Further Results on Acyclic Chromatic Number*

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ABSTRACT
An acyclic coloring of a graph is a proper vertex coloring such that the union of any two color classes induces a disjoint collection of trees. The purpose of this paper is to derive exact values of acyclic chromatic number of some graphs.

Keywords: Acyclic Coloring; Acyclic Chromatic Number; Central Graph; Middle Graph; Total Graph

1. Introduction
Graph coloring is a branch of graph theory which deals with such partitioning problems. For example, suppose that we have world map and we would like to color the countries so that if two countries share a boundary line, then they need to get different colors. We can translate the map to graph by letting countries be represented by vertices and two vertices are made adjacent if and only if the corresponding countries share a boundary line. Then the problem of map coloring is equivalent to vertex coloring of the corresponding graph. Hence the original map coloring now reduces to vertex coloring of the associated graph.

A vertex coloring of a graph is an assignment of colors to its vertices so that no two vertices have the same color. The chromatic number \( \chi(G) \) of a graph \( G \) is the minimum number of colors needed to label the vertices, so that adjacent vertices receive different colors.

A proper vertex coloring of a graph is acyclic if every cycle uses at least three colors [1]. The acyclic chromatic number of \( G \), denoted by \( a(G) \), is the minimum number of colors required for its acyclic coloring.

2. Acyclic Coloring of Central Graph of \( C_n \)

2.1. Central Graph [2]
Let \( G \) be a finite undirected graph with no loops and multiple edges. The central graph of a graph \( G \), \( C(G) \) is obtained by subdividing each edge of \( G \) exactly once and joining all the non-adjacent vertices of \( G \).

2.2. Structural Properties of Central Graphs
Let \( G=(p,q) \) be any undirected simple graph, then by the definition of \( C(G) \) of a graph.
- The number of vertices in the central graph of \( G \) is \( p(C(G)) = p+q \).
- For any \( (p,q) \) graph there exists exactly \( p \) vertices of degree \( p-1 \) and \( q \) vertices of degree \( 2 \) in \( C(G) \).
- The central graph of two isomorphic graphs is also isomorphic.
- The maximum degree in \( C(G) \) is \( \Delta = p-1 \).
- Central graph of any graph is connected.
- If \( G \) is any graph with odd \( p \) then \( C(G) \) is Eulerian.

2.3. Theorem
The acyclic coloring of central graph of cycle, \( a(C(C_n)) = n-2, \) for \( n > 4 \).

Proof
Consider the graph \( C_n \) with vertex set \( V = \{ v_i / 1 \leq i \leq n \} \). Let \( G = C(C_n) \) be the central graph of \( C_n \), which is obtained by subdividing each edge of \( C_n \) exactly once and joining non adjacent vertices of \( C_n \). Let the newly introduced vertices be \( v_{h,k}, k = 1,2,3,\ldots,n \) with \( h < k \). Consider the color class \( C = \{ c_i / 1 \leq i \leq n-2 \} \). Now assign a proper coloring to the vertices as follows. The coloring is in such a way that the sub graph induced by any two color is a forest containing at most the path \( P_4 \). The vertices \( v_i \) are as-
signed the colour $c_i$ for $i = 1, 2$; $c_{i+4}$ for $i = 3, 4$; $c_{i+2}$ for $5 \leq i \leq n$.

Case 1: When $n > 5$.

The newly $v_{2,1}, v_{4,5}$ are assigned the colors $c_i$ and $c_4$ respectively and all others are colored properly.

Case 2: When $n = 5$.

$v_{2,1} = v_{4,5} = c_1$, all others are assigned so that the coloring is proper. Now the coloring is obviously acyclic, by the very arrangement of the colors. It is also minimum, because if we replace any color by an already used color, it will become either improper or cyclic (Figures 1 and 2).

2.4. Note

$a(C(C_i)) = 3$, for $n = 3, 4$.

3. Acyclic Coloring of Line Graph of Central Graph of $K_n$

3.1. Definition

Let $G$ be a finite undirected graph with no loops and multiple edges, the line graph of $G$, denoted by $L(G)$, is the intersection graph $\Omega(X)$. Thus the points of $L(G)$ are the lines of $G$, with two points of $L(G)$ are adjacent whenever the corresponding lines of $G$ are.

