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Existence and Uniqueness of Positive Solutions for a Coupled System of Nonlinear Fractional Differential Equations

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ABSTRACT

In this paper, we research the existence and uniqueness of positive solutions for a coupled system of fractional differential equations. By means of some standard fixed point principles, some results on the existence and uniqueness of positive solutions for coupled systems are obtained.

Keywords: Caputo Fractional Derivative; Fractional Differential Equations; Coupled System; Fixed Point Theorem; Positive Solutions

1. Introduction

Fractional differential equations can describe many phenomena in various fields of engineering and scientific disciplines such as control theory, physics, chemistry, biology, economics, mechanics and electromagnetic. In recent years, there are a large number of papers dealing with the existence of positive solutions of boundary value problems for nonlinear differential equations of fractional order. We refer readers to the monographs such as Kilbas etal. [8], Miller and Ross [20], Podlubny [21], and the papers [1,3-5,12-19,28-33] and references therein.

In [12], Li, Luo and Zhou considered the existence of positive solutions of the following boundary value problem of fractional order differential equations:

$$D_{0+}^{\alpha}u(t)+f(t,u(t))=0, 0<t<1,$$
$$u(0)=0, D_{0+}^{\beta}u(1)=aD_{0+}^{\beta}u(\xi),$$

where D_{0+}^{α} is the standard Riemann-Liouville fractional derivative of order $1<\alpha\le 2,\ 0<\beta\le 1,\ 0\le a\le 1,\ \xi\in(0,1),$ $a\xi^{\alpha-\beta-2}\le 1-\beta,\ 0\le\alpha-\beta-1$ and $f:[0,1]\times[0,\infty)\to[0,\infty)$ satisfies Caratheodory type conditions.

In [30], Yang, Wei and Dong investigated the following existence of positive solutions of fractional order differential equations:

$$^{c}D_{0+}^{\alpha}u(t)=f(t,u(t),u'(t)), 0<t<1,$$
$$u(0)+u'(0)=0, u(1)+u'(1)=0,$$

where $^{c}D_{0+}^{\alpha}$ is the Caputo fractional derivative of order $1<\alpha\le 2$ and $f\in C([0,1]\times[0,\infty)\times R, R^{+})$.

In addition, recently some authors also pay close attention to the existence of solutions for coupled systems of fractional differential equations (see[2,3,25,26]).

In [26], Su studied the existence of solutions for a coupled system of fractional differential equations:

$$\begin{cases} D_{0+}^{\alpha}u(t)=f(t,v(t),D_{0+}^{\mu}v(t)),\ 0<t<1,\\ D_{0+}^{\beta}v(t)=g(t,u(t),D_{0+}^{\nu}u(t)),\ 0<t<1,\\ u(0)=u(1)=v(0)=v(1),\end{cases}$$

where $1<\alpha,\ \beta<2,\ \mu,\nu>0,\ \alpha-\nu\ge 1,\ \beta-\mu\ge 1,\ f,g:[0,1]\times R^{2}\to R$ are given functions and D_{0+} is the standard Riemann-Liouville fractional derivative.

In [25], Sun, Liu and Liu considered the following systems of fractional differential equations with antiperiodic boundary conditions:

$$\begin{cases} ^{c}D_{0+}^{\alpha}u(t)=f_{1}(t,u(t),v(t),{}^{c}D_{0+}^{p}u(t),{}^{c}D_{0+}^{q}v(t)),\\ t\in J=[0,T],\\ ^{c}D_{0+}^{\beta}v(t)=f_{2}(t,u(t),v(t),{}^{c}D_{0+}^{p}u(t),{}^{c}D_{0+}^{q}v(t)),\\ t\in J=[0,T],\\ u(0)=-u(T),u'(0)=-u'(T),\\ v(0)=-v(T),v'(0)=-v'(T),\end{cases} \quad (1.1)$$

where $^{c}D_{0+}^{\alpha}$ denotes the Caputo fractional derivative, $1<\alpha,\ \beta\le 2,\ 0<p,q<1,\ f_{1},f_{2}\in C(J\times R^{4},R)$.

