The Number of Canalyzing Functions over Any Finite Set*

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ABSTRACT

In this paper, we extend the definition of Boolean canalyzing functions to the canalyzing functions of multi-state case. Namely, \( f : Q^n \to Q \), where \( Q = \{a_1, a_2, \ldots, a_q\} \). We obtain its cardinality and the cardinalities of its various subsets (They may not be disjoint). When \( q = 2 \), we obtain a combinatorial identity by equating our result to the formula in [1]. For a better understanding to the magnitude, we obtain the asymptotes for all the cardinalities as either \( n \to \infty \) or \( q \to \infty \).

Keywords: Canalyzing Function; Inclusion and Exclusion Principle

1. Introduction

The idea of canalization was initiated from Waddington, C. H. [2]. When comparing the class of canalyzing functions to other classes of functions with respect to their evolutionary plausibility as emergent control rules in genetic regulatory systems, it is informative to know the number of canalyzing functions with a given number of input variables [1]. However, the Boolean network modeling paradigm is rather restrictive, with its limit to two possible functional levels, ON and OFF, for genes, proteins, etc. Many discrete models of biological networks therefore allow variables to take on multiple states. Common used discrete multi-state model types are so-called logical models [3], Petri nets [4], and agent-based models [5].

In this paper, we generalize the concept of Boolean canalyzing rules to the multi-state case. By generalizing the results in [1], we provide formulas for the cardinalities of various subsets of canalyzing functions. We also obtain the asymptotes of these cardinalities as either \( n \to \infty \) or \( q \to \infty \). We obtain a combinatorial identity by equating our result to the formula in [1].

2. Preliminaries

In this section we introduce the definition of a canalyzing function.

Let \([n] = \{1, 2, \ldots, n\}\), \(Q = \{a_1, a_2, \ldots, a_q\}\) and \(f : Q^n \to Q\).

A function is canalyzing if there is a variable \(x_i\) and an element \(a \in Q\) so that the value of the function is fixed once variable \(x_i\) is fixed at \(a\). More precisely, we have the following definitions.

**Definition 2.1**

1) The function \(f(x_1, x_2, \ldots, x_n)\) is \(\langle i : a : b \rangle\) canalyzing if \(f(x_1, \ldots, x_{i-1}, a, x_{i+1}, \ldots, x_n) = b\), for all \(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n\).

2) The function \(f(x_1, x_2, \ldots, x_n)\) is \(\langle i : a \rangle\) canalyzing if there exists \(b \in Q\) such that \(f(x_1, \ldots, x_{i-1}, a, x_{i+1}, \ldots, x_n) = b\), for all \(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n\).

3) The function \(f(x_1, x_2, \ldots, x_n)\) is \(\langle - : a : b \rangle\) canalyzing if there exists \(i \in [n]\) such that \(f(x_1, \ldots, x_{i-1}, a, x_{i+1}, \ldots, x_n) = b\), for all \(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n\).

4) The function \(f(x_1, x_2, \ldots, x_n)\) is \(\langle i : - b \rangle\) canalyzing if there exists \(a \in Q\) such that \(f(x_1, \ldots, x_{i-1}, a, x_{i+1}, \ldots, x_n) = b\), for all \(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n\).

5) The function \(f(x_1, x_2, \ldots, x_n)\) is \(\langle - a : - \rangle\) canalyzing if there exist \(i \in [n], b \in Q\) such that \(f(x_1, \ldots, x_{i-1}, a, x_{i+1}, \ldots, x_n) = b\), for all \(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n\).

6) The function \(f(x_1, x_2, \ldots, x_n)\) is \(\langle f : - \rangle\)
such that \( \Phi \). There are \( \{x_1, x_2, \ldots, x_n\} \) to stand for its cardinality. We use \( \Phi \) to stand for an empty set.

By the definitions, we immediately have the following propositions.

