H- and H₂-Cordial Labeling of Some Graphs

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ABSTRACT

In this paper we prove that the join of two path graphs, two cycle graphs, Ladder graph and the tensor product $P_n \otimes P_2$ are H₂-cordial labeling. Further we prove that the join of two wheel graphs $W_n$ and $W_m$, $n + m = 0 \pmod{4}$ admits a H-cordial labeling.

Keywords: H-Cordial; H₂-Cordial; Join of Two Graphs

1. Introduction

All graphs considered here are finite, simple and undirected. The origin of graph labelings can be attributed to Rosa. Several types of graph labeling have been investigated both from a purely combinatorial perspective as well as from an application point of view. Any graph labeling will have the following three common characteristics. A set of numbers from which vertex labels are chosen, $v_i$ number of vertices of $G$ having label $i$ under $f$, $e_j(i)$ = number of edges of $G$ having label $i$ under $f$.

The concept of cordial labeling was introduced by I. Cahit, who called a graph $G$ cordial if there is a vertex labeling $f : V(G) \to \{0, 1\}$ such that the induced labeling $f^* : E(G) \to \{0, 1\}$ defined by

$$f^*(xy) = |f(x) - f(y)|,$$

for all edges $xy \in E(G)$ and with the following inequalities holding

$$|v_i(0) - v_i(1)| \leq 1 \quad \text{and} \quad |e_j(0) - e_j(1)| \leq 1.$$

In [1] introduced the concept of H-cordial labeling. [1] calls a graph H-cordial if it is possible to label the edges with the numbers from the set $\{1, -1\}$ in such a way that, for some $k$, at each vertex $v$ the sum of the labels on the edges incident with $v$ is either $k$ or $-k$ and the inequalities

$$|v_i(k) - v_i(-k)| \leq 1 \quad \text{and} \quad |e_j(i) - e_j(-i)| \leq 1$$

are also satisfied where $v(i)$ and $e(j)$ are respectively, the number of vertices labeled with $i$ and the number of edges labeled with $j$. He calls a graph Hₐ-cordial if it is possible to label the edges with the numbers from the set $\{\pm 1, \pm 2, \ldots, \pm n\}$ in such a way that at each vertex $v$, the sum of the labels on the edges incident with $v$ is in the set $\{\pm 1, \pm 2, \ldots, \pm n\}$ and the inequalities

$$|v_i(i) - v_i(-i)| \leq 1 \quad \text{and} \quad |e_j(i) - e_j(-i)| \leq 1$$

are also satisfied for each $i$ with $1 \leq i \leq n$.

In [1] proved that $k_{n, a}$ is H-Cordial if and only if $n > 2$ and “$n$” is even; and $k_{n, a}, m \neq n$ is H-cordial if and only if $n \equiv 1 \pmod{4}$, $m$ is even and $m > 2$, $n > 2$.

In [2] proved that $k_n$ is H-Cordial if and only if $n \equiv 0$ or $3 \pmod{4}$ and $n \neq 3$.

$W_n$ is H-cordial if and only if $n$ is odd. $k_n$ is not H₂-cordial if $n \equiv 1 \pmod{4}$. Also [2] proved that every wheel has an H₂-cordial labeling.

In [3] several variations of graph labeling such as graceful, bigraceful, harmonious, cordial, equitable, humming etc. have been introduced by several authors. For definitions and terminologies in graph theory we refer to [4].

1.1. Definition: The join $G = G_1 + G_2$ of graph $G_1$ and $G_2$ with disjoint point sets $V_1$ and $V_2$ and edge sets $E_1$ and $E_2$ denoted by $G = G_1 \cup G_2$ is the graph union $G_1 \cup G_2$ together with all the edges joining $v_1, v_2$. If $G_1$ is ($p_1, q_1$) graph and $G_2$ is ($p_2, q_2$) graph then $G_1 + G_2$ is ($p_1 + p_2, q_1 + q_2 + p_1 + p_2$).

