Edge Colorings of Planar Graphs without 6-Cycles with Two Chords

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ABSTRACT

It is proved here that if a planar graph has maximum degree at least 6 and any 6-cycle contains at most one chord, then it is of class 1.

Keywords: Edge Coloring; Planar Graph; Cycle; Class 1

1. Introduction

All graphs considered here are finite and simple. Let G be a graph with the vertex set V(G) and edge set E(G). If v ∈ V(G), then its neighbor set N(v) (or simply N(v)) is the set of the vertices in G adjacent to v and the degree d(v) of v is |N(v)|. We denote the maximum degree of G by Δ(G). For G′ ⊆ V(G), we denote N(G′) = ∪u∈G′ N(u). A vertex of degree k, a k-vertex, k⁹ -vertex is a vertex of degree k, at least k. A k (or k⁹) -vertex adjacent to a vertex x is called a k(or k⁹)-neighbor of x. Let d(x), d⁺(x) denote the number of k-neighbors, k⁹-neighbors of x. A k-cycle is a cycle of length k. Two cycles sharing a common edge are said to be adjacent. Given a cycle C of length k in G, an edge xy ∈ E(G) \\ E(C) is called a chord of C if x, y ∈ V(C). Such a cycle C is also called a chordal-k-cycle.

A graph is k-edge-colorable, if its edges can be colored with k colors in such a way that adjacent edges receive different colors. The edge chromatic number of a graph G, denoted by χ′(G), is the smallest integer k such that G is k-edge-colorable. In 1964, Vizing showed that for every simple graph G, Δ(G) ≤ χ′(G) ≤ Δ(G) + 1. A graph G is said to be of class 1 if χ′(G) = Δ(G), and of class 2 if χ′(G) = Δ(G) + 1. A graph G is critical if it is connected and of class 2 and χ′(G − e) < χ′(G) for any edge e of G. A critical graph with maximum degree Δ is called a Δ-critical graph. It is clear that every critical graph is 2-connected.

For planar graphs, more is known. As noted by Vizing [1], if C₄, C₅, the octahedron, and the icosahedron have one edge subdivided each, class 2 planar graphs are produced for Δ ∈ {2, 3, 4, 5}. He proved that every planar graph with Δ ≥ 8 is of class 1 (There are more general results, see [2] and [3]) and then conjectured that every planar graph with maximum degree 6 or 7 is of class 1. The case Δ = 7 for the conjecture has been verified by Zhang [4] and, independently, by Sanders and Zhao [5]. The case Δ = 6 remains open, but some partial results are obtained. Theorem 16.3 in [1] stated that a planar graph with the maximum degree Δ and the girth g is of class 1 if Δ ≥ 3 and g ≥ 8, or Δ ≥ 4 and g ≥ 5, or Δ ≥ 5 and g ≥ 4. Lam, Liu, Shiu and Wu [6] proved that a planar graph G is of class 1 if Δ ≥ 6 and no two 3-cycles of G sharing a common vertex. Zhou [7] obtained that every planar graph with Δ ≥ 6 and without 4 or 5-cycles is of class 1. Ni [9] extended the result that every planar graph with Δ ≥ 6 and without chordal 6-cycles is of class 1. In the note, we improve the above result by proving that every planar graph with Δ ≥ 6 and without 6-cycles with two chords is of class 1.

2. The Main Result and Its Proof

To prove our result, we will introduce some known lemmas.

Lemma 1. (Vizing’s Adjacency Lemma [1]). Let G be a Δ-critical graph, and let u and v be adjacent vertices of G with d(v) = k.

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1) If \( k < \Delta \), then \( u \) is adjacent to at least \( \Delta - k + 1 \) vertices of degree \( \Delta \).

2) If \( k = \Delta \), then \( u \) is adjacent to at least two vertices of degree \( \Delta \).

From the Vizing’s Adjacency Lemma, it is easy to get the following corollary.