**Proposition 2.2** If \( b_1 \neq b_2 \), then \( \{i : a : b_1\} \cap \{i : a : b_2\} = \Phi \).

**Proposition 2.3** If \( b_1 \neq b_2 \) and \( i \neq i_2 \), then \( \{i : - : b_1\} \cap \{i : - : b_2\} = \Phi \).

By the definitions, we have

\[
\{i : - : b\} = \bigcup_{a \in Q} \{i : - : a\} \bigcup_{i = a}[i : - : a],
\]

\[
\{i : : b\} = \bigcup_{a \in Q} \{i : : a\} \bigcup_{i = a}[i : : a],
\]

\[
\{i : a : b\} = \bigcup_{a \in Q} \{i : a : b\} \bigcup_{i = a}[i : a : b].
\]

For any set \( S \), we use \( |S| \) to stand for its cardinality. We use \( C(n, k) = \frac{n!}{k!(n-k)!} \) to stand for the binomial coefficients. As usual, \( C(n, k) \) should be explained as zero once \( k > n \).

Obviously, for the above notations, the cardinality are same for different values of \( i, a \) and \( b \). In other words, we have \( |i : a : b| = |i : a : b| \), \( |i : : b| = |i : : c| \), \( |i : a : b| = |i : a : c| \) and etc.

### 3. Enumeration

**Theorem 3.1** Given \( i \in [n] \), \( a, b \in Q \), the number of \( \{i : a : b\} \) canalyzing functions is \( q^{a-b} q^{a-1} \). In other words, we have \( |i : a : b| = q^{a-b} q^{a-1} \).

**Proof:** A function in the set \( \{i : a : b\} \) is uniquely determined by its value on inputs \( (x_1, \ldots, x_n) \), such as \( x_i \neq a \). There are \((q-1)^{a-b} q^{a-1} \) such inputs, and the function can take \( q \) different values. Thus \( |i : a : b| = q^{a-b} q^{a-1} \).

Because \( |i : a : : | = \bigcup_{b \in Q} \{i : a : b\} \), by Proposition 2.2, we get

\[ q^{a-b} q^{a-1} = q^{a-b} q^{a-1}. \]

**Lemma 3.3** We have \( \bigcap_{j=1}^k \{i : a_j : b\} = q^{a_k - a_{k-1}} \) for any \( \{a_1, a_2, \ldots, a_k\} \in Q \).

**Proof:** A function in the set \( \bigcap_{j=1}^k \{i : a_j : b\} \) is uniquely determined by it values on inputs \( (x_1, \ldots, x_k) \) with \( x_j \neq a_j \) for \( j \leq k \). There are \((q-k)^{a_k - a_{k-1}} \) such inputs.

**Theorem 3.4** Given \( i \in [n] \) and \( b \in Q \), the number of \( \{i : - : b\} \) canalyzing functions is \( q^{a} - (q^{a-1} - 1)^q \). In other words, we have \( |i : - : b| = q^a - (q^{a-1} - 1)^q \).

**Proof:** By Inclusion and Exclusion Principle, we have

\[
\{i : - : b\} = \bigcup_{a \in Q} \{i : a : b\} - \sum_{a_1, a_2 \in Q} \{i : a_1 : b\} \bigcap \{i : a_2 : b\} - \cdots + (-1)^{k-1} \sum_{a_1, a_2, \ldots, a_k \in Q} \bigcap_{j=1}^k \{i : a_j : b\} + \cdots + (-1)^{q-1} = C(q, 1) q^{a-n} - C(q, 2) q^{a-2n} - \cdots + (-1)^{k-1} C(q, k) q^{a-kn} + \cdots + (-1)^{q-1} = \sum_{k=1}^q (-1)^{k-1} C(q, k) q^{a-kn} = q^a \sum_{k=1}^q \left(1 - (q^{a-k} - 1)^q \right) = q^{a} - (q^{a-1} - 1)^q. \]

Similar to Lemma 3.3, we have

**Lemma 3.5** If \( \{i_1, i_2, \ldots, i_k\} \subset [n] \), then

\[
\{i_1, i_2, \ldots, i_k\} \in [n].
\]

Based on this lemma, we can get the following result.