1.2. Definition: Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. The Cartesian product of $G_1$ and $G_2$ which is denoted by $G_1 \times G_2$ is the graph with vertex set $V = V_1 \times V_2$ consisting of vertices $V = \{u \times v_1 : u \in V_1, v_1 \in V_2\}$ and $v$ and $v'$ are adjacent in $G_1 \times G_2$ whenever $u_1, u_2$ are adjacent to $v_2$ or $u_1$ adjacent to $v_2$ and $u_2 = v_2$.

1.3. Definition: The tensor product $G = G_1 \otimes G_2$ of graphs $G_1$ and $G_2$ with disjoint point sets $V_1$ and $V_2$ and edge sets $E_1$ and $E_2$ is the graph with vertex set $V_1 \times V_2$ such that $(u_1, u_2)$ is adjacent to $(v_1, v_2)$ whenever $u_1, v_1 \in E_1$ and $u_2, v_2 \in E_2$. If $G_1$ is ($p_1, q_1$) graph and $G_2$ is ($p_2, q_2$) graph, then $G_1 \otimes G_2$ is a ($p_1p_2, 2q_1q_2$).

In this paper we have investigated some results on H- and H₂-cordial labeling for join of two graphs, Cartesian
product and tensor product of some graphs.

2. Main Results

2.1. Theorem: The join of two path graphs \( P_n \) and \( P_m \) admits a \( H_2 \)-cordial labeling when \( n + m \equiv 1, 2 \pmod{4} \).

Proof: Let \( v_1, v_2, \cdots, v_n \) and \( u_1, u_2, \cdots, u_m \) are the two vertex sets of the path graphs \( P_n \) and \( P_m \). The edge set \( E_1 \) and \( E_2 \) is the graph union of \( P_n \) and \( P_m \) together with all the edges joining the vertex sets \( v_i \) and \( u_i \), \( i = 1, 2, \cdots, n \).

Define the edge labeling

\[ f : E(G) \to \{1, -1\} \]

The edge matrix of \( P_n + P_m \) is given in Table 1.

In view of the above labeling pattern we give the proof as follows:

1) When \( n + m \equiv 1 \pmod{4} \)

Consider the join of two path graphs \( P_n \) and \( P_m \).

Using Table 1 the edge label matrix of \( P_n + P_m \) is given by

\[
\begin{bmatrix}
  u_1 & u_2 & \cdots & u_m \\
  v_1 & -1 & \cdots & -1 \\
  v_2 & -1 & 1 & \cdots & -1 \\
  v_3 & -1 & -1 & \cdots & -1 \\
  \vdots & \vdots & \ddots & \ddots & \vdots \\
  v_r & -1 & \cdots & -1 & \cdots & -1 \\
  u_1 & u_2 & \cdots & u_m & u_m & u_m \\
\end{bmatrix}
\]

with respect to the above labeling total number of vertices labeled with \( 1^s, -1^s, 2^s \) and \( -2^s \) are given by

\[ v_r (1) = n - 4, \ v_r (-1) = n - 4, \ v_r (2) = n - 3 \text{ and } v_r (-2) = n - 4. \]

\[ v_r (1) - v_r (-1) = \left| v_r (2) - v_r (-2) \right| = 1, \text{ differ by one.} \]

The total number of edges labeled with \( 1^s, -1^s, 2^s \) and \( -2^s \) are given by

\[ e_r (1) = \frac{n-1}{2}, \ e_r (-1) = \frac{n+1}{2}, \ e_r (2) = e_r (-2) = 0 \]

\[ \left| e_r (1) - e_r (-1) \right| + \left| e_r (2) - e_r (-2) \right| = 1, \text{ differ by one.} \]

Hence the join of two path graphs \( P_n \) and \( P_m \) admits a \( H_2 \)-cordial labeling.

2) When \( n + m \equiv 2 \pmod{4} \)

Consider the join of two path graphs \( P_n \) and \( P_m \).

