4-Cycle Decompositions of Graphs

Teresa Sousa

Departamento de Matemática and Centro de Matemática e Aplicações, Faculdade de Ciências e Tecnologia, Universidade Nova de Lisboa, Lisbon, Portugal

Email: tmjs@fct.unl.pt

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ABSTRACT

In this paper we consider the problem of finding the smallest number \( \phi \) such that any graph \( G \) of order \( n \) admits a decomposition into edge disjoint copies of \( C_4 \) and single edges with at most \( \phi \) elements. We solve this problem for \( n \) sufficiently large.

Keywords: Graph Decomposition; 4-Cycle Packing; Graph Packing

1. Introduction

All graphs in this paper are finite, undirected and simple. For notation and terminology not discussed here the reader is referred to [1].

Given two graphs \( G \) and \( H \), an \( H \)-decomposition of \( G \) is a partition of the edge set of \( G \) such that each part is either a single edge or forms an \( H \)-subgraph, i.e., a graph isomorphic to \( H \). We allow partitions only, that is, every edge of \( G \) appears in precisely one part. Let \( \phi_H(G) \) be the smallest possible number of parts in an \( H \)-decomposition of \( G \). For non-empty \( H \), let \( p_H(G) \) be the maximum number of pairwise edge-disjoint \( H \)-subgraphs that can be packed into \( G \) and \( e(G) \) the number of edges in \( G \). It is easy to see that

\[
\phi_H(G) = e(G) - p_H(G)(e(H)-1).
\] (1.1)

Here we study the function

\[
\phi_H(n) = \max\{\phi_H(G) : e(G) = n\},
\]

which is the smallest number, such that, any graph \( G \) of order \( n \) admits an \( H \)-decomposition with at most \( \phi_H(n) \) elements.

The function \( \phi_H(n) \) was first studied by Erdös, Goodman and Pósa [2], who proved that \( \phi_{K_r}(n) = t_r(n) \), where \( K_r \) denotes the complete graph (clique) of order \( r \) and \( t_r(n) \) is the maximum size of an \( r \)-partite graph on \( n \) vertices. A decade later, this result was extended by Bollobás [3], who proved that

\[
\phi_{K_r}(n) = t_{r-1}(n), \quad \text{for all } n \geq r \geq 3.
\]

Recently, Pikhuirk and Sousa [4] studied \( \phi_H(n) \) for arbitrary graphs \( H \).

Theorem 1.1. (See Theorem 1.1 from [4]) Let \( H \) be any fixed graph of chromatic number \( r \geq 3 \). Then,

\[
\phi_H(n) = t_{r-1}(n) + o(n^2).
\]

Let \( \text{ex}(n, H) \) denote the maximum number of edges in a graph of order \( n \), that does not contain \( H \) as a subgraph. Recall that \( \text{ex}(n, K_r) = t_{r-1}(n) \). Pikhurko and Sousa [4] also made the following conjecture.

Conjecture 1. For any graph \( H \) with chromatic number at least 3, there is \( n_0 = n_0(H) \) such that \( \phi_H(n) = \text{ex}(n, H) \), for all \( n \geq n_0 \).

The exact value of the function \( \phi_H(n) \) is far from being known. Sousa determined it for a few special edge-critical graphs, namely for clique-extensions of order \( r \geq 4 \) \( (n \geq r) \) [5] and the cycles of length 5 \( (n \geq 6) \) and 7 \( (n \geq 10) \) [6,7]. Later, Özkahya and Person [8] determined it for all edge-critical graphs. Moreover, the only graph attaining \( \phi_H(n) \) is the Turán graph \( T_{r-1}(n) \).

Recently, Allen, Böttcher and Person [9] improved the error term obtained by Pikhurko and Sousa in Theorem 1.1.

The case when \( H \) is a bipartite graph has been less studied. Pikhurko and Sousa [4] determined \( \phi_H(n) \) for any fixed bipartite graph with an \( O(1) \) additive error. For a non-empty graph \( H \), let \( \gcd(H) \) denote the greatest common divisor of degrees of \( H \). For example, \( \gcd(K_{3,4}) = 2 \) while for any tree \( T \) with at least 2 vertices we have \( \gcd(T) = 1 \). They proved the following result.
**Theorem 1.3.** (See Theorem 1.3 from [4]) Let $H$ be a bipartite graph with $m$ edges and let $d = \gcd(H)$. Then there is $n_0 = n_0(H)$ such that for all $n \geq n_0$ the following statements hold.