However, the research on the systems of positive solutions of fractional differential equations hasn't received remarkable attention. In this paper, we shall concern with the existence and uniqueness of positive solutions for a coupled system of nonlinearfractional differential equations. More precisely, we will consider the following problem:

$$\begin{cases} {}^cD_{0+}^{\alpha}u(t) = f_1(t,u(t),v(t),{}^cD_{0+}^{p}u(t),{}^cD_{0+}^{q}v(t)), \\ t\in J=[0,T], \\ {}^cD_{0+}^{\beta}v(t) = f_2(t,u(t),v(t),{}^cD_{0+}^{p}u(t),{}^cD_{0+}^{q}v(t)), \\ t\in J=[0,T], \\ u(0)+\lambda_1 u'(0)=0,\ \ u(1)+\lambda_2{}^cD_{0+}^{p}u(1)=0, \\ v(0)+\lambda_1 v'(0)=0,\ \ v(1)+\lambda_2{}^cD_{0+}^{q}v(1)=0, \end{cases}$$

where ${}^cD_{0+}^{\gamma}$ denotes the Caputo fractional derivative for $\gamma>0,\ 1<\alpha,\ \beta\le 2,\ 0<p,\ q\le 1,\ f_1,\ f_2\in C([0,1]\times R_+^2\times R^2, R_+)$.

This paper is organized as follows: In section 2, we introduce some preliminary results, including basic definitions of fractional integrals and derivatives, some properties and a fixed point theorems. In section 3, by applying some standard fixed point principles, we prove the existence and uniqueness of positive solutions for a coupled system of nonlinear fractional differential equations.

2. Preliminaries

Let us introduce a space: $X=\{u(t)\,|\,u(t)\in C^1([0,1])\}$ endowed with the norm

$$\|u\|_X=\max_{t\in[0,1]}|u(t)|+\max_{t\in[0,1]}|u'(t)|.$$

Indeed, $(X,\|.\|_X)$ is a Banach space. Obviously, the product space $(X\times X,\|.\|_{X\times X})$ is also a Banach space with $\|(u,v)\|_{X\times X}=\|u\|_X+\|v\|_X$. For the convenience of the readers, we first present some useful definitions and fundamental facts of fractional calculus theory, which can be found in [8,21].

Definition 2.1. For $\gamma>0$, the integral

$$I_{0+}^{\gamma}f(t)=\frac{1}{\Gamma(\gamma)}\int_0^t\frac{f(s)}{(t-s)^{1-\gamma}}ds, \tag{2.1}$$

is called the Riemann-Liouville fractional integral.

Definition 2.2. For a function $f(t)$ given in the interval $[0,\infty)$, the expression

$$ {}^LD_{0+}^{\gamma}f(t)=\frac{1}{\Gamma(n-\gamma)}(\frac{d}{dt})^n\int_0^t\frac{f(s)}{(t-s)^{\gamma-n+1}}ds, \quad n=[\gamma]+1 \tag{2.2}$$

is called the Riemann-Liouville fractional derivative of order $\gamma>0$, where $[\gamma]$ denotes the integer part of real number γ.

Definition 2.3 [6]. The Caputo's derivative of order γ for a function $f\in C^n([0,\infty),R)$ can be written as

$$ {}^cD_{0+}^{\gamma}f(t)={}^LD^{\gamma}[f(t)-\sum_{k=0}^{n-1}\frac{t^kf^{(k)}(0)}{k!}], \quad n-1<\gamma<n. \tag{2.3}$$

Lemma 2.4 [8,21]. Let $u\in C^{n-1}[0,1]$ and $q\in(n-1,n]$, $n\in N$. Then for $t\in[0,1]$,

$$I^{q\,c}D_{0+}^{q}u(t)=u(t)-\sum_{k=0}^{n-1}\frac{t^k}{k!}u^{(k)}(0). \tag{2.4}$$

Lemma 2.5. Let $\varphi\in C[0,1]$. If $H:1\le\lambda_1<1+\frac{\lambda_2}{\Gamma(2-p)}$, $\lambda_2>0$, then $u(t)$ is a solution of the following fractional differential equations:

$$\begin{cases} {}^cD_{0+}^{\alpha}u(t)=\varphi(t),\ t\in[0,1], 1<\alpha<2, \\ u(0)+\lambda_1u'(0)=0, u(1)+\lambda_2{}^cD_{0+}^{p}u(1)=0, 0<p<1, \end{cases} \tag{2.5}$$

if and only if $u(t)$ is a solution of the fractional integral equations

$$u(t)=\int_0^1G_\alpha(t,s)\varphi(s)ds, \tag{2.6}$$

where

$$G_\alpha(t,s)=\begin{cases} \dfrac{\Lambda(t-s)^{\alpha-1}+(\lambda_1-t)(1-s)^{\alpha-1}}{\Lambda\Gamma(\alpha)} \\ +\dfrac{\lambda_2(\lambda_1-t)(1-s)^{\alpha-p-1}}{\Lambda\Gamma(\alpha-p)}, \\ 0\le s\le t\le 1, \\ \dfrac{(\lambda_1-t)(1-s)^{\alpha-1}}{\Lambda\Gamma(\alpha)}+\dfrac{\lambda_2(\lambda_1-t)(1-s)^{\alpha-p-1}}{\Lambda\Gamma(\alpha-p)}, \\ 0\le t\le s\le 1, \end{cases} \tag{2.7}$$

$\Lambda=1+\dfrac{\lambda_2}{\Gamma(2-p)}-\lambda_1.$

Furthermore, if the assumption H holds, then $G_\alpha(t,s)\in C([0,1)\times[0,1))$ and $G_\alpha(t,s)>0$, for any $t,s\in(0,1)$.

