Periodic Solution of Impulsive Lotka-Volterra Recurrent Neural Networks with Delays

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ABSTRACT

In this paper, periodic solution of impulsive Lotka-Volterra recurrent neural networks with delays is studied. Using the continuation theorem of coincidence degree theory and analysis techniques, we establish criteria for the existence of periodic solution of impulsive Lotka-Volterra recurrent neural networks with delays.

Keywords: Lotka-Volterra; Delays; Periodic Solution; Impulsive

1. Introduction

In recent years, applications of theory differential equations in mathematical ecology have been developed rapidly. Various mathematical models have been proposed in the study of population dynamics. The Lotka-Volterra competition system is the most famous models for dynamics of population. Owing to its theoretical and practical significance, the Lotka-Volterra systems have been studied extensively [1,2]. The Lotka-Volterra type neural networks, derived from conventional membrane dynamics of competing neurons, provide a mathematical basis for understanding neural selection mechanisms. Recently, periodic solutions of impulsive Lotka-Volterra recurrent neural networks have been reported.

It is well known that delays are important phenomenon in neural networks [3]. Thus, studying the dynamic properties of neural networks with delays has interesting implications in both theory and applications [4-7]. In this paper, we will study the following impulsive Lotka-Volterra recurrent neural networks system with delays:

\[
\begin{align*}
x_i'(t) &= x_i(t) [h_i(t) - x_i(t)] + \sum_{j=1}^{n} a_{ij}(t)x_j(t) \\
&\quad + c_i(t)x_i(t - \tau_i(t)) \quad t \neq t_k, \\
\Delta x_i(t_k) &= h_{ik}x_i(t_k), \quad i, k = 1, 2, \ldots, n,
\end{align*}
\]

where each \( x_i(t) \) denotes the activity of neuron, \( A = (a_{ij})_{n \times n} \) is real \( n \times n \) matrices, each of their elements denotes a synaptic weight and represents the strength of the synaptic connection from neuron \( j \) to neuron \( i \), \( h_i(t) \) denotes external inputs. The variable delays \( \tau_i(t) \) for \( i = 1, 2, \ldots, n \) are nonnegative continuous functions satisfying \( 0 \leq \tau_i(t) \leq \tau \) for \( t \geq 0 \), where \( \tau \geq 0 \) is a constant. \( h_i(t), a_{ij}(t), c_i(t) \) are all positive periodic continuous functions with period \( T > 0 \).

2. Existence of Positive Periodic Solutions

Lemma 1 [8] Let \( X \) and \( Y \) be two Banach spaces. Consider an operator equation \( Lx = \lambda Nx \) where \( L : \text{Dom } L \cap X \rightarrow Y \) is a Fredholm operator of index zero and \( \lambda \in [0,1] \) is a parameter. Let \( P \) and \( Q \) denote two projectors such that \( P : X \rightarrow \text{Ker } L \) and \( Q : Y \rightarrow \text{Im } L \). Assume that \( N : \Omega \rightarrow Y \) is \( L \)-compact on \( \bar{\Omega} \), where \( \Omega \) is open bounded in \( X \). Furthermore, assume that

(a) For each \( \lambda \in (0,1), \quad x \in \partial \Omega \cap \text{Dom } L, \quad Lx \neq \lambda Nx, \)

(b) For each \( x \in \partial \Omega \cap \text{Ker } L, \quad QNx \neq 0, \)

(c) \( \text{deg} \{ JQN, \Omega \cap \text{Ker } L, 0 \} \neq 0 \),

where \( J : \text{Im } Q \rightarrow \text{Ker } L \) is an isomorphism and \( \text{deg} \{* \} \) represents the Brouwer degree.

Then the equation \( Lx = Nx \) has at least one solution in \( \bar{\Omega} \cap \text{Dom } L \).

For the sake of convenience, we introduce the following notation:

\[
\begin{align*}
\Pi &= \frac{1}{T} \int_0^T u(t) dt, \quad g_i^* = \min_{t \in [0,T]} g_i^*(t), \\
g_i^* &= \max_{t \in [0,T]} g_i^*(t), \quad (i = 1, 2, \ldots, n),
\end{align*}
\]

\[
\begin{align*}
PC(J,R) &= \left\{ x : J \rightarrow R \mid x(t) \text{ is continuous with respect to } t \neq t_1, \ldots, t_p, \right. \\
&\quad \left. x(t') \text{ exist at } t_1, \ldots, t_p, \right. \\
&\quad \left. \text{and } x(t_k) = x(t_k^*), k = 1, 2, \ldots, p \right\}
\end{align*}
\]
where \( u(t), g(t) \) are \( T \)-periodic functions.