Using Table 1, the edge label matrix of \( P_n + P_m \) is given by

\[
\begin{bmatrix}
  u_1 & u_2 & \cdots & u_m \\
  v_1 & -1 & \cdots & -1 \\
  v_2 & 1 & 1 & \cdots & -1 \\
  v_3 & -1 & 1 & 1 & -1 \\
  \vdots & \vdots & \ddots & \ddots & \vdots \\
  v_r & -1 & \cdots & -1 & \cdots & -1 \\
  u_1 & u_2 & \cdots & u_m & u_m & u_m \\
\end{bmatrix}
\]

In view of the above labeling pattern the total number of edges labeled with \( 1^s, -1^s, 2^s \) and \( -2^s \) are given by

\[ e_r (1) = \frac{n}{2}, \ e_r (-1) = \frac{n}{2}, \ e_r (2) = 0, \ e_r (-2) = 0 \]

\[ \left| e_r (1) - e_r (-1) \right| = \left| e_r (2) - e_r (-2) \right| = 0, \text{ differ by zero.} \]

The total number of vertices labeled with \( 1^s, -1^s, 2^s \) and \( -2^s \) are given by

\[ v_r (1) = n - 4, \ v_r (-1) = n - 4, \ v_r (2) = n - 5 \text{ and } v_r (-2) = n - 5. \]

\[ v_r (1) - v_r (-1) = \left| v_r (2) - v_r (-2) \right| = 1, \text{ differ by zero.} \]

Thus in each cases we have

\[ \left| v_r (1) - v_r (-1) \right| + \left| v_r (2) - v_r (-2) \right| \leq 1 \text{ and } \left| e_r (1) - e_r (-1) \right| + \left| e_r (2) - e_r (-2) \right| \leq 1. \]

Hence the join of two path graphs \( P_n \) and \( P_m \) admits a \( H_2 \)-cordial labeling.

2.2. Theorem: The join of two cycle graphs \( C_n \) and \( C_m \) admits a \( H_2 \)-cordial labeling when

\[
P_1 : \begin{array}{cccc}
  & 1 & -1 & -1 \\
V_1 & V_2 & V_3 & V_4 \\
\end{array}
\]

\[
P_2 : \begin{array}{cccc}
  & 1 & u & u \\
u & V_1 & V_2 & V_3 \\
\end{array}
\]

Figure 1. \( H_2 \)-cordial labeling on \( P_4 + P_2 \).

\[
\begin{array}{cccc}
  v_1 & -1 & u_1 & -1 \\
  v_2 & 1 & 1 & 1 \\
  v_3 & 1 & -1 & -1 \\
  v_4 & -1 & 1 & -1 \\
\end{array}
\]

Table 1. Edge matrix of \( P_n + P_m \).

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\( n + m = 1,3 \pmod{4}, n, m \geq 3 \).

**Proof:** Let \( v_1, v_2, \ldots, v_m \) and \( u_1, u_2, \ldots, u_m \) are the vertex set of cycles \( C_n \) and \( C_m \), respectively. The edge sets \( E_1 \) and \( E_2 \) is the graph union of \( C_n \) and \( C_m \) together with all the edges joining the vertex sets \( v_1, v_2, \ldots, v_n \) and \( u_1, u_2, \ldots, u_m \).

We note that \( |V(G)| = p_1 + p_2 \) and
\[
|E(G)| = q_1 + q_2 + p_1 p_2.
\]

Define \( f : E(G) \to \{1, -1\} \).

The edge matrix table of \( C_n + C_m \) is given in Table 2.

In view of the above labeling pattern we give the proof as follows.

Case (1) when \( n + m = 3 \pmod{4}, n, m \geq 3 \).

Consider the join of two cycle graphs \( C_n \) and \( C_m \).

Using Table 2 the edge label matrix of \( C_n \) and \( C_m \) is given by
\[
\begin{pmatrix}
u_1 & u_2 & u_3 & u_4 \\
u_1 & 1 & 1 & -1 & 1 & -11 \\
u_2 & 1 & -1 & -1 & 1 & -11 \\
u_3 & -1 & 1 & -1 & 1 & 1 \\
-11 & -11 & 1 & 1 & 1 & 1
\end{pmatrix}
\]

In view of the above labeling pattern the total number of edges labeled with \( 1', \ldots, -1', 2'' \) and \(-2''\) are given by
\[
e_f(1) = \frac{n+1}{2}, e_f(-1) = \frac{n+2}{2}, e_f(2) = 0, e_f(-2) = 0.
\]

Thus in each cases we have
\[
|v_f(1) - v_f(-1)| + |v_f(2) - v_f(-2)| \leq 1 \quad \text{and} \quad
|e_f(1) - e_f(-1)| + |e_f(2) - e_f(-2)| \leq 1.
\]

Hence the join of two cycle graphs \( C_n \) and \( C_m \) admits a H2-cordial labeling.