3.2. Structural Properties of Line Graph of Central Graph of $K_n$

Line graph of central graph of $K_n$ is denoted by $L(G(C(K_n)))$.

- Number of vertices in $L(G(C(K_n))) = n(n - 1)$.
- Maximum Degree of vertices = Minimum Degree of vertices = $n - 1$.
- $L(G(C(K_n)))$ contains $n$ copies of vertex disjoint $K_{n,4}$.
- There is a cycle $C'$ of length $2n$ with alternate edges from each of the complete graph $K_{n,4}$.

3.3. Theorem

For any complete graph $K_n$, $a[L(G(C(K_n)))] = n - 1$.

**Proof:**

Let $K_n$ be the complete graph on $n$ vertices. Consider its line graph of central graph $G = L(G(C(K_n)))$, it contains $n$ copies of vertex disjoint sub graphs $K_{4,n}$, $j = 1, 2, \ldots, n$ and which are marked in anti-clockwise direction. Let

$$V[L(G(C(K_n)))] = \{u_1^i, u_2^i, u_3^i, \ldots, u_{n+1}^i\}$$

where $j = 1, 2, 3, \ldots, n$; so that the total number of vertices in $G$ is $n(n - 1)$. Here there exists a unique bridge between each pair of sub graphs $K_{4,n}$. The bridge in the consecutive pairs of sub graph $(K_{4,n}, K_{4,n+1})$, is given by for $2i < n$, it is $(u_{2i}^i, u_{2i+1}^i)$ and for $2i \geq n$, it is $(u_{2i}^i, u_{2i+1}^i)$, $2i = x \mod (n-1)$ only for $x = 1, 2, \ldots, n - 1$, form a bridge in the sub graph $(K_{4,n}, K_{4,n+1})$. In a similar manner bridges are formed in non consecutive pairs also. Consider the color class $C = \{c_1, c_2, c_3, \ldots, c_{n+1}\}$. Assign the color $c_i$ to the vertex $u_{j}^i$ for $j = 1, 2, 3, \ldots, n$. Next we prove that the coloring is acyclic. That is the coloring does not induce a bi-chromatic cycle. Clearly for each complete sub graph $K_{4,n}$, the coloring is acyclic (it never induce a bi-chromatic cycle). Now exactly two pairs of sub graphs $K_{4,n}$, $j = 1, 2, 3, \ldots, n$, never allow to induce a bi-chromatic cycle for any pair $c_j$, as there is only a unique bridge between each pair of sub graphs $K_{4,n}$. Note that bi-chromatic cycle is possible only for even cycles. The coloring is in such a way that more than three sub graphs $K_{4,n}$, never allow to induce a bi-chromatic cycle for any pair $c_j$. The maximum number of times a color will occur in any bi-chromatic path in this coloring is three. So the above said coloring acyclic. Also the coloring is minimum, as $G = L(G(C(K_n)))$, contains the subgraph $K_{4,n}$, minimum $n - 1$ colors are required for its proper coloring (Figure 3).
4. Acyclic Coloring of Middle Graph of $C_n$

4.1. Middle Graph [3]

Let $G$ be a graph with vertex set $V(G)$ and edge set $E(G)$. The middle graph of $G$, denoted by $M(G)$, is defined as follows. The vertex set of $M(G)$ is $V(G) \cup E(G)$. Two vertices $x, y$ in the vertex set of $M(G)$ are adjacent in $M(G)$ in case one of the following holds:

1) $x, y$ are in $E(G)$ and $x, y$ are adjacent in $G$; 2) $x$ is in $V(G)$, $y$ is in $E(G)$, and $x, y$ are incident in $G$.

4.2. Theorem

The acyclic chromatic number of the middle graph of $C_n$ is $a(M(C_n))=3$, for $n \geq 3$.