Proof. Assume $u(t)$ satisfies (2.5). By (2.4), (2.5), we have

$$u(t)=\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\varphi(s)ds+u(0)+u'(0)t.$$

Hence,

$$u'(t)=\int_0^t\frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}\varphi(s)ds+u'(0).$$

By definition 2.3 together with the facts that

$$ {}^LD_{0+}^{p}t=\frac{t^{1-p}}{\Gamma(2-p)},\quad {}^LD_{0+}^{p}I^{\alpha}u(t)=I^{\alpha-p}u(t)$$

and the linearity of fractional differential, we get

$$ {}^cD_{0+}^{p}u(t)=\int_0^t\frac{(t-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}\varphi(s)ds+u'(0)\frac{t^{1-p}}{\Gamma(2-p)}.$$

Applying the boundary conditions

$$u(0)+\lambda_1u'(0)=0,\ u(1)+\lambda_2{}^cD_{0+}^{p}u(1)=0,$$

we obtain that

$$u(0)=\frac{\lambda_1}{\Lambda\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}\varphi(s)ds+\frac{\lambda_1\lambda_2}{\Lambda\Gamma(\alpha-p)}\times\int_0^1(1-s)^{\alpha-p-1}\varphi(s)ds,$$

$$u'(0)=-\frac{1}{\Lambda\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}\varphi(s)ds-\frac{\lambda_2}{\Lambda\Gamma(\alpha-p)}\times\int_0^1(1-s)^{\alpha-p-1}\varphi(s)ds.$$

Consequently,

$$\begin{aligned}u(t)&=\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\varphi(s)ds+\frac{\lambda_1-t}{\Lambda\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}\varphi(s)ds\\&\quad+\frac{\lambda_2(\lambda_1-t)}{\Lambda\Gamma(\alpha-p)}\int_0^1(1-s)^{\alpha-p-1}\varphi(s)ds\\&=\int_0^1G_\alpha(t,s)\varphi(s)ds.\end{aligned}$$

Conversely, assume $u(t)$ is a solution of fractional integral equations (2.6). Using the definition of Caputo's derivative (2.3) and the fact that ${}^LD_{0+}^pC=\frac{Ct^{2-p}}{\Gamma(1-p)}$ and ${}^LD_{0+}^\alpha$ is the left inverse of I_{0+}^α we get (2.5).

Observing the expression of $G_\alpha(t,s)$ in (2.6), we easily obtain $G_\alpha(t,s)\in C([0,1)\times[0,1))$. Let

$$g_1(t,s)=\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}+\frac{(\lambda_1-t)(1-s)^{\alpha-1}}{\Lambda\Gamma(\alpha)}+\frac{\lambda_2(\lambda_1-t)(1-s)^{\alpha-p-1}}{\Lambda\Gamma(\alpha-p)},0\le s\le t\le1, \tag{2.8}$$

$$g_2(t,s)=\frac{(\lambda_1-t)(1-s)^{\alpha-1}}{\Lambda\Gamma(\alpha)}+\frac{\lambda_2(\lambda_1-t)(1-s)^{\alpha-p-1}}{\Lambda\Gamma(\alpha-p)},\quad 0\le t\le s\le1. \tag{2.9}$$

By H, we have $\Lambda>0$ and

$$g_2(t,s)=\frac{(\lambda_1-t)(1-s)^{\alpha-1}}{\Lambda\Gamma(\alpha)}+\frac{\lambda_2(\lambda_1-t)(1-s)^{\alpha-p-1}}{\Lambda\Gamma(\alpha-p)}>0,\quad\forall s,t\in(0,1),$$

which also implies $g_1(t,s)>0$ by (2.8). Hence $G_\alpha(t,s)>0$ for all $s,t\in(0,1)$. The proof is completed.

Remark 2.6. If we make use of

$$H':\lambda_1\ge1,\ \lambda_2<-\frac{\Gamma(\alpha-p)}{\Gamma(\alpha)}$$

instead of H, we may similarly consider the problem (1.1). We omit it here.