**Lemma 2** \( z(t) \) is an \( T \)-periodic solution of (1) if and only if \( \ln \{ z_i(t) \} \) is an \( T \)-periodic solution of

\[
\begin{cases}
    z_i'(t) = b_i(t) - \exp \{ z_i(t) \} + \sum_{j=1}^{n} a_{ij}(t) \exp \{ z_j(t) \} + c_i(t) \exp \{ z_i(t) - \tau_i(t) \}, & t \neq t_k, \\
    \Delta x_i(t_k) = \ln(1 + b_{ik}), & i = 1, 2, \ldots, n, k = 1, 2, \ldots, n.
\end{cases}
\]

(2)

where \( \ln \{ z_i(t) \} = \{ \ln \{ z_i(t) \}, \ln \{ z_2(t) \}, \ldots, \ln \{ z_n(t) \} \} \).

Now we are ready to state and prove the main results of the present paper.

**Theorem** Assume that \( \alpha_i + \beta_i < 1 \), then system (1) has at least one \( T \)-periodic solution.

**Proof.** To complete the proof, we only need to search for an appropriate open bounded subset verifying all the requirements in Lemma 1.

Let

\[
Z = (z_1(t), z_2(t), \ldots, z_n(t))^T,
\]

\[
Z = \{ z \in PC(R, R^n) \mid z(t + T) = z(t) \},
\]

\[
Y = Z \times \mathbb{R}^2
\]

then it is standard to show that both \( Z \) and \( Y \) are Banach space when they are endowed with the norms

\[
\| z \|_i = \sup_{t \in [0, T]} | z(t) |
\]

and

\[
\| (z_1, c_1, \ldots, c_p) \| = (\| z \|_1^2 + \| c_1 \|^2 + \cdots + \| c_p \|)^{1/2}.
\]

Set \( L : \text{Dom} L \to Y \) as

\[
(Lz)(t) = (z'(t), \Delta z_1(t), \ldots, \Delta z_p(t)),
\]

where \( \text{Dom} L = Z = \{ z \in Z \mid z(t) \in PC(R, R^n) \} \).

At the same time, we denote

\[
(\mathcal{N}z)(t) = ((h_i(t) - \exp \{ z(t) \} + \sum_{j=1}^{n} a_{ij}(t) \exp \{ z_j(t) \})
\]

\[
+ c_i(t) \exp \{ z_i(t) - \tau_i(t) \}, (I, \ldots, I_p))
\]

It is easily to prove that \( L \) is a Fredholm mapping of index zero.

Consider the operator equation

\[
Lz = \lambda \mathcal{N}z \quad \lambda \in (0,1).
\]

(3)

Integrating (3) over the interval \([0, T]\), we obtain

\[
\overline{h}_T = -\sum_{k=1}^{p} \ln(1 + b_k) + \int_0^T \exp \{ z(t) \} dt
\]

\[
- \int_0^T \sum_{j=1}^{n} a_{ij}(t) \exp \{ z_j(t) \} dt
\]

\[
- \int_0^T c_i(t) \exp \{ z_i(t) - \tau_i(t) \} dt,
\]

(4)

\((i = 1, 2, \ldots, n). \)

Then, we can derive

\[
\int_0^T | z_i'(t) | \leq 2\overline{h}_T + \sum_{k=1}^{p} \ln(1 + b_k), (i = 1, 2, \ldots, n).
\]

Since \( z_i(t) \in PC([0, T], R^n) \), there exist \( \xi_i \in [0, T] \)

\[
\cap \{ t_1', t_2', \ldots, t_p' \}, \quad \text{such that}
\]

\[
z_i(\xi_i) = \inf_{t \in [0, T]} z_i(t), \quad z_i(t_i) = \sup_{t \in [0, T]} z_i(t), \quad (i = 1, 2, \ldots, n).
\]