**Corollary 2.** Let \( G \) be a \( \Delta \)-critical graph. Then

1) Every vertex is adjacent to at most one \( 2 \)-vertex and at least two \( \Delta \)-vertices;

2) The sum of the degree of any two adjacent vertices is at least \( \Delta + 2 \);

3) If \( uv \in E(G) \) and \( d(u) + d(v) = \Delta + 2 \), then every vertex of \( N\{\{u,v\}\}\setminus\{u,v\} \) is a \( \Delta \)-vertex.

**Lemma 3** [4]. Let \( G \) be a \( \Delta \)-critical graph, \( uv \in E(G) \) and \( d(u) + d(v) = \Delta + 2 \). Then

1) every vertex of \( N\{\{u,v\}\}\setminus\{u,v\} \) is of degree at least \( \Delta - 1 \);

2) if \( d(u),d(v) < \Delta \), then every vertex of \( N\{\{u,v\}\}\setminus\{u,v\} \) is a \( \Delta \)-vertex.

**Lemma 4** [5]. No \( \Delta \)-critical graph has distinct vertices \( x,y,z \) such that \( x \) is adjacent to \( y \) and \( z \), \( d(z) < \Delta - d(x) - d(y) + 2 \) and \( xz \) is in at least \( d(x) + d(y) - \Delta - 2 \) triangles not containing \( y \).

To be convenient, for a plane graph \( G \), let \( F(G) \) be the face set of \( G \). A face of a graph is said to be incident with all edges and vertices in its boundary. Two faces sharing an edge \( e \) are said to be adjacent at \( e \). A degree of a face \( f \) is denoted by \( d_f(f) \), which is the number of edges incident with \( f \), where each cut edge is counted twice. A \( k,k' \)-face is a face of degree \( k \), at least \( k \). A \( k \)-face of \( G \) is denoted by \( [v_1,v_2,\ldots,v_k] \) if it is incident with \( v_1,v_2,\ldots,v_k \) along its boundary. A 3-face \( [x,y,z] \) of \( G \) is called an \((i,j,k)\)-face if \( d(x) = i \leq d(y) = j \leq d(z) = k \). For a vertex \( x \in V(G) \), we denote by \( f_k(x) \) the number of \( k \)-faces incident with \( v \).

**Lemma 5** [4,5]. If \( G \) is a planar graph with \( \Delta(G) \geq 7 \), then \( G \) is of class 1.

**Lemma 6** [8]. If \( G \) is a graph of class 2, then \( G \) contains a \( k \)-critical subgraph for each \( k \) satisfying \( 2 \leq k \leq \Delta(G) \).

**Theorem 7.** Let \( G \) be a planar graph with \( \Delta \geq 6 \). If any 6-cycle contains at most one chord, then \( G \) is of class 1.

**Proof.** Suppose that \( G \) is a counterexample to our theorem with the minimum number of edges and suppose that \( G \) is embedded in the plane. Then \( G \) is a 6-critical graph by Lemmas 5 and 6, and it is 2-connected. By Euler’s formula \( \left| F(G) \right| - \left| E(G) \right| + \left| V(G) \right| = 2 \), we have

\[
\sum_{v \in V(G)} (d(v) - 4) + \sum_{f \in F(G)} (d(f) - 4) = -8 < 0
\]

We define \( ch \) to be the initial charge. Let \( ch(x) = d(x) - 4 \) for each \( x \in V \cup F \). So \( \sum_{x \in V \cup F} ch(x) < 0 \). In the following, we will reassign a new charge denoted by \( ch'(x) \) to each \( x \in V \cup F \) according to the discharging rules. Since our rules only move charges around, and do not affect the sum. If we can show that \( ch'(x) \geq 0 \) for each \( x \in V \cup F \), then we get an obvious contradiction \( 0 \leq \sum_{x \in V \cup F} ch(x) = \sum_{x \in V \cup F} ch(x) < 0 \). which completes our proof.

The discharging rules are defined as follows.