1. If $d = 1$, then $\phi_H(n) = \left\lfloor \frac{n(n-1)}{2m} \right\rfloor + C$, where $C = m-1$ or $C = m-2$.
2. If $d \geq 2$, then $\phi_H(n) = \frac{nd}{2m} \left\lfloor \frac{n}{d} \right\rfloor - \frac{1}{2} n(d-1) + O(1)$.

Moreover, there is a procedure running in polynomial in $\log n$ time which determines $\phi_H(n)$ and describes a family $D$ of $n$-sequences such that a graph $G$ of order $n$ satisfies $\phi_H(G) = \phi_H(n)$ if and only if the degree sequence of $G$ belongs to $D$. (It will be the case that $|D| = O(1)$ and each sequence in $D$ has $n - O(1)$ equal entries, so $D$ can be described using $O(\log n)$ bits.)

Here we will determine the exact value of $\phi_H(n)$ for $n$ sufficiently large.

**Theorem 1.4.** There is $n_0 = n_0(C_4)$ such that for all $n \geq n_0$ the following statements hold.

1. If $n$ is even then $\phi_{C_4}(n) = \frac{n^2}{8} + \frac{n}{4} + 1$.
2. If $n = 1 \mod 8$ then $\phi_{C_4}(n) = \frac{n^2}{8} + \frac{n}{8} + \frac{14}{8}$.
3. If $n = 3 \mod 8$ then $\phi_{C_4}(n) = \frac{n^2}{8} + \frac{n}{8} + \frac{3}{2}$.
4. If $n = 5 \mod 8$ then $\phi_{C_4}(n) = \frac{n^2}{8} + \frac{n}{8} + \frac{10}{8}$.
5. If $n = 7 \mod 8$ then $\phi_{C_4}(n) = \frac{n^2}{8} + \frac{n}{8} + 2$.

**2. Proof of Theorem 1.4**

In this section we will prove Theorem 1.4, but first we need to introduce the tools. We start with the following easy result about $H$-decompositions.

**Lemma 5.** (Lemma 1.3) For any non-empty graph $H$ with $m$ edges and any integer $n$, we have

$$\phi_H(n) \leq \frac{1}{m} \left\lfloor \frac{n}{2} \right\rfloor + \frac{m-1}{m} \text{ex}(n,H).$$

(2.1)

In particular, if $H$ is a fixed bipartite graph with $m$ edges and $n \to \infty$, then

$$\phi_H(n) = \left(\frac{1}{m} + O(1)\right)\left\lfloor \frac{n}{2} \right\rfloor.$$  

(2.2)

The following result is the well known Erdős-Gallai theorem that gives a necessary and sufficient condition for a finite sequence to be the degree sequence of a simple graph.

**Theorem 2.6.** (Erdős-Gallai Theorem [10]) Let $0 \leq d_1 \leq \cdots \leq d_n$ be a sequence of integers. There is a graph with degree sequence $d_1, \cdots, d_n$ if and only if

1. $d_1 + \cdots + d_n$ is even;
2. for each $1 \leq k \leq n$

$$\sum_{i=0}^{k} d_i \leq k(k-1) + \sum_{i=1}^{k} \min\{d_i, k\}.$$  

(2.3)

The following results appearing in Alon, Caro and Yuster [11, Theorem 1.1, Corollary 3.4, Lemma 3.5] which follow with some extra work from the powerful decomposition theorem of Gustavsson [12] are essential to the proof of Theorem 1.4.

**Lemma 2.7.** For any non-empty graph $H$ with $m$ edges, there are $\gamma > 0$ and $n_0$ such that the following holds. Let $d = \gcd(H)$. Let $G$ be a graph of order $n \geq n_0$ and of minimum degree $\delta(G) \geq (1-\gamma)n$. If $d = 1$, then

$$p_H(G) = \left\lfloor \frac{e(G)}{m} \right\rfloor.$$  

(2.4)

If $d \geq 2$, let $\alpha_u = \deg(u) / d$ for $u \in V(G)$ and let $X$ consist of all vertices whose degree is not divisible by $d$. If $|X| \geq \frac{n}{10d^2}$, then

$$p_H(G) = \left\lfloor \frac{1}{2m \sum_{u \in V(G)} \alpha_u} \right\rfloor.$$  

(2.5)

If $|X| < \frac{n}{10d^2}$, then

$$p_H(G) \geq \frac{1}{m} \left( e(G) - \frac{n}{5d^2} \right).$$  

(2.6)

One can extract the following result from the proof of Theorem 1.2 from [4].