For (4) we can see

\[
\overline{h}_T \leq -\sum_{k=1}^{p} \ln(1 + b_k) + \int_0^T \exp \{ z(t) \} dt
\]

\[
- \int_0^T \sum_{j=1}^{n} a_{ij}(t) \exp \{ z_j(t) \} dt
\]

\[
- \int_0^T c_i(t) \exp \{ z_i(t - \tau_i(t)) \} dt
\]

\leq -\sum_{k=1}^{p} \ln(1 + b_k) + \int_0^T \exp \{ z(t) \} dt
\]

\[
- \int_0^T \sum_{j=1}^{n} a_{ij}(t) \exp \{ z_j(t) \} dt
\]

\[
- \int_0^T c_i(t) \exp \{ z_i(t) \} dt
\]

\leq -\sum_{k=1}^{p} \ln(1 + b_k) - (\overline{a}_k + \overline{c}_k - 1) \exp \{ z(t) \} T,
\]

which implies

\[
z_i(t) \geq \ln \left[ \frac{\overline{h}_i + \frac{1}{T} \sum_{k=1}^{p} \ln(1 + b_k)}{1 - \overline{a}_i - \overline{c}_i} \right] := A
\]

Thus, \( \forall t \in [0, T], \) we have

\[
z_i(t) \geq z_i(t) + \sum_{k=1}^{p} \ln(1 + b_k) - \int_0^T | z_i'(t) | dt \geq A - 2\overline{h}_T := M.
\]

Similarly, according to (4), we have

\[
\overline{h}_T \geq -\sum_{k=1}^{p} \ln(1 + b_k) + \int_0^T \exp \{ z(t) \} dt
\]

\[
- \int_0^T \sum_{j=1}^{n} a_{ij}(t) \exp \{ M \} dt
\]

\[
- \int_0^T c_i(t) \exp \{ z_i(t - \tau_i(t)) \} dt
\]

\[
\geq -\sum_{k=1}^{p} \ln(1 + b_k) + \int_0^T \exp \{ z(t, \xi_i) \} dt
\]

\[
- \int_0^T \sum_{j=1}^{n} a_{ij}(t) \exp \{ M \} dt
\]

\[
- \int_0^T c_i(t) \exp \{ z_i(t, \xi_i) \} dt
\]

\geq -\sum_{k=1}^{p} \ln(1 + b_k) - \sum_{j=1}^{n} \overline{a}_{ij}(t) \exp \{ M \} T
\]

\[
+ (1 - \overline{c}_k) \exp \{ z_i(t, \xi_i) \} T,
\]
which implies,
\[
z_i(\xi_0) \leq \ln \left[ \frac{\bar{h}_i + \frac{\gamma}{T} \sum_{k=1}^{p} \ln(1 + b_k) + \sum_{j=1}^{s} \bar{a}_j \exp(M)}{1 - c_i} \right]
\]
\[
:= B.
\]
Thus, \( \forall t \in [0, T] \), we have
\[
z_i(t) \leq z_i(\xi_0) + \sum_{k=1}^{p} \ln(1 + b_k) + \int_0^T |z_i'(t)| dt
\]
\[
\leq B + 2\bar{h}T + 2\sum_{k=1}^{p} \ln(1 + b_k) := N.
\]

Now, we can derive
\[
|z_i(t)| \leq \max \{|M|, |N|\} := M_i.
\]

Obviously, \( M_i \) is independent of \( \lambda \). Then, there exists a constant \( F > 0 \), such that \( \max \{|z_i|\} \leq F \). Let \( r > M_i + F \), \( \Omega = \{z \in Z : ||z|| < r\} \), then it is clear that \( \Omega \) satisfies condition (a) of Lemma 1 and \( N \) is \( L \)-compact on \( \overline{\Omega} \). Let \( J : \text{Im} \Omega \to x, (d, 0, \ldots, 0) \to d \), a direct computation gives
\[
\text{deg}\{JQN, \Omega \cap \text{Ker}L, 0\} \neq 0.
\]

By now we have proved that \( \Omega \) satisfies all the requirements in Mawhin’s continuation theorem (Lemma 1). Hence, system (2) has at least one \( T \)-periodic solution \( z(t) = (z_1(t), z_2(t), \ldots, z_p(t))^T \) in \( \text{Dom} L \cap \overline{\Omega} \). The proof is completes.

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