Lemma 2.7 [22]. Let E be a Banach space. Assume that $T:E\to E$ is a completely continuous operator and the set $V=\{u\in E\mid u=\mu Tu,0\le\mu\le1\}$ is bounded. Then T has a fixed point in E.

3. Main Result

For the sake of convenience, we set

$$M_1=\frac{\Lambda+\lambda_1+\Lambda\alpha+1}{\Lambda\Gamma(\alpha+1)}+\frac{\lambda_2(\lambda_1+1)}{\Lambda\Gamma(\alpha-p+1)},\quad M_2=\frac{\Lambda^*+\lambda_1^*+\Lambda^*\alpha+1}{\Lambda^*\Gamma(\alpha+1)}+\frac{\lambda_2^*(\lambda_1^*+1)}{\Lambda^*\Gamma(\alpha-p+1)}, \tag{3.1}$$

$$M_3=\max\left\{[M_1c_1+M_2d_1+(M_1c_3+M_2d_3)\frac{1}{\Gamma(2-p)}],\ [M_1c_2+M_2d_2+(M_1c_4+M_2d_4)\frac{1}{\Gamma(2-q)}]\right\}. \tag{3.2}$$

We denote

$$\Lambda=1+\frac{\lambda_2}{\Gamma(2-p)}-\lambda_1,\Lambda^*=1+\frac{\lambda_2^*}{\Gamma(2-q)}-\lambda_1^*$$

and give the following assumption

$$H^*:1\le\lambda_1<1+\tfrac{\lambda_2}{\Gamma(2-p)},\ \lambda_2>0,\quad 1\le\lambda_1^*<1+\tfrac{\lambda_2^*}{\Gamma(2-q)},\ \lambda_2^*>0.$$

Define the operator $T:X\times X\to X\times X$ as

$$\begin{aligned}&T(u,v)(t)\\&=(T_1(u,v)(t),T_2(u,v)(t))\\&=(\int_0^1G_\alpha(t,s)f_1(s,u(s),v(s),{}^cD_{0+}^pu(s),{}^cD_{0+}^qv(s))ds,\\&\quad\int_0^1G_\beta(t,s)f_2(s,u(s),v(s),{}^cD_{0+}^pu(s),{}^cD_{0+}^qv(s))ds),\end{aligned} \tag{3.3}$$

which implies

$$\begin{aligned}&(T(u,v))'(t):=((T_1(u,v))'(t),(T_2(u,v))'(t))\\&=(\int_0^t\frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}f_1(s,u(s),{}^cD_{0+}^pu(s),{}^cD_{0+}^qv(s))ds\\&\quad-\frac{1}{\Lambda\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}f_1(s,u(s),{}^cD_{0+}^pu(s),{}^cD_{0+}^qv(s))ds\\&\quad-\frac{\lambda_2}{\Lambda\Gamma(\alpha-p)}\int_0^1(1-s)^{\alpha-p-1}f_1(s,u(s),{}^cD_{0+}^pu(s),{}^cD_{0+}^qv(s))ds,\\&\quad\int_0^t\frac{(t-s)^{\beta-2}}{\Gamma(\beta-1)}f_2(s,u(s),{}^cD_{0+}^pu(s),{}^cD_{0+}^qv(s))ds\\&\quad-\frac{1}{\Lambda^*\Gamma(\beta)}\int_0^1(1-s)^{\beta-1}f_2(s,u(s),{}^cD_{0+}^pu(s),{}^cD_{0+}^qv(s))ds\\&\quad-\frac{\lambda_2^*}{\Lambda^*\Gamma(\beta-p)}\int_0^1(1-s)^{\beta-p-1}f_2(s,u(s),{}^cD_{0+}^pu(s),{}^cD_{0+}^qv(s))ds).\end{aligned} \tag{3.4}$$

Lemma 3.1. The operator $T:X\times X\to X\times X$ is completely continuous.

Proof. Firstly, we show that the operator $T:X\times X\to X\times X$ is continuous.