**Theorem 3.6** We have

\[ |i : : : | = \sum_{i \in S} (-1)^{k-1} C(n, k) q^{a-i} q^{a-k}. \]

**Proof:** By Inclusion and Exclusion Principle, we have
\[
\begin{align*}
|\langle \neg a : b \rangle| &= \left| \bigcup_{i \in [a]} \{i : a : b\} \right| \\
&= \sum_{i : a : b \in \mathcal{Q}} \left| \{i : a : b\} \right| - \sum_{i : a : b \in \mathcal{Q}} \left| \{i : a : b\} \cap \{j : a : b\} \right| \\
&\quad + \cdots + (-1)^{k-1} \sum_{i : a : b \in \mathcal{Q}} \left[ \left| \{i : a : b\} \right| - \sum_{j : a : b \in \mathcal{Q}} \left| \{i : a : b\} \cap \{j : a : b\} \right| \right] \\
&= C(n, 1) q^{-k} - C(n, 2) q^{-2} + \cdots + \left( -1 \right)^{k-1} C(n, k) q^{-k} + \cdots + \left( -1 \right)^{n} q^{-n} \\
&= \sum_{i : a : b \in \mathcal{Q}} \left( -1 \right)^{k-1} C(n, k) q^{-k} q^{-n}.
\end{align*}
\]

From the above theorem, we can get the following result.

**Theorem 3.7** We have

\[
|\langle \neg a : \neg b \rangle| = q \sum_{i : a : b \in \mathcal{Q}} (-1)^{k-1} C(n, k) q^{-k} q^{-n}.
\]

Proof: Because \(\langle \neg a : \neg b \rangle = \bigcup_{b \in \mathcal{Q}} \langle \neg a : b \rangle\), by Theorem 3.6, we just need to show \(\langle \neg a : b \rangle \cap \langle \neg a : b \rangle = \emptyset\) if \(b_1 \neq b_2\). Suppose \(f \in \langle \neg a : b \rangle \cap \langle \neg a : b \rangle\), then there exist \(i_1 \) and \(i_2 \in \mathcal{N}\) such that \(f \in \langle i_1 : a : b \rangle \cap \langle i_2 : a : b \rangle\) since \(\langle \neg a : b \rangle = \bigcup_{i : a : b \in \mathcal{Q}} \langle i : a : b \rangle\).

If \(i_1 = i_2\), we get a contradiction by Proposition 2.2. If \(i_1 \neq i_2\), we get a contradiction by Proposition 2.3 since \(\langle i_1 : a : b \rangle \cap \langle i_2 : a : b \rangle\) and \(\langle i_1 : a : b \rangle \cap \langle i_2 : a : b \rangle\).

Now, we are going to find the formula for the number of all the canalizing functions with given canalized value \(b\). In other words, the formula of \(|\langle \neg a : \neg b \rangle|\).

Let \(S_b = \{ \langle i : a : b \rangle \mid i \in [n], a \in \mathcal{Q} \}\) for any \(b \in \mathcal{Q}\). By Inclusion and Exclusion Principle, we have

\[
|\langle \neg a : \neg b \rangle| = \left| \bigcup_{i : a : b \in \mathcal{Q}} \{i : a : b\} \right| = \sum_{i : a : b \in \mathcal{Q}} (-1)^{k-1} N_k.
\]

where

\[
N_k = \sum_{i : a : b \in \mathcal{Q}} \left| \bigcap_{T \in \mathcal{S}} \right|.
\]

In order to evaluate \(N_k\), we write all the members in \(S_b\) as the following \(n \times q\) matrix.

\[
A = \begin{pmatrix}
\langle 1 : a : b \rangle & \langle 1 : a : b \rangle & \cdots & \langle 1 : a : b \rangle \\
\langle 2 : a : b \rangle & \langle 2 : a : b \rangle & \cdots & \langle 2 : a : b \rangle \\
\vdots & \vdots & \ddots & \vdots \\
\langle n : a : b \rangle & \langle n : a : b \rangle & \cdots & \langle n : a : b \rangle
d\end{pmatrix}
\]

For any \(s \subseteq S_b\) with \(|s| = k\), we will choose \(k\) elements from the above matrix to form \(s\).