Proof

$V(C_n) = \{v_1, v_2, \ldots, v_n\}$ and $E(C_n) = \{e_1, e_2, \ldots, e_n\}$ in which $e_i = v_{i+1}$ with $v_{n+1} = v_1$. Let $G = M(C_n)$ be the middle graph of the $n$-cycle. By the definition of middle graph

$$V(M(C_n)) = \{v_1, v_2, \ldots, v_n\} \cup \{e_1, e_2, \ldots, e_n\},$$

and

$$E(M(C_n)) = \{e_1, e_{n+1} / 1 \leq i \leq n-1\} \cup e_v e_i \cup \{v_{i+1} / 1 \leq i \leq n\} \cup e_v v_i \cup \{v_{i+1} / 1 \leq i \leq n\}.$$

Then in the middle graph, there are $n$-vertices of degree 2 and another $n$-vertices of degree 4. Let $C^*_n$ be the cycle of length $n$ in $G$ with degree of each vertex 4 and $C'^*_n$ be the cycle of length $n$ in $G$ with degree of vertices alternately 2 and 4. The cycle $C^*_n$ are assigned the colors $c_1$ and $c_2$ alternately with last vertex preceding to $v_n$ by $c_1$. All other vertices except vertices adjacent to $v'_n$ (which are colored as $c_2$) are colored as $c_1$. The coloring is minimum, as for any cycle minimum 3 colors needed for its acyclic coloring. The coloring is acyclic (Figure 4).

5. Acyclic Coloring of Total Graph of $C_n$

5.1. Total Graph [3]

Let $G$ be a graph with vertex set $V(G)$ and edge set $E(G)$. The total graph of $G$, denoted by $T(G)$ is defined as follows. The vertex set of $T(G)$ is $V(G) \cup E(G)$. Two vertices $x, y$ in the vertex set of $T(G)$ are adjacent in $T(G)$ in case one of the following holds:

1) $x, y$ are in $V(G)$ and $x$ is adjacent to $y$ in $G$; 2) $x, y$ are in $E(G)$ and $x, y$ are adjacent in $G$; 3) $x$ is in $V(G)$, $y$ is in $E(G)$, and $x, y$ are incident in $G$.

5.2. Some Structural Properties of Total Graph of $C_n$

- Every cycle has a 4-regular total graph.
- The number of vertices in the total graph of $C_n$ is 2 times the number of vertices in the cycle $C_n$.
- The number of edges in the total graph of $C_n$ is 4 times the number of edges in the cycle $C_n$.
- The total graph of $C_n$ is Eulerian.
- The total graph of $C_n$ is Hamiltonian.

5.3. Theorem

The acyclic chromatic number of the total graph of $C_n$ is $a(T(C_n))=4$, for $n \geq 4$.

Proof

Let $V(T(C_n)) = \{v_1, v_2, \ldots, v_n\}$ and $E(T(C_n)) = \{e_1, e_2, \ldots, e_n\}$ in which $e_i = v_{i+1}$ with $v_{n+1} = v_1$. Let $G = T(C_n)$ be the total graph of the $n$-cycle. By the definition of total graph

$$V(T(C_n)) = \{v_1, v_2, \ldots, v_n\} \cup \{e_1, e_2, \ldots, e_n\},$$

and

$$E(T(C_n)) = \{e_1, e_{n+1} / 1 \leq i \leq n-1\} \cup \{v_{i+1} / 1 \leq i \leq n-1\} \cup e_v e_i \cup \{v_{i+1} / 1 \leq i \leq n-1\} \cup e_v v_i \cup \{v_{i+1} / 1 \leq i \leq n-1\} \cup v_i v_{i+1}.$$
By Menger’s theorem as, there are four pair wise vertex-independent paths between any two non adjacent vertices, the total graph of \( C_n \) is 4-connected. To prove that \( a(T(C_n)) = 4 \), if possible consider the color class \( C = \{c_1, c_2, \ldots, c_m\} \) with \( m < 4 \), such that the coloring is acyclic. Then there exist no pair \( (v_h, v_k) \) such that they induce a bi-chromatic cycle. \( i.e., \) there exist a three vertex cut in \( T(C_n) \). This is a contradiction to the fact that \( T(C_n) \) is 4-connected. Also acyclic chromatic number is can’t be 5, as in this case we can replace a color by an already used color.

Therefore \( a(T(C_n)) = 4, \) for \( n \geq 4 \) (Figure 5).

5.4. Note

\[ a(T(C_n)) = 5. \]

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REFERENCES