For $0<p<1,\{u_n,v_n\}\subseteq X\times X$ such that $(u_n,v_n)\to(u_0,v_0)$ in $X\times X$, we have

$$\begin{aligned}&\max_{t\in[0,1]}|{}^c D_{0+}^p u_n(t)-{}^c D_{0+}^p u_0(t)|\\&=\max_{t\in[0,1]}\Big|\frac{1}{\Gamma(1-p)}\int_0^t(t-s)^{-p}u'_n(s)ds-\frac{1}{\Gamma(1-p)}\\&\quad\times\int_0^t(t-s)^{-p}u_0(s)ds\Big|\\&=\max_{t\in[0,1]}\Big|\frac{1}{\Gamma(1-p)}\int_0^t(t-s)^{-p}[u'_n(s)-u_0(s)]ds\Big|\\&=\frac{1}{\Gamma(2-p)}\max_{t\in[0,1]}|u'_n(t)-u_0(t)|\\&=\frac{1}{\Gamma(2-p)}\|u_n-u_0\|_X .\end{aligned}$$

By $\|u_n-u_0\|_X\to 0$, we get the sequence ${}^c D_{0+}^p u_n(t)$ converges uniformly on $[0,1]$ with

$$\lim_{n\to\infty}{}^c D_{0+}^p u_n(t)={}^c D_{0+}^p u_0(t).$$

Similarly, by $\|v_n-v_0\|_X\to 0$, we get the sequence ${}^c D_{0+}^q v_n(t)$ converges uniformly on $[0,1]$ with $\lim_{n\to\infty}{}^c D_{0+}^q v_n(t)={}^c D_{0+}^q v_0(t)$.

Since

$$\begin{aligned}&\|T(u_n,v_n)-T(u_0,v_0)\|_{X\times X}\\&=\max_{t\in[0,1]}|T_1(u_n,v_n)(t)-T_1(u_0,v_0)(t)|\\&\quad+\max_{t\in[0,1]}|(T_1(u_n,v_n))'(t)-(T_1(u_0,v_0))'(t)|\\&\quad+\max_{t\in[0,1]}|T_2(u_n,v_n)(t)-T_2(u_0,v_0)(t)|\\&\quad+\max_{t\in[0,1]}|(T_2(u_n,v_n))'(t)-(T_2(u_0,v_0))'(t)|.\end{aligned}$$

Combining (3.3),(3.4) with the continuity of f_1, f_2, we can get

$$\|T(u_n,v_n)-T(u_0,v_0)\|_{X\times X}\to 0,((u_n,v_n)\to(u_0,v_0)).$$

Thus T is continuous in $X\times X$.

Let $\Omega\subset X\times X$ be bounded. Then there exist positive constants $L_i>0$ such that

$$|f_i(t,u(t),v(t),{}^c D_{0+}^p u(t),{}^c D_{0+}^q v(t))|\le L_i\,(i=1,2)\quad \forall(u,v)\in\Omega.$$

Thus, for any $(u,v)\in\Omega$, we have

$$\begin{aligned}|T_1(u,v)(t)|&\le\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}|f_1(s,u(s),v(s),{}^c D_{0+}^p u(s),{}^c D_{0+}^q v(s))|ds\\&\quad+\frac{|\lambda_1-t|}{\Lambda\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}|f_1(s,u(s),v(s),{}^c D_{0+}^p u(s),{}^c D_{0+}^q v(s))|ds\\&\quad+\frac{\lambda_2|\lambda_1-t|}{\Lambda\Gamma(\alpha-p)}\int_0^1(1-s)^{\alpha-p-1}|f_1(s,u(s),v(s),{}^c D_{0+}^p u(s),{}^c D_{0+}^q v(s))|ds\\&\le\frac{L_1}{\Gamma(\alpha+1)}+\frac{L_1\lambda_1}{\Lambda\Gamma(\alpha+1)}+\frac{\lambda_1\lambda_2L_1}{\Lambda\Gamma(\alpha-p+1)}\\&=\frac{L_1(\Lambda+\lambda_1)}{\Lambda\Gamma(\alpha+1)}+\frac{\lambda_1\lambda_2L_1}{\Lambda\Gamma(\alpha-p+1)}\end{aligned}$$

$$\begin{aligned}|(T_1(u,v))'(t)|&\le\frac{1}{\Gamma(\alpha-1)}\int_0^t(t-s)^{\alpha-2}|f_1(s,u(s),v(s),{}^c D_{0+}^p u(s),{}^c D_{0+}^q v(s))|ds\\&\quad+\frac{1}{\Lambda\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}|f_1(s,u(s),v(s),{}^c D_{0+}^p u(s),{}^c D_{0+}^q v(s))|ds\\&\quad+\frac{\lambda_2}{\Lambda\Gamma(\alpha-p)}\int_0^1(1-s)^{\alpha-p-1}|f_1(s,u(s),v(s),{}^c D_{0+}^p u(s),{}^c D_{0+}^q v(s))|ds\\&\le\frac{L_1}{\Gamma(\alpha)}+\frac{L_1}{\Lambda\Gamma(\alpha+1)}+\frac{\lambda_2L_1}{\Lambda\Gamma(\alpha-p+1)}\\&=\frac{L_1(\Lambda\alpha+1)}{\Lambda\Gamma(\alpha+1)}+\frac{\lambda_2L_1}{\Lambda\Gamma(\alpha-p+1)}.\end{aligned}$$

Hence

$$\|T_1(u,v)\|_X\le\frac{L_1(\Lambda+\lambda_1+\Lambda\alpha+1)}{\Lambda\Gamma(\alpha+1)}+\frac{\lambda_2L_1(\lambda_1+1)}{\Lambda\Gamma(\alpha-p+1)}=M_1L_1 \tag{3.5}$$

where M_1 is given by (3.1).