Suppose \(k_1\) of its elements are from the first row (there are \(C(q, k_1)\) ways to do so). Let these \(k_1\) elements be \(\{1 : a_1 : b_1\}, \{1 : a_2 : b_2\}, \ldots, \{1 : a_{k_1} : b_{k_1}\}\).

Suppose \(k_2\) of its elements are from the second row (there are \(C(q, k_2)\) ways to do so). Let these \(k_2\) elements be \(\{2 : a_1 : b_1\}, \{2 : a_2 : b_2\}, \ldots, \{2 : a_{k_2} : b_{k_2}\}\).

\[
\vdots
\]

Suppose \(k_n\) of its elements are from the last row (there are \(C(q, k_n)\) ways to do so). Let these \(k_n\) elements be \(\{n : a_1 : b_1\}, \{n : a_2 : b_2\}, \ldots, \{n : a_{k_n} : b_{k_n}\}\).

\[
k_1 + k_2 + \cdots + k_n = k, 0 \leq k_i \leq q, i = 1, 2, \ldots, n.
\]

Similar to Lemma 3.3, we have

**Lemma 3.8** Let \(s\) be the subset of \(S_b\) as mentioned above, then

\[
\left| \bigcap_{T \in \mathcal{S}} \right| = q^{-(q-k)(q-k)}.
\]

Hence,

\[
N_k = \sum_{k_1 + \cdots + k_n = k} C(q, k_1) \cdots C(q, k_n) q^{-(q-k)(q-k)}.
\]

We get

**Theorem 3.9** For any \(b \in \mathcal{Q}\), we have

\[
|\langle \neg a : \neg b \rangle| = \sum_{k_1 + \cdots + k_n = k} \left( \prod_{k = 1}^q \left( -1 \right)^{k-1} \left( C(q, k) q^{-k} \right) \right).
\]

In order to evaluate \(\langle i : \neg a : \neg b \rangle\), we need two more lemmas. Their proofs are similar to that of Lemma 3.3 and we omit them.

**Lemma 3.10** If \(\{a_1, a_2, \ldots, a_k\} \subseteq \mathcal{Q}\) and \(\{b_1, b_2, \ldots, b_k\} \subseteq \mathcal{Q}\), then

\[
\left| \bigcap_{j=1}^k \{a_j : b_j\} \right| = q^{-(q-k)(q-k)}.
\]

**Lemma 3.11** If \(a_1, a_2, a_3, a_4, \ldots, a_k, a_{k+1}, \ldots, a_{k+l}\) are \(k_1, \ldots, k_l\) distinct elements of \(\mathcal{Q}\), \(\{b_1, \ldots, b_k\} \subseteq \mathcal{Q}\). Then,

\[
\left| \bigcap_{j=1}^k \{a_j : b_j\} \right| = \left| \bigcap_{j=1}^{k_1} \{a_j : b_j\} \right| \left| \bigcap_{j=1}^{k_2} \{a_j : b_j\} \right| \cdots \left| \bigcap_{j=1}^{k_l} \{a_j : b_j\} \right| = q^{-(q-k)(q-k)}.
\]

Now, we are ready to find the cardinality of \(\langle i : \neg a : \neg b \rangle\).

**Theorem 3.12** We have

\[
|\langle i : \neg a : \neg b \rangle| = q \sum_{k=1}^{n} \sum_{i : a : b} \left( \left| \bigcap_{T \in \mathcal{S}} \right| \right).
\]