**Lemma 2.8.** Let $H$ be a bipartite graph with $m$ edges and let $\gcd(H) = d \geq 2$. Then, there is $n_0 = n_0(H)$ such that if $G$ is a graph of order $n \geq n_0$ with $\phi_H(G) = \phi_H(n)$ then the following holds:

1. Let $d_1, \cdots, d_n$ be the degree sequence of $G$, then

$$\phi_H(G) = \frac{1}{2} \sum_{i=1}^{n} d_i - (m-1) \left( \frac{1}{2m} \sum_{i=1}^{n} \left\lfloor \frac{d_i}{d} \right\rfloor \right).$$  

(2.7)

2. Let $n =qd + r$ with $0 \leq r \leq d-1$ and $d_i = qd + r_i$ with $0 \leq r_i \leq d-1$. Then, for $1 \leq i \leq n$ exactly one of the following holds:

(a) $d_i = qd - 1$;
(b) $i \in C_1 \iff d_i = d - 1$ and $d_i < qd - 1$;
(c) $i \in C_2 \iff d_i = n - 1$ if $n - 1 \neq R$ and
result from [4] (Lemma 3.1) states that there is a constant $C$, such that all but at most $C$ vertices of $G[X_i]$ can be covered by edge disjoint copies of $H-y$ each of them having vertex disjoint sets $A$. Therefore, all but at most $C$ edges between $x_i$ and $X_i$ can be decomposed into copies of $H$. All other edges incident to $x_i$ are removed as single edges. Let $G_{i-1}$ consist of the remaining edges of $G_r-x_i$ (that is, those edges that do not belong to an $H$-subgraph of the above $x_i$ decomposition). This finishes the description of the case $\deg_{G_{i-1}}(x_i) > an$.

Consider the sets $S = \{x_1, \ldots, x_{n+1}\}$, $S_i = \{x_i \in S | \deg_{G_i}(x_i) \leq an\}$, and $S_2 = S/S_i$. Let their sizes be $s$, $s_1$, and $s_2$ respectively, so $s = s_1 + s_2$.

Let $F$ be the graph with vertex set $V(G_{n+1}) \cup S_2$, consisting of the edges coming from the removed $H$-subgraphs when we processed the vertices in $S_2$. We have

\[
\phi_H(G) \leq \phi_H(G_{n+1}) + \frac{e(F)}{m} + s_1an + s_2C + \left(\frac{s}{2}\right). \tag{2.9}
\]

We know that $\phi_H(G_{n+1}) = e(G_{n+1}) - p_H(G_{n+1}(m-1))$, furthermore, $\delta(G_{n+1}) \geq (1-\gamma)(n-s)$. Thus, $p_H(G_{n+1})$ can be estimated using Lemma 2.7.

If (2.6) holds, some calculations show that there exists a graph $G'$ such that $\phi_H(G') < \phi_H(G')$, which contradicts the optimality of $G$. Therefore, (2.5) must hold. It follows that $p_H(G)$ and thus $\phi_H(G)$, depends only on the degree sequence $d_1, \ldots, d_n$ of $G$. Namely, the packing number

$\ell = p_H(G)$ equals $\left(\frac{1}{2m}\sum_{i=1}^{n}r_i\right)$, where $r_i = d_i/d$.

is the largest multiple of $d$ not exceeding $d_i$.

Therefore,

\[
\phi_H(G) = \frac{1}{2} \sum_{i=1}^{n}d_i - (m-1) \left(\frac{1}{2m}\sum_{i=1}^{n}d_i/d\right), \tag{2.10}
\]

where $d_1, \ldots, d_n$ is the degree sequence of $G$.