In the same way ,we can verify that

$$\|T_2(u,v)\|_X\le\frac{L_2(\Lambda^*+\lambda_1^*+\Lambda^*\alpha+1)}{\Lambda^*\Gamma(\beta+1)}+\frac{L_2\lambda_2^*(\lambda_1^*+1)}{\Lambda^*\Gamma(\beta-q+1)}=M_2L_2 \tag{3.6}$$

where M_2 is given by (3.1). Thus,

$$\|T(u,v)\|_{X\times X}\le M_1L_1+M_2L_2:=M$$

which implies that the operator T is uniformly bounded.

Next we show that T is equicontinuous.

For any $0\le t_1\le t_2\le 1$,

$$\begin{aligned}&|T_1(u(t_2),v(t_2))-T_1(u(t_1),v(t_1))|\\&\le\int_0^{t_1}|[G_\alpha(t_2,s)-G_\alpha(t_1,s)]f_1(s,u(s),v(s),{}^c D_{0+}^p u(s),{}^c D_{0+}^q v(s))|ds\\&\quad+\int_{t_1}^{t_2}|[G_\alpha(t_2,s)-G_\alpha(t_1,s)]f_1(s,u(s),v(s),{}^c D_{0+}^p u(s),{}^c D_{0+}^q v(s))|ds\\&\quad+\int_{t_2}^{1}|[G_\alpha(t_2,s)-G_\alpha(t_1,s)]f_1(s,u(s),v(s),{}^c D_{0+}^p u(s),{}^c D_{0+}^q v(s))|ds\\&\le L_1\Big[\frac{t_2-t_1}{\Lambda\Gamma(\alpha+1)}+\frac{t_2^\alpha-t_1^\alpha}{\Gamma(\alpha+1)}+\frac{\lambda_2(t_2-t_1)}{\Lambda\Gamma(\alpha-p+1)}\Big]\end{aligned}$$

$$|(T_1(u,v))'(t_2)-(T_1(u,v))'(t_1)|$$
$$=\frac{1}{\Gamma(\alpha-1)}\Big|\int_0^{t_2}(t_2-s)^{\alpha-2}f_1(s,u(s),v(s),{}^cD_{0+}^pu(s),{}^cD_{0+}^qv(s))ds$$
$$-\int_0^{t_1}(t_1-s)^{\alpha-2}f_1(s,u(s),v(s),{}^cD_{0+}^pu(s),{}^cD_{0+}^qv(s))ds\Big|$$
$$\le\frac{L_1}{\Gamma(\alpha-1)}\Big|\int_0^{t_2}(t_2-s)^{\alpha-2}ds-\int_0^{t_1}(t_1-s)^{\alpha-2}ds\Big|$$
$$\le\frac{L_1}{\Gamma(\alpha)}(t_2^{\alpha-1}-t_1^{\alpha-1}).$$

Analogously, we can obtain the following inequalities:

$$|T_2(u(t_2),v(t_2))-T_2(u(t_1),v(t_1))|$$
$$\le L_2\Big[\frac{t_2-t_1}{\Lambda^*\Gamma(\beta+1)}+\frac{t_2^\beta-t_1^\beta}{\Gamma(\beta+1)}+\frac{\lambda_2^*(t_2-t_1)}{\Lambda^*\Gamma(\beta-q+1)}\Big],$$
$$|(T_2(u,v))'(t_2)-(T_2(u,v))'(t_1)|$$
$$\le\frac{L_1}{\Gamma(\beta)}(t_2^{\beta-1}-t_1^{\beta-1}).$$

Since the functions $t^\alpha, t^\beta, t^{\alpha-1}, t^{\beta-1}$ are uniformly continuous on the interval $[0,1]$, we can conclude that $T(u,v)$ is equicontinuous on $[0,1]$.

Thus, the operator $T: X\times X\to X\times X$ is completely continuous. The proof is completed.

