}$ for which the inductive hypothesis is true. Let $k \in \mathbb{N}$ so $k > j+1$. Then

$$\lambda(j+1) \lambda(k+2 - (j+1)) - \lambda(j) \lambda(k+1 - (j+1))$$

$$= \lambda(j) \lambda(k+1 + j) - \lambda(j) \lambda(k - j)$$

$$= - \lambda(j-1) \lambda(k+1 - j)$$

$$= \lambda(j) \lambda(k+1 - j) - \lambda(j) \lambda(k+1 - j)$$

$$= \lambda(j) \lambda(k+1 - j).$$

Define a new sequence by $t_i = t_i - \frac{1}{r-2}$ into the previous relation to get

$$\forall k \geq 2, \quad t_k = r \cdot t_{k-1} - t_k.$$
By Lemma 15, this equals \(-\lambda(k+1)\). \(\square\)

**Lemma 17** Let \(k \in \mathbb{N}, \ r \in \mathbb{R}\) and \(j \in \text{Ind}(k)\). Assume \(r \neq -2\). Then
\[
\sum_{i=1}^{\text{Ind}(k)} M[r,k]_{i,j} = \sum_{i=1}^{\text{Ind}(k)} M[r,k]_{ij}
\]
equals
\[
\lambda(-r,k+1-j) + \lambda(-r,j) - \lambda(-r,k+1) + r + 2
\]

**Proof.** Put \(M = M[r,k]\) and, for each index \(i\), \(\lambda(i) = \lambda(-r,i)\). Split the sum from \(i=1\) to \(k\) of \(M_{i,j}\) at index \(j\):
\[
\sum_{i=1}^{\text{Ind}(k)} M_{i,j} = \sum_{i=1}^{\text{Ind}(k)} \lambda(i) \lambda(k+1-i) + \sum_{i=1}^{\text{Ind}(k)} \lambda(j) \lambda(k+1-i)
\]
In the previous line, the first sum is determined by Lemma 14. Recall the parameter is \(-r\), not \(r\)
\[
\lambda(k+1-j) \sum_{i=1}^{\text{Ind}(k)} \lambda(i) = \frac{\lambda(k+1-j)}{r-2} (\lambda(j) - \lambda(j+1))
\]
In the second sum, change index to \(p = k+1-i\). One can use the same Lemma.
\[
\lambda(j) \sum_{p=1}^{\text{Ind}(k)} \lambda(p) = \frac{\lambda(j)}{r+2} (\lambda(k-j) - \lambda(k+1-j) + 1).
\]
Add the two terms to get
\[
\sum_{i=1}^{\text{Ind}(k)} M_{i,j} = \frac{1}{r+2} (\lambda(k+1-j) \lambda(j) - \lambda(k+1-j) \lambda(j+1) + \lambda(k+1-j) + \lambda(j))
\]
By Lemma 15, this is the stated formula. \(\square\)

At last, we introduce weights. Define \(\omega_j\) as in the statement of Lemma 2.

**Corollary 18** Let \(k \in \mathbb{N}, \ r \in \mathbb{R}\). Assume \(r \geq 2\), and let \(\{\omega_j\}\) be the weight system for \(r,k\).
(A) If \( r = 2 \) and \( k \) is odd and \( j \) is even, then \( \omega_j = 0 \).

(B) If \( r = 2 \) and either \( k \) is even or \( j \) is odd, then \( \omega_j > 0 \).

(C) If \( r > 2 \), then \( \omega_j > 0 \).

(D) Let \( x \in \mathbb{R}^n \). Expand \( L[r,k] \cdot x \) as \( (b_1, \ldots, b_n)^T \). Then

\[
\sum(x) = \frac{1}{(r+2)\lambda(r,k+1)} \sum_{j=1}^{k} \omega_j \cdot b_j. \tag{11}
\]

**Proof.** We start with Part (D), as that is our motivation. Given

\[ x = (x_1, \ldots, x_n) \quad \text{and} \quad b = L[r,k] \cdot x = (b_1, \ldots, b_n), \]

it follows that

\[ x = L[r,k]^{-1} b. \]

By Lemma 16, for each \( 1 \leq i \leq k \),

\[ x_i = \frac{1}{-\lambda(-r,k+1)} \sum_{j=1}^{k} M[k,r]_{i,j} \cdot b_j. \]