To conclude the proof we need to estimate the values that the degrees of $G$ can attain. To do that we need to prove an upper bound on $\phi_H(G)$ by estimating $\phi_{\max}$, the maximum of

\[
\phi(d_1, \ldots, d_n) = \frac{1}{2} \sum_{i=1}^{n}d_i - (m-1) \left(\frac{1}{2m}\sum_{i=1}^{n}d_i/d\right), \tag{2.11}
\]

over all (not necessarily graphical) sequences $d_1, \ldots, d_n$ of integers with $0 \leq d_i \leq n-1$.

Let $d_1, \ldots, d_n$ be an optimal sequence attaining the value $\phi_{\max}$. For $i = 1, \ldots, n$ let $d_i = q_id + r_i$ with $0 \leq r_i \leq d-1$. Then, $\ell = \left(\frac{q_1 + \ldots + q_n}{2m}\right)$. **Copyright © 2012 SciRes.**
Let \( n = qd + r \) with \( 0 \leq r \leq d - 1 \) and \( q = \lfloor n/d \rfloor \).
Define \( R = qd - 1 \) to be the maximum integer which is at most \( n - 1 \) and is congruent to \( d - 1 \) modulo \( d \).
Let \( C_1 = \{ i \in [n] : r_i = d - 1 \text{ and } d_i \neq R \} \) and
\( C_2 = \{ i \in [n] : d_i = n - 1 \} \) if \( n - 1 \neq R \) and \( C_2 = \emptyset \) otherwise.

Since \( d_1, \ldots, d_n \) is an optimal sequence, we have that if \( r_i \neq d - 1 \) then \( d_i = n - 1 \) for all \( i \in [n] \).
To conclude the proof it remains to show that \( |C_1| \leq \frac{2m}{d} - 1 \) and \( |C_2| \leq 2m - 1 \).
Suppose first that \( |C_1| \geq \frac{2m}{d} \).
Consider the new sequence of integers
\[
d_i' = \begin{cases} 
  d_i + d, & \text{if } i \in C_1, \\
  d_i, & \text{if } i \notin C_1.
\end{cases}
\]
Then, \( \ell' = \ell + 1 \) and \( \phi' = \phi_{\text{max}} + 1 \) which contradicts our assumption on \( \phi_{\text{max}} \).

Now suppose that \( |C_2| \geq 2m \) and consider the new sequence of integers \( d_1', \ldots, d_n' \) obtained from \( d_1, \ldots, d_n \) by replacing \( 2m \) values of \( n - 1 \) by \( R \).
Then, \( \ell' = \ell - d \) and \( \phi' \geq \phi_{\text{max}} + m - d > \phi_{\text{max}} \), which contradicts our assumption on \( \phi_{\text{max}} \) and the proof is concluded.

We now have all the tools needed to prove Theorem 1.4.

**Proof of Theorem 1.4.** Let \( n_0 \) be given by Lemma 2.8.
Let \( G \) be a graph of order \( n \geq n_0 \) with \( \phi_{\text{max}}(G) = \phi_{\text{cs}}(n) \) and degree sequence \( d_1, \ldots, d_n \).
For \( i = 1, 2, \ldots, n \) let \( d_i = 2q_i + r_i \) with \( 0 \leq r_i \leq 1 \).
Let \( R = 2\lfloor n/2 \rfloor + 1 \) and let the sets \( C_1 \) and \( C_2 \) be as in Lemma 2.8.
Let \( n = 2q + r \) with \( 0 \leq r \leq 1 \) and \( q = \lfloor n/2 \rfloor \).
From (2.7) we obtain
\[
\phi_{\text{cs}}(n) = n(q - 1) + \frac{n}{2} + \frac{1}{2}[C_1] - \sum_{i \in C_1} (q - 1 - q_i) = n(q - 1) + \frac{n}{2} - \frac{1}{4}(C_1 - \sum_{i \in C_1} (q - 1 - q_i)) 
\]
In what follows let \( \alpha = |C_1| \) and \( \beta = \sum_{i \in C_1} (q - 1 - q_i) \).