**R1:** Every \( 5 \)-face \( f \) sends \( \frac{d(f) - 4}{d(f)} \) to each incident vertex.

**R2:** Every 2-vertex receives 1 from each adjacent vertex.

**R3:** Every 3-vertex receives \( \frac{1}{3} \) from each adjacent vertex.

**R4:** Let \( f \) be a 3-face \( [x,y,z] \) with \( d(x) \leq d(y) \leq d(z) \).

- If \( 2 \leq d(x) \leq 4 \) and \( \min\{d(y),d(z)\} \leq 5 \), then \( f \) receives \( \frac{1}{2} \) from \( y \), \( \frac{1}{2} \) from \( z \); If \( d(x) = d(y) = 4 \) and \( d(z) = 6 \) then \( z \) sends 1 to \( f \); If \( \min\{d(x),d(y),d(z)\} \geq 5 \), then \( x, y, z \) sends \( \frac{1}{3} \) to \( f \), respectively.

**R5:** If a 5-vertex \( v \) is adjacent to a 6-vertex \( x \) and incident with a \( (3,5,6) \)-face \([u,v,w]\) such that \( u \notin E(G) \) and \( w \neq x \), then \( x \) sends \( \frac{1}{5} \) to \( v \).

Now, let’s begin to check \( ch'(x) \geq 0 \) for all \( x \in V \cup F \) of \( G \). Then \( d(f) \geq 3 \). If \( d(f) \geq 5 \), then \( ch'(f) \geq ch(f)+\frac{d(f) - 4}{d(f)} \geq 0 \) by R1. If \( d(f) = 4 \), then \( ch'(f) = ch(f) = 0 \). If \( d(f) = 3 \), then \( ch'(f) \geq ch(f) + \frac{d(f) - 4}{3} = \frac{2}{3} \geq 0 \) by R4. Let \( w \in V(G) \). Then \( d(w) \geq 2 \). If \( d(w) = 2 \), then \( ch'(w) = ch(w) + 2 \geq 2 \geq 0 \) by R2. If \( d(w) = 3 \), then \( w \) is adjacent to three \( 5 \)-vertices by Corollary 2, and it follows that \( ch'(w) = ch(w) \geq 3 \times \frac{1}{3} = 1 \geq 0 \) by R3. If \( d(w) = 4 \), then \( ch'(w) = ch(w) = 0 \).

Since any 6-cycle of \( G \) contains at most one chord, we have the following claim.

**Claim 1.** Let \( f, f', f'' \) be three faces incident with \( w \) such that \( f' \) is adjacent to \( f \) and \( f'' \). If \( f \) and \( f'' \) are 3-faces, then \( f' \) must be a \( 5 \)-face.

Suppose that \( d(w) = 5 \). We have \( ch(w) = 1 \), \( f_1(w) \leq 3 \), \( \min\{d(u)\mid u \in N\{w\}\} \geq 3 \), \( d_1(w) \leq 1 \) and \( d_2(w) \geq 2 \), \( w_0, w_1, \ldots, w_5 \) be neighbors of \( w \) and \( f_0, f_1, \ldots, f_5 \) be faces incident with \( w \) such that \( f_i \) is incident with \( w_i \) and \( w_{i+1} \) for all \( i \in \{0,1,\ldots,4\} \), where \( w_5 = w_0 \). If all neighbors of \( w \) are \( 5 \)-vertices, then \( ch'(w) \geq ch(w) + 3 \times \frac{1}{3} = 1 \geq 0 \) by R4. Suppose that \( d(w) \geq 4 \). If \( f_1(w) \leq 2 \), then
are 3-faces. Then $f_1$ and $f_3$ are 5-faces by Claim 1. By Lemma 4, $d_2(v) = 1$. So $w$ sends at most $\left(\frac{2}{5}\times\frac{1}{2}\right)$ to its adjacent 3-faces. At the same time, $w$ receives at least $2\times\frac{1}{5}$ from $f_1$ and $f_3$ by R1, and it follows that