**Theorem:** The join of two wheel graphs \( W_n \) and \( W_m \) admits a H-cordial labeling when \( n + m = 0 \pmod{4} \).

**Proof:** Let \( v_1, v_2, \ldots, v_n \) and \( u_1, u_2, \ldots, u_m \) are the vertex set of the wheel graph \( W_n \) and \( W_m \). The edge set \( E_1 \) and \( E_2 \) is the graph union of \( W_n \) and \( W_m \) together with all the edges joining the vertex set \( v_1, v_2, \ldots, v_n \) and \( u_1, u_2, \ldots, u_m \). We note that \( |V(G)| = p_1 + p_2 \) and
\[
|E(G)| = q_1 + q_2 + p_1 p_2.
\]

Define \( f : E(G) \to \{1, -1\} \).

The edge matrix is given in Table 3.

In the view of the above labeling pattern we give the proof as follows:

when \( n + m = 0 \pmod{4} \)

Consider the join of two wheel graphs \( W_n \) and \( W_m \). Using Table 3 the edge label matrix of \( W_n + W_m \) is given by

\[
\begin{pmatrix}
u_1 & u_2 & u_3 & u_4 \\
u_1 & 1 & 1 & -1 & 1 & 11 \\
u_2 & 1 & -1 & -1 & 1 & 1 \\
u_3 & -1 & 1 & -1 & 1 & 1 \\
-1 & 1 & 1 & 1 & 1 & 11
\end{pmatrix}
\]
Figure 2. $H_2$-cordial labeling on $C_5 + C_4$.

Table 3. Edge matrix of $W_n + W_m$.

<table>
<thead>
<tr>
<th></th>
<th>$u_1$</th>
<th>$u_2$</th>
<th>...</th>
<th>$u_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v_1$</td>
<td>$v_1u_1$</td>
<td>$v_1u_2$</td>
<td>...</td>
<td>$v_1u_n$</td>
</tr>
<tr>
<td>$v_2$</td>
<td>$v_2u_1$</td>
<td>$v_2u_2$</td>
<td>...</td>
<td>$v_2u_n$</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td></td>
<td>...</td>
</tr>
<tr>
<td>$v_n$</td>
<td>$v_nu_1$</td>
<td>$v_nu_2$</td>
<td>...</td>
<td>$v_nu_n$</td>
</tr>
</tbody>
</table>

$M = [M_{ij}]_{n \times m}$
In view of the above labeling pattern we give the proof as follows.

The total number of edges labeled with $1^\prime$ and $-1^\prime$ are given by $e_f(1) = n/2$, $e_f(-1) = n/2$
\Rightarrow |e_f(1) - e_f(-1)| = 0, differ by zero. The total number of vertices labeled with $1^\prime$ and $-1^\prime$ are given by $v_f(1) = n/2$ and $v_f(-1) = n/2$
\Rightarrow |v_f(1) - v_f(-1)| = 0, differ by zero.

Thus in each cases we have $|v_f(1) - v_f(-1)| \leq 1$ and $|e_f(1) - e_f(-1)| \leq 1$.

Hence the join of two wheel graphs $w_4$ and $w_4$ admits a H-cordial labeling.

In Figure 3 illustrates the H-cordial labeling on $W_4 + W_4$. Among the twenty eight edges, fourteen edges receive the label +1 and the other fourteen edges receive the label –1. The induced vertex labels are shown in the figure.

2.4 Theorem: $L_n = P_n \times P_2$ (also known as ladder graph) is H$_2$-Cordial labeling for even $n$.

Proof: Let $G$ be the graph $P_n \times P_2$ where $n$ is even and $V(G) = \{V_{ij} \mid i = 1, 2, \ldots, n \text{ and } j = 1, 2\}$ be the vertices of $G$.

We note that $|V(G)| = 2n$ and $|E(G)| = 3n - 2$.