We consider first the case when \( n \) is even. Then \( C_2 = \emptyset \) and we have
\[
\phi_{\text{cs}}(n) = n(q - 1) + n/2 - \beta - 3 \left( \frac{1}{2} n(q - 1) - \frac{\beta}{4} \right) 
\]
\[
= n(q - 1) + n/2 - \frac{3}{2} n(q - 1) - \beta - 3 \left( \frac{\beta}{4} \right) 
\]
\[
= n(q - 1) + n/2 - 3q(q - 1)/2 - \beta - 3 \left( \frac{\beta}{4} \right) = n(q - 1) + n/2 - 3q(q - 1)/2 - \beta - 3 \left( \frac{\beta}{4} \right) 
\]
(2.13)

**Claim 1.** Let \( d_1, \ldots, d_n \) be the degree sequence of a graph. Then,

\[
-\beta - 3 \left( \frac{\beta}{4} \right) \leq 1. 
\]

**Proof.** Routine calculations show that for \( \beta \neq 1 \) we have \( -\beta - 3 \left( \frac{\beta}{4} \right) \leq 1. \) Suppose \( \beta = 1. \) Then \( C_i \) has exactly one element, thus the sequence \( (d_i)_{i=1,\ldots,n} \) has exactly one element equal to \( n - 3 \) and all the others equal to \( n - 1. \) But this is not a degree sequence of a graph since condition (2.3) of Theorem 2.6 does not hold for \( k = n - 2. \) \( \square \)

Therefore, using the estimate of Claim 1 in (2.13) it follows that
\[
\phi_{\text{cs}}(n) \leq \frac{n^2}{8} + \frac{n + 1}{4}. 
\]

To prove the lower bound consider the graph \( L_5 \) obtained from \( K_n \) after the deletion of the edges of a \( C_5 \). Using (1.1) and (2.5) we show that
\[
\phi_{\text{cs}}(L_5) = \frac{n^2}{8} + \frac{n + 1}{4}. 
\]

We now consider the case when \( n \) is an odd number.

**Case 1:** Let \( n = 8t + 1 \) and \( q = 4t \).
From (2.12) we obtain
\[
\phi_{\text{cs}}(n) = \frac{n}{2}(n - 3) + \frac{n}{2}(8t^2 - 1) 
\]
\[
+ \frac{1}{2} \alpha - \beta - 3 \left( \frac{\alpha - \beta - 1}{4} \right) 
\]
(2.14)

**Claim 2.** Let \( d_1, \ldots, d_n \) be the degree sequence of a graph. Then,
\[
\frac{1}{2} \alpha - \beta - 3 \left( \frac{\alpha - \beta - 1}{4} \right) \leq \frac{5}{2}. 
\]

**Proof.** Routine calculations show that the result follows if \( \alpha \neq 0 \) or \( \beta \neq 0. \) If \( \alpha = 0 \) and \( \beta = 0 \) then \( d_i = n - 2 \) for all \( 1 \leq i \leq n. \) This is not a degree sequence of a graph since \( \sum_{i \in \mathcal{C}} d_i \) is not even. \( \square \)

Therefore, using the estimate of Claim 2 in (2.14) we prove that
\[
\phi_{\text{cs}}(n) \leq \frac{n^2}{8} + \frac{n + 14}{8}. 
\]

As for the lower bound consider the graph \( L' \) with all vertices of degree \( n - 2 \) except one of degree \( n - 3. \)
Using (1.1) and (2.5) we show that
\[
\phi_{\text{cs}}(L') = \frac{n^2}{8} + \frac{n + 14}{8}. 
\]

**Case 2:** Let \( n = 8t + 3 \) and \( q = 4t + 1 \).
From (2.12) we obtain
\[
\phi_{c_i}(n) = \frac{n}{2}(n-3) + \frac{n}{2} - 3(8t^2 + 3t) + \frac{1}{2} \alpha - \beta - 3 \left[ \frac{\alpha - \beta}{4} \right].
\] (2.15)

Claim 3. Let \(d_1, \ldots, d_n\) be the degree sequence of a graph. Then,

\[
\frac{1}{2} \alpha - \beta - 3 \left[ \frac{\alpha - \beta}{4} \right] \leq \frac{3}{2}.
\]

Proof. It follows from routine calculations for all values of \(\alpha\) and \(\beta\) except when \(\alpha = 0\) and \(\beta = 1\).