Proof: First, we have \(\langle i : \neg a : \neg b \rangle = \bigcup_{s \subseteq S_b} \{i : a : b\}\).
Let $S_i = \{(i:a:b) | a, b \in Q\}$, we get
\[
\binom{q^2}{i:-:-} = \sum_{k=1}^{q^2} (-1)^{k-1} N_k.
\]
where
\[
N_k = \sum_{s \subset S_i, |s| = k} \left| \bigcap T_s \right|.
\]
In order to evaluate $N_k$, we write all the elements in $S_i$ as the following $q \times q$ matrix.
\[
B = \begin{bmatrix}
(i:a_1:b_1) & (i:a_2:b_1) & \cdots & (i:a_q:b_1) \\
(i:a_1:b_2) & (i:a_2:b_2) & \cdots & (i:a_q:b_2) \\
\vdots & \vdots & \ddots & \vdots \\
(i:a_1:b_q) & (i:a_2:b_q) & \cdots & (i:a_q:b_q)
\end{bmatrix}
\]
For any $s \subset S_i$ with $|s| = k$, we will choose $k$ elements from the above matrix to form $s$.

Suppose $k_1$ of its elements are from the first row (There are $C(q,k_1)$ ways to do so). Let these $k_1$ elements be $(i:a_1:b_1), (i:a_2:b_1), \cdots, (i:a_{k_1}:b_1)$.

Suppose $k_2$ of its elements are from the second row, we must choose these elements from different columns, otherwise the intersection will be $\emptyset$ by Proposition 2.2 (There are $C(q-k_1,k_2)$ ways to do so). Let these $k_2$ elements be $(i:a_1:b_2), (i:a_2:b_2), \cdots, (i:a_{k_2}:b_2)$
\[
\vdots
\]
Suppose $k_q$ of its elements are from the last row (There are $C(q-k_{q-1},k_q)$ ways to do so). Let these $k_q$ elements be $(i:a_1:b_q), (i:a_2:b_q), \cdots, (i:a_{k_q}:b_q)$.

where $k_1 + k_2 + \cdots + k_q = k, 0 \leq k_i \leq q, i = 1, 2, \cdots, q$. We have
\[
N_k = \sum_{s \subset S_i, |s| = k} \left| \bigcap T_s \right| = \sum_{k_1 + \cdots + k_q = k, 0 \leq k_i \leq q} C(q,k_1) C(q-k_1,k_2) \cdots C(q-k_{q-1},k_q) I_{k_1 \cdots k_q}
\]
where
\[
I_{k_1 \cdots k_q} = \left| \bigcap_{j=1}^{k_1} \{i:a_j:b_j\} \bigcap_{j=1}^{k_2} \{i:a_j:b_2\} \cdots \bigcap_{j=1}^{k_q} \{i:a_j:b_q\} \right|
\]

By Lemma 3.11, we know
\[
I_{k_1 \cdots k_q} = (q^{q-k_1-\cdots-k_q})^{q-1} = q^{(q-k)(q-1)}
\]
This number is zero if $k > q$.

A straightforward computing shows that
\[
C(q,k_1) C(q-k_1,k_2) \cdots C(q-k_{q-1},k_q)
\]
\[
= \frac{q!}{k_1! k_2! \cdots k_q! (q-k)!}
\]
Hence, we get
\[
\binom{q^2}{i:-:-} = \sum_{k=1}^{q^2} (-1)^{k-1} N_k = \sum_{k=1}^{q^2} (-1)^{k-1} \frac{q!}{k_1! k_2! \cdots k_q! (q-k)!}
\]
\[
= q \sum_{k=1}^{q^2} (-1)^{k-1} \frac{q!}{(q-k)!} \sum_{k_1 + \cdots + k_q = k, 0 \leq k_i \leq q} \prod_{i=1}^{q} k_i!
\]
Now we begin to evaluate $\binom{-:-:-}{-}$.