From Lemma 17,

\[
\sum(x) = \frac{1}{-\lambda(-r,k+1)} \sum_{j=1}^{k} M[k,r] \cdot b_j = \frac{1}{(r+2)\lambda(-r,k+1)} \sum_{j=1}^{k} \lambda(-r,k+1) - \lambda(-r,k+1-j) - \lambda(-r,j) \cdot b_j.
\]

Now replace each \( \lambda(-r,i) \) by \( (-1)^{i+1} \lambda(r,i) \). The \( b_j \)-coefficient becomes \( \omega_j/((r+2)\lambda(r,k+1)) \).

Recall Lemma 12. Then \( \{\lambda(r,i)\}_i \) is a non-negative and convex sequence, and \( \lambda(r,0) = 0 \). Convexity implies that

\[
\lambda(r,k+1) + (-1)^{i+1} \lambda(r,k+1-j) + (-1)^{k+j} \lambda(r,j)
\]

is positive unless

(12a) \( j + 1 \) and \( k + j \) are both odd, and

(12b) \( \{\lambda(r,i)\}_i \) is not strictly convex.

This remark establishes all our conclusions except in the case when \( r = 2 \), \( k \) is odd and \( j \) is even. Assume these parameters, and we know \( \lambda(2,i) = i \) for all \( i \), and (A) follows. \( \square \)

This corollary proves Lemma 2.

**Corollary 19** Let \( k \in \mathbb{N} \), \( r \in \mathbb{R} \). Assume \( r \geq 2 \), and let \( \{\omega_j\} \) be the overlap weights for \( (k,r) \). Let \( \hat{1} \) be the vector in which every entry is 1, that is \( (1,1, \ldots, 1) \). Then

\[
\sum(L[r,k]^{-1} \cdot \hat{1}) = \mu(r,k),
\]

where \( \mu(r,k) \) is defined in Definition 3.

---

**Proposition 20** For \( k, n \geq 4 \),

\[
\mu(2,k)n \leq \gamma_c(Z \cap P_1 \cap P_2) \leq \mu(2,k)(n+2).
\]

**Proof.** There is \( d \in \mathbb{N} \) and \( Z \subseteq P \cap C_d \) such that

(14a) \( n+2 \) divides \( d \),

(14b) \( Z \) is a totally dominant subset of \( P \cap C_d \),

(14c) \( |Z| = \mu(2,k) \cdot d \).

Partition \( P \cap C_n \) into subsets \( Y_1, \ldots, Y_m \) where each \( Y_i \) consists of \( n+2 \) successive columns. For at least one index \( i \), \( |Y_i \cap Z| \leq \mu(2,k)(n+2) \). Choose such an index. Identify \( P \cap P_n \) with \( Y' \), the subgraph of columns 2 through \( n+1 \) of \( Y' \). Let \( Z = Z \cap Y' \). Any member of \( Y' \) which is not dominated by \( Z \) is dominated by exactly one member of \( Y \) in either the 1st or
\( n + 2 \) column; furthermore, each member of either column dominates just one member of \( Y' \). Consequently, we can expand \( Z_i \) to a totally dominant \( Z_2 \) for \( Y' \) of size \( |Y \cap Z| \).

6. Extended Functigraphs

Our lower bound uses only a few aspects of the graphs \( P_s \square H \). Consequently, the calculation applies to a slightly larger family of graphs.

Fix \( k, r, n \in \mathbb{N} \) with \( n, k, r > 2 \). Let \( H_1, \ldots, H_k \) be a list of \( r \)-regular graphs, each with \( n \) vertices. For each \( 1 \leq i < k \), let \( h_i : V(H_i) \rightarrow V(H_{i+1}) \) be a bijection. Define the extended functigraph on this data to be \( G \) in which

(15a) \( V(G) \) is the (disjoint) union \( \bigcup_{i=1}^{k} V(H_i) \), and

(15b) \( E(G) \) is union of \( \bigcup_{i=1}^{k} E(H_i) \) with \( \{vh_i(v) : 1 \leq i < k \land v \in V(H_i) \} \).

Then the assertions of Theorem 4, and its Corollary, apply to \( G \).

REFERENCES