Define $f : E(G) \rightarrow \{1, -1\}$ as follows

Case (1) When $n \equiv 0 \mod 4$
For $1 \leq i, k \leq n - 1$
$f(v_{i1}, v_{k1}) = 1$

For $n - 1 < i, k \leq n$
$f(v_{i1}, v_{k1}) = -1$

For $1 \leq i, k \leq n - 1$
$f(v_{i2}, v_{k2}) = 1$

For $n - 1 < i, k \leq n$
$f(v_{i2}, v_{k2}) = -1$

Case (2) when $n \equiv 2 \mod 4$
For $1 \leq i, k \leq n - 2$
$f(v_{i1}, v_{k1}) = 1$

$f(v_{i2}, v_{k2}) = 1$

For $n - 2 < i, k \leq n$
$f(v_{i1}, v_{k1}) = -1$

$f(v_{i2}, v_{k2}) = -1$

For $1 \leq i, k \leq n - 2$
$f(v_{i1}, v_{k1}) = 1$

$f(v_{i2}, v_{k2}) = 1$
For $n - 2 < i \leq n$,  
\[ f(v_i, v_3) = 1 \]

In view of the above defined labeling pattern we give the proof as follows.

The total number of edges labeled with $1^*, -1^*, 2^*$ and $-2^*$ are given by  
\[ e_j(1) = n/2, \quad e_j(-1) = n/2, \quad e_j(2) = e_j(-2) = 0. \]

Case (1). When $n$ is even  
\[ f(u_i, v_j) = \begin{cases} 1, & \text{if } i = 1 \pmod{2} \\ -1, & \text{if } i = 1 \pmod{2} \end{cases} \]

Case (2) When $n$ is odd  
\[ f(u_i, v_j) = \begin{cases} 1, & \text{if } i = 1, 3 \pmod{4} \\ -1, & \text{if } i = 0, 2 \pmod{4} \end{cases} \]

In view of the above defined labeling pattern we give the proof as follows.

The total number of edges labeled with $1^*, -1^*, 2^*$ and $-2^*$ are given by  
\[ e_j(1) = n/2, \quad e_j(-1) = n/2, \quad e_j(2) = e_j(-2) = 0. \]

Thus in each cases we have  
\[ |v_j(1) - v_j(-1)| + |v_j(2) - v_j(-2)| = 0 \text{ differ by zero.} \]

Hence the ladder graph $P_n \times P_2$ admits a $H_2$-cordial labeling.

In Figure 4 illustrates the $H_2$-cordial labeling on $P_4 \times P_2$. Among the ten edges, five edges receive the label $+1$ and other five edges receive the label $-1$. The induced vertex labels are shown in the figure.

2.5 Theorem: The tensor product $P_n \otimes P_2$ is $H_2$-cordial labeling.

Proof: Let $G$ be the graph $P_n \otimes P_2$ and  
\[ V(G) = \{u_i, v_j \mid i, j = 1, 2, \ldots, n \} \]

We note that  
\[ |V(G)| = 2n \quad \text{and} \quad |E(G)| = 2n - 2 \]

Define $f: E(G) \rightarrow \{1, -1\}$ two cases are to be considered.
\[ |v_f(1) - v_f(-1)| + |v_f(2) - v_f(-2)| = 0 \text{ differ by zero.} \]

Thus in each cases we have
\[ |v_f(1) - v_f(-1)| + |v_f(2) - v_f(-2)| \leq 1 \]
\[ |e_f(1) - e_f(-1)| + |e_f(2) - e_f(-2)| \leq 1 \]

Hence the tensor product \( P_n \otimes P_2 \) admits a H\(_2\)-cordial labeling.

In Figure 5 illustrates the H\(_2\)-cordial labeling on \( P_5 \otimes P_2 \). Among the eight edges four edges receive the label +1 and the other four edges receive the label -1. The induced vertex labels are shown in the figure.

3. Concluding Remarks

Here we investigate H- and H\(_2\)-cordial labeling for join of path graphs, cycle graphs, wheel graphs, Cartesian product and tensor product. Similar results can be derived for other graph families and in the context of different graph labeling problem is an open area of research.

REFERENCES