**Theorem 3.13** We have
\[
\binom{-:-:-}{-} = \sum_{k=1}^{q^2} (-1)^{k-1} U_k + \sum_{k=1}^{q^2} (-1)^{k-1} V_k
\]
where
\[
U_k = n \sum_{k_1 + \cdots + k_q = k, 0 \leq k_i \leq q} \frac{q!}{(q-k)!} \frac{q^{(q-k)(q-1)}}{q^{(q-k)(q-1)}}
\]
\[
= n q! \frac{q^{(q-k)(q-1)}}{(q-k)!} \sum_{k_1 + \cdots + k_q = k, 0 \leq k_i \leq q} \prod_{i=1}^{q} k_i!
\]
and
\[
V_k = q \sum_{k_1 + \cdots + k_q = k, 0 \leq k_i \leq q} \left( \prod_{i=1}^{q} C(q,k_i) \right) q^{(q-k)(q-2)}
\]

Proof: Let
\[
S = \{(i:a:b) | a, b \in Q, i \in [n]\}
\]
We have
\[
\binom{-:-:-}{-} = \bigcup_{i \geq a \in Q} \bigcup_{b \in Q} \binom{i:-:-}{i:a:b}
\]
and
\[
\binom{-:-:-}{-} = \sum_{k=1}^{nq^2} (-1)^{k-1} N_k
\]
where
\[
N_k = \sum_{s \subset S_i, |s| = k} \left| \bigcap T_s \right|
\]
We write all the $nq^2$ elements of $S$ as the following $n$ matrices.
two cases are not disjoint.

Suppose we choose \( k \) elements from
\[
M_i, i = 1, 2, \cdots, n, k_i + k_2 + \cdots + k_n = k,
\]
\[
0 \leq k_i \leq k, i = 1, 2, \cdots, n
\]
If there exist \( i \) such that \( k_i = k \), then \( k_j = 0, \forall j \neq i \).

This implies the intersection looks like the one in Lemma 3.11 and \( k \leq q \).
If \( 0 \leq k \leq k - 1, \forall i \in [n] \), then the intersection looks like the one in Lemma 3.8 and \( k \leq nq \).

The above two cases are disjoint now. By Lemma 3.11 and Lemma 3.8, we get
\[
N_k = \sum_{x=1}^{q} \left| \bigcap_{i=1}^{k} T_i \right| = \sum_{\forall k \leq q, \forall x} 
\]
\[
= \sum_{\forall k \leq q} \sum_{\forall x} \left| \bigcap_{i=1}^{k} T_i \right| = U_k + V_k
\]
where (Note: there are \( n \) matrices \( M_1, M_2, \cdots, M_n \) and \( q \) columns of \( M \))
\[
U_k = n\sum_{k_1 + \cdots + k_n = k} \frac{q!}{(q-k)!} \frac{1}{t_1!t_2! \cdots t_q!} \frac{\prod_{i=1}^{q} C(q, k_i)}{\prod_{i=1}^{q} (q-k_i)}
\]
\[
V_k = q\sum_{k_1 + \cdots + k_n = k} \left( \prod_{i=1}^{q} C(q, k_i) \right) g^{q-k}
\]
Hence,
\[
\left| \cap_{i=1}^{k} T_i \right| = \sum_{k=1}^{nq} (-1)^{k-1} N_k
\]
\[
= \sum_{k=1}^{nq} (-1)^{k-1} (U_k + V_k) = \sum_{k=1}^{nq} (-1)^{k-1} U_k + \sum_{k=1}^{nq} (-1)^{k-1} V_k,
\]
In the following, we will reduce the formula
\[
\left| \cap_{i=1}^{k} T_i \right|
\]
when \( q = 2 \) and compare it with the one in [1]. We have
\[
\left| \cap_{i=1}^{k} T_i \right| = \sum_{k=1}^{2} (-1)^{k-1} U_k + \sum_{k=1}^{2n} (-1)^{k-1} V_k,
\]
where
\[
U_k = n\sum_{k_1 + \cdots + k_n = k} \frac{2!}{t_1!t_2! (2-k)!} \frac{1}{2^{2-k} t_1!t_2!}
\]
\[
V_k = 2\sum_{k_1 + \cdots + k_n = k} \left( \prod_{i=1}^{n} C(2, k_i) \right) 2^{k-1}
\]
A simple calculation shows that
\[
U_1 = 4n2^{2-n} = C(n, 1) 2^{2-n}
\]
and
\[ U_2 = 4n. \]