Theorem 3.2. Assume that there exist positive constants $c_i, d_i (i=0,1,2,3,4)$, $c_0>0$, $d_0>0$, c_i, $d_i\ge 0$ $(i=1,2,3,4)$ such that

$$\forall(t,x_1,x_2,x_3,x_4)\in[0,1]\times R_+^2\times R^2, t\in[0,1],$$
$$|f_1(t,x_1,x_2,x_3,x_4)|\le c_0+c_1|x_1|+c_2|x_2|+c_3|x_3|+c_4|x_4|,$$
$$|f_2(t,x_1,x_2,x_3,x_4)|\le d_0+d_1|x_1|+d_2|x_2|+d_3|x_3|+d_4|x_4|. \tag{3.7}$$

In addition, assume that

$$M_1c_1+M_2d_1+(M_1c_3+M_2d_3)\frac{1}{\Gamma(2-p)}<1, M_1c_2+M_2d_2+(M_1c_4+M_2d_4)\frac{1}{\Gamma(2-q)}<1,$$

where M_1 and M_2 are defined by (3.1).

Then the problem (1.1) has at least one positive solution.

Proof. Let us verify that the set

$$V=\{(u,v)\in X\times X\mid (u,v)=\mu T(u,v), 0\le\mu\le 1\}$$

is bounded. Let $(u,v)\in V$, then $(u,v)=\mu T(u,v)$. For any $t\in[0,1]$, we have

$$|u(t)|=\mu|T_1(u,v)(t)|$$
$$\le\frac{1}{\Gamma(\alpha)}\int_0^t\mu(t-s)^{\alpha-1}|f_1(s,u(s),v(s),{}^cD_{0+}^pu(s),{}^cD_{0+}^qv(s))|ds$$
$$+\frac{\lambda_1-t}{\Lambda\Gamma(\alpha)}\int_0^1\mu(1-s)^{\alpha-1}|f_1(s,u(s),v(s),{}^cD_{0+}^pu(s),{}^cD_{0+}^qv(s))|ds$$
$$+\frac{\lambda_2(\lambda_1-t)}{\Lambda\Gamma(\alpha-p)}\int_0^1\mu(1-s)^{\alpha-p-1}|f_1(s,u(s),v(s),{}^cD_{0+}^pu(s),{}^cD_{0+}^qv(s))|ds$$
$$\le[c_0+c_1|u(t)|+c_2|v(t)|+c_3|{}^cD_{0+}^pu(t)|+c_4|{}^cD_{0+}^qv(t)|]\times\Big(\frac{\Lambda+\lambda_1}{\Lambda\Gamma(\alpha+1)}+\frac{\lambda_1\lambda_2}{\Lambda\Gamma(\alpha-p+1)}\Big)$$
$$\le[c_0+c_1\|u\|_X+c_2\|v\|_X+c_3\frac{1}{\Gamma(2-p)}\|u\|_X+c_4\frac{1}{\Gamma(2-q)}\|v\|_X]\times\Big(\frac{\Lambda+\lambda_1}{\Lambda\Gamma(\alpha+1)}+\frac{\lambda_1\lambda_2}{\Lambda\Gamma(\alpha-p+1)}\Big),$$
$$|u'(t)|=\mu|(T_1(u,v))'(t)|$$
$$=\frac{1}{\Gamma(\alpha-1)}\int_0^t\mu(t-s)^{\alpha-2}|f_1(s,u(s),v(s),{}^cD_{0+}^pu(s),{}^cD_{0+}^qv(s))|ds$$
$$-\frac{1}{\Lambda\Gamma(\alpha)}\int_0^1\mu(1-s)^{\alpha-1}|f_1(s,u(s),v(s),{}^cD_{0+}^pu(s),{}^cD_{0+}^qv(s))|ds$$
$$\frac{\lambda_2}{\Lambda\Gamma(\alpha-p)}\int_0^1(1-s)^{\alpha-p-1}|f_1(s,u(s),v(s),{}^cD_{0+}^pu(s),{}^cD_{0+}^qv(s))|ds$$
$$\le[c_0+c_1|u(t)|+c_2|v(t)|+c_3|{}^cD_{0+}^pu(t)|+c_4|{}^cD_{0+}^qv(t)|]\Big(\frac{\Lambda\alpha+1}{\Lambda\Gamma(\alpha+1)}+\frac{\lambda_2}{\Lambda\Gamma(\alpha-p+1)}\Big)$$
$$\le[c_0+c_1\|u\|_X+c_2\|v\|_X+c_3\frac{1}{\Gamma(2-p)}\|u\|_X+c_4\frac{1}{\Gamma(2-q)}\|v\|_X]\times\Big(\frac{\Lambda\alpha+1}{\Lambda\Gamma(\alpha+1)}+\frac{\lambda_2}{\Lambda\Gamma(\alpha-p+1)}\Big).$$