Suppose that \(\alpha = 0\) and \(\beta = 1\). Then \(C_2 = \emptyset\) and \(C_1\) has exactly one element, thus the sequence \((d_i)_{i=1, \ldots, n}\) has exactly one element equal to \(n-2\) and all the others equal to \(n-1\). But this is not a degree sequence of a graph since \(\sum d_i\) is not even. □

Therefore, using the estimate of Claim 3 in (2.15) we prove that

\[
\phi_{c_i}(n) \leq \frac{n^2}{8} + \frac{n}{8} + \frac{3}{2}.
\]

As for the lower bound consider the graph \(L\) with degree sequence \(d_1 = d_2 = n-4\), \(d_3 = \cdots = d_{n-1} = n-2\) and \(d_n = n-1\) (the existence of \(L\) can be proved directly or by Erdös-Gallai theorem, Theorem 2.6). Using (1.1) and (2.5) we have that

\[
\phi_{c_i}(L) = \frac{n^2}{8} + \frac{n}{8} + \frac{3}{2}.
\]

Case 3: Let \(n = 8t + 5\) and \(q = 4t + 2\). From (2.12) we obtain

\[
\phi_{c_i}(n) = \frac{n}{2}(n-3) + \frac{n}{2} - 3(8t^2 + 7t) + \frac{1}{2} \alpha - \beta - 3 \left[ \frac{\alpha - \beta}{4} \right].
\] (2.16)

Claim 4. Let \(d_1, \ldots, d_n\) be the degree sequence of a graph. Then,

\[
\frac{1}{2} \alpha - \beta - 3 \left[ \frac{\alpha - \beta + 5}{4} \right] \leq \frac{5}{2}.
\]

Proof. Routine calculations show that

\[\alpha/2 - \beta - 3 \left[ \frac{\alpha - \beta + 5}{4} \right] \leq -5/2\] for all values of \(\alpha\) and \(\beta\) except for \(\alpha = 2\) and \(\beta = 0\) or \(\alpha = 0\) and \(\beta = 2\).

Suppose first that \(\alpha = 2\) and \(\beta = 0\). Then the sequence \((d_i)_{i=1, \ldots, n}\) has two elements equal to \(n-1\) and all the others equal to \(n-2\). This is not a degree sequence of a graph since \(\sum d_i\) is not even. □

Suppose now that \(\alpha = 0\) and \(\beta = 2\). If \(|C_1| = 2\) then the sequence has two elements equal to \(n-4\) and all the others equal to \(n-2\) and this is not a degree sequence of a graph since \(\sum d_i\) is not even. Finally, if \(|C_1| = 1\) then we have one element equal to \(n-6\) and all the others equal to \(n-2\). Again, this is not a degree sequence of a graph since \(\sum d_i\) is not even. □

Therefore, using the estimate of Claim 4 in (2.16) we prove that

\[
\phi_{c_i}(n) \leq \frac{n^2}{8} + \frac{n}{8} + \frac{10}{8}.
\]

As for the lower bound consider the graph \(K_n-I\) obtained from \(K_n\) by deleting the edges of a maximum matching. Using (1.1) and (2.5) we show that

\[
\phi_{c_i}(K_n-I) = \frac{n^2}{8} + \frac{n}{8} + \frac{10}{8}.
\]

Case 4: Let \(n = 8t + 7\) and \(q = 4t + 3\). From (2.12) we obtain

\[
\phi_{c_i}(n) = \frac{n}{2}(n-3) + \frac{n}{2} - 3(8t^2 + 1t) + \frac{1}{2} \alpha - \beta - 3 \left[ \frac{\alpha - \beta + 14}{4} \right].
\] (2.17)

Claim 5. Let \(d_1, \ldots, d_n\) be the degree sequence of a graph. Then,

\[
\frac{1}{2} \alpha - \beta - 3 \left[ \frac{\alpha - \beta + 14}{4} \right] \leq -\frac{17}{2}.
\]

Proof. It follows directly from simple calculations. □

Therefore, using the estimate of Claim 5 in (2.17) we prove that

\[
\phi_{c_i}(n) \leq \frac{n^2}{8} + \frac{n}{8} + 2.
\]

Furthermore, using (1.1) and (2.5) we have

\[
\phi_{c_i}(K_n-I) = \frac{n^2}{8} + \frac{n}{8} + 2,
\]

so the equality follows and the proof is now complete. □

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REFERENCES