We have the following notation

**Definition 3.14** We call \( f(x) \equiv g(x) \) if 
\[
\lim_{x \to \infty} \frac{f(x)}{g(x)} = 1.
\]

We will give a proof of the last row, the others are similar and easier.

\[
\begin{align*}
\left\lfloor -:::\right\rfloor = & \sum_{k=1}^{n} (-1)^{k+1} C(n,k) 2^{k+1} 2^{2k-4} \\
+ & \sum_{3 \leq k \leq 2n} (-1)^{k-1} \sum_{\lfloor t \rfloor \leq k/2} C(n,t) C(n-t,k-2t) 2^{k-2t+1}
\end{align*}
\]

When \( n=1,2,3,4, \) one can obtain (without calculator) the sequence 4, 14, 120, 3514. These results are consistent with those in [1]. By [1], the cardinality of \( \left\lfloor -:::\right\rfloor \) should be 
\[
\left\lfloor -:::\right\rfloor = 2((-1)^n - n) + \sum_{1 \leq k \leq n} (-1)^{k+1} C(n,k) 2^{k+1} 2^{2k-4}.
\]

So, we obtain the following combinatorial identity for any positive integer \( n \).
\[
\sum_{3 \leq k \leq 2n} (-1)^{k-1} \sum_{\lfloor t \rfloor \leq k/2} C(n,t) C(n-t,k-2t) 2^{k-2t+1} = 2((-1)^n + n)
\]

The left sum should be explained as 0 if \( n=1 \). As usual, \( C(n,k) \) is 0 if \( k > n \).

From Theorem 3.1, we know \( \left\lfloor i:a:b \right\rfloor = [q^{x} - q^{x-1}] \), since \( -:::- = \bigcup_{i=1}^{k} \bigcup_{a \in S_i} \bigcup_{b \in F_i} (i:a:b) \), we obtain
\[
\left\lfloor -:::\right\rfloor \leq nq^2 q^{(q-1)p^{x-1}}.
\]

In order to get an intuitive idea about the magnitude of all the cardinality numbers, We will find their asymptotes as \( n \to \infty \) or \( q \to \infty \).

\[
V_1 = 0 \quad \text{since the condition of the sum is not satisfied.}
\]

\[
V_2 = 2 \sum_{k_1 + \cdots + k_s = 2} C(2,k_1) 2^{2k_1} \leq C(n,2) 2^n 2^{2n-2}.
\]

Note, \( C(n,2) \) is the number of solutions of the equation \( k_1 + \cdots + k_s = 2, 0 \leq k_i \leq 1 \).

When \( 3 \leq k \leq 2n \),
\[
V_2 = 2 \sum_{k_1 + \cdots + k_s = 2} C(2,k_1) 2^{2k_1} + \sum_{1 \leq t \leq \lfloor k/2 \rfloor} C(n,t) C(n-t,k-2t) 2^{k-2t+1}
\]

Note, \( C(n,t) C(n-t,k-2t) \) is the number of solutions of the equation \( k_1 + \cdots + k_s = 2, 0 \leq k_i \leq 2 \) with exactly \( t \) components equal to 2.

hence, when \( q = 2 \),
\[
\left\lfloor -:::-\right\rfloor = -4n + \sum_{1 \leq k \leq n} (-1)^{k+1} C(n,k) 2^{k+1} 2^{2k-4} \\
+ \sum_{3 \leq k \leq 2n} (-1)^{k-1} \sum_{\lfloor t \rfloor \leq k/2} C(n,t) C(n-t,k-2t) 2^{k-2t+1}
\]

When \( n=1,2,3,4, \) one can obtain (without calculator) the sequence 4, 14, 120, 3514. These results are consistent with those in [1]. By [1], the cardinality of \( \left\lfloor -:::-\right\rfloor \) should be 
\[
\left\lfloor -:::-\right\rfloor = 2((-1)^n - n) + \sum_{1 \leq k \leq n} (-1)^{k+1} C(n,k) 2^{k+1} 2^{2k-4}.
\]

So, we obtain the following combinatorial identity for any positive integer \( n \).
\[
\sum_{3 \leq k \leq 2n} (-1)^{k-1} \sum_{\lfloor t \rfloor \leq k/2} C(n,t) C(n-t,k-2t) 2^{k-2t+1} = 2((-1)^n + n)
\]

The left sum should be explained as 0 if \( n=1 \). As usual, \( C(n,k) \) is 0 if \( k > n \).