$$ch'(w)\geq 1 + \frac{2}{5} \left(\frac{2}{3}\times\frac{1}{2}\right) > 0.$$ 

Suppose that $d_3(w) = 1$, without loss of generality, assume that $d(w_1) = 3$. Then $d_6(w) = 4$ by Lemma 1. If $f_3(w) \leq 1$, or $f_3(w) = 2$ and $ww_i$ is not incident with a 3-face, then

$$ch'(w)\geq 1 + \frac{2}{5} \left(\frac{2}{3}\times\frac{1}{2}\right) = 0,$$

by R3 and R4; Otherwise, $f_3(w) \geq 2$ and $w$ is incident with a 5-face. If $f_3(w) = 2$, then

$$ch'(w)\geq 1 + \frac{2}{5} \left(\frac{2}{3}\times\frac{1}{2}\right) > 0;$$

Otherwise, $w$ is incident with two 5-faces. If $ww_i$ is not incident with a 3-face, then $ch'(w)\geq 1 + 2\times\frac{1}{5} \left(\frac{2}{3}\times\frac{1}{2}\right) > 0$ by R3 and R4; Otherwise, $w$ receives at least $2\times\frac{1}{5}$ from its neighbors by R5, and it follows that

$$ch'(w)\geq 1 + 4\times\frac{1}{5} \left(\frac{2}{3}\times\frac{1}{2}\right) > 0.$$

In the following we check the case that $d(w) = 6$. Thus we have $ch(w) = 2$, $f_1(w) = 4$, $d_2(w) = 1$ and $d_6(w) = 2$ by Lemma 1.

**Case 1.** $w$ sends positive charge to some adjacent 5-vertex $v$ (ref. R5).

Suppose that $v$ is incident with a (3,5,6)-face $[u,v,x]$ such that $wu \notin E(G)$ and $w \neq x$ (see R5). Then $x$ may sends $\frac{1}{5}$ to $v$ by R5. At the same time, $w$ is adjacent to five 6-vertices by Lemma 3, that is,

$$d_6(w) = 5.$$ 

Since $f_3(w) \leq 4$, $ch'(w) = 2 - \frac{1}{5} - 4\times\frac{1}{3} > 0.$

**Case 2.** $w$ sends no charge to its adjacent 5-vertices.

Let $k = \min \{d(u) \mid u \in N(w)\}$. If $k \geq 5$, then

$$ch'(w)\geq 2 - 4\times\frac{1}{3} > 0.$$ 

Suppose that $k = 4$. Then $d_6(w) \geq 3$ by Lemma 1 and $w$ may be incident with a (4,4,6)-face. If $f_3(w) \leq 3$, then

$$ch'(w)\geq 2 - \frac{1}{2} - \frac{1}{3} > 0;$$

Otherwise, $f_3(w) \geq 2$ and it follows that

$$ch'(w)\geq 2 + \frac{1}{2} - \frac{1}{3} - \frac{1}{2} > 0.$$ 

Suppose that $k = 3$. Then $d_6(w) \geq 6 - 3 + 1 = 4$ by Lemma 1. If $d_3(w) \geq 5$, then

$$ch'(w)\geq 2 + \frac{1}{2} - \frac{1}{3} - \frac{1}{2} > 0;$$

Otherwise, $w$ is incident with two 4-vertices $u,v$, then $u$ and $v$ are incident with at most one 3-face by Lemma 4 since $d(u) + d(v) + d(w) \leq 3 + 4 + 6 < 14$. So $f_3(w) \leq 3$ and it follows that

$$ch'(w)\geq 2 - 2\times\frac{1}{2} - 2\times\frac{1}{3} > 0$$

by R3 and R4.

Suppose that $k = 2$, that is, $w$ is adjacent to a 2-vertex $v$. Then $d_3(w) = 5$ by Lemma 1. If $f_3(w) = 4$, then $f_5(w) = 2$ and it follows that

$$ch'(w)\geq 2 + \frac{1}{2} - \frac{1}{3} - \frac{1}{2} - \frac{1}{3} = 0.$$ 

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