From Theorem 3.1, we know \( \left\lfloor i:a:b \right\rfloor = [q^{x} - q^{x-1}] \), since \( -:::- = \bigcup_{i=1}^{k} \bigcup_{a \in S_i} \bigcup_{b \in F_i} (i:a:b) \), we obtain
\[
\left\lfloor -:::-\right\rfloor \leq nq^2 q^{(q-1)p^{x-1}}.
\]

In order to get an intuitive idea about the magnitude of all the cardinality numbers, We will find their asymptotes as \( n \to \infty \) or \( q \to \infty \).
\[ V_n = q \sum_{k_1 + \ldots + k_j = k \atop 0 \leq k_i \leq 1, 2, \ldots, n} \left( \prod_{j=1}^n C(q,k_j) \right) \prod_{i=1}^n (q^{-1} k_i) \]
\[ \leq q \sum_{0 \leq k_i \leq 1, 2, \ldots, n} (q!)^n q^{(q-1)n} q^{-2} \]
\[ = q(q+1)^n (q!)^n q^{(q-3)n} q^{-2} . \]

Note, \( \prod_{j=1}^n (q-k_j) \leq (q-1)^n q^{-2} \). Hence,
\[ \left| \sum_{j=1}^n (-1)^{j-1} V_k \right| \leq nqq(q+1)^n (q!)^n q^{(q-3)j} q^{-2} . \]

We obtain
\[ \left| \sum_{j=1}^n (-1)^{j-1} V_k \right| \leq \frac{nqq(q+1)^n (q!)^n q^{(q-3)j} q^{-2} }{U_1} \]
\[ \leq \frac{nqq(q+1)^n (q!)^n q^{(q-3)j} q^{-2} }{q^{-q^2(q+1)q^{-1}}} \]
\[ = \frac{(q+1)^n (q!)^n q^{(q+1)n} q^{(q-3)j} q^{-2} }{q^{-q^2(q+1)q^{-1}}} \leq \frac{q^{(2q)n}}{q^{q^2(q+1)q^{-1}}} . \]

Hence,
\[ \lim_{n \to \infty} \frac{\left| \sum_{j=1}^n (-1)^{j-1} V_k \right|}{U_1} = 0. \]

In summary, we obtain
\[ \lim_{n \to \infty} \frac{\left| \sum_{j=1}^n (-1)^{j-1} \right|}{nqq^2 q^{q(q-1)q^{-1}}} = 1. \]

In other words,
\[ \left| \sum_{j=1}^n (-1)^{j-1} \right| \leq nqq^2 q^{q(q-1)q^{-1}} . \]

From the above proof, it is also clear that we have
\[ \lim_{q \to \infty} \frac{\left| \sum_{j=1}^n (-1)^{j-1} \right|}{nqq^2 q^{q(q-1)q^{-1}}} = 1. \]

In other words,
\[ \left| \sum_{j=1}^n (-1)^{j-1} \right| \leq nqq^2 q^{q(q-1)q^{-1}} . \]

When \( q = 2 \), the first equation of the last row in the above theorem has been obtained in [1].

### 4. Conclusion

In this paper, we generalized the definition of Boolean canalyzing functions to the functions of multi-state case. Using Inclusion and Exclusion Principle, we get formulas for the cardinality all such functions and the cardinalities of its various subsets. When \( q = 2 \), we derive an interesting combinatorial identity by equating our formula to the one in [1]. Finally, for a better understanding to the magnitudes, we provide all the asymptotes of these cardinalities as either \( n \to \infty \) or \( q \to \infty \).

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### REFERENCES