Hence,

$$\|u\|_X\le M_1[c_0+c_1\|u\|_X+c_2\|v\|_X+c_3\frac{1}{\Gamma(2-p)}\|u\|_X+c_4\frac{1}{\Gamma(2-q)}\|v\|_X],$$

where M_1 is defined by (3.1). Similarly, we can get

$$\begin{aligned}\|v\|_X &\le M_2[d_0+d_1\|u\|_X+d_2\|v\|_X\\ &+d_3\frac{1}{\Gamma(2-p)}\|u\|_X+d_4\frac{1}{\Gamma(2-q)}\|v\|_X],\end{aligned}\quad(3.9)$$

where M_2 is defined by (3.1). Combining (3.8) with (3.9), we obtain

$$\begin{aligned}&\|(u,v)\|_{X\times X}\\ &=\|u\|_X+\|v\|_X\\ &\le[M_1c_1+M_2d_1+(M_1c_3+M_2d_3)\frac{1}{\Gamma(2-p)}]\|u\|_X\\ &+[M_1c_2+M_2d_2+(M_1c_4+M_2d_4)\frac{1}{\Gamma(2-q)}]\|v\|_X\\ &+M_1c_0+M_2d_0\\ &\le M_3(\|u\|_X+\|v\|_X)+M_1c_0+M_2d_0.\end{aligned}$$

As a result

$$\|(u,v)\|_{X\times X}\le\frac{M_1c_0+M_2d_0}{1-M_3},$$

for any $t\in[0,1]$, where M_3 is given by (3.2). So the set V is bounded.

Thus, by Lemma 2.7, the operator T has at least one fixed point. Hence the problem (1.1) has at least one positive solution.The proof is completed.

Theorem 3.3. Assume that both f_1 and

$$f_2:[0,1]\times R_+^2\times R^2\to R$$

are continuous functions and there exist constants $n_i,n_i^*\ge 0\ (i=1,2,3,4)$, such that

$$\forall t\in[0,1],(u_1,u_2,u_3,u_4)\in R_+^2sR^2,\ (v_1,v_2,v_3,v_4)\in R_+^2\times R^2,$$

$$\begin{aligned}&|f_1(t,u_1,u_2,u_3,u_4)-f_1(t,v_1,v_2,v_3,v_4)|\\ &\le n_1|u_1-v_1|+n_2|u_2-v_2|+n_3|u_3-v_3|\\ &+n_4|u_4-v_4|\\ &|f_2(t,u_1,u_2,u_3,u_4)-f_2(t,v_1,v_2,v_3,v_4)|\\ &\le n_1^*|u_1-v_1|+n_2^*|u_2-v_2|+n_3^*|u_3-v_3|\\ &+n_4^*|u_4-v_4|.\end{aligned}$$

In addition, assume that

$$n_1+n_2+n_3\frac{1}{\Gamma(2-p)}+n_4\frac{1}{\Gamma(2-q)}\le\frac{1}{4M_1},$$

$$n_1^*+n_2^*+n_3^*\frac{1}{\Gamma(2-p)}+n_4^*\frac{1}{\Gamma(2-q)}\le\frac{1}{4M_2}.$$

where M_1,M_2 are defined by (3.1). Then the problem (1.1) has a unique solution.

Proof. Define $\sup_{t\in[0,1]}|f_1(t,0,0,0,0)|=N_1<\infty$,

$$\sup_{t\in[0,1]}|f_2(t,0,0,0,0)|=N_2<\infty,$$

such that $r\ge 4\max\{M_1N_1,M_2N_2\}$. We show that $TB_r\subset B_r$, where $B_r=\{(u,v)\in X\times X\ \|\,\|(u,v)\|\le r\}$.

For $(u,v)\in B_r$, we have

$$\begin{aligned}&|T_1(u,v)(t)|\\ &\le\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}|f_1(s,u(s),v(s),{}^cD_{0+}^pu(s),{}^cD_{0+}^qv(s))|ds\\ &+\frac{|\lambda_1-t|}{\Lambda\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}|f_1(s,u(s),v(s),{}^cD_{0+}^pu(s),\\ &\qquad{}^cD_{0+}^qv(s))|ds\\ &+\frac{\lambda_2|\lambda_1-t|}{\Lambda\Gamma(\alpha-p)}\int_0^1(1-s)^{\alpha-p-1}|f_1(s,u(s),v(s),\\ &\qquad{}^cD_{0+}^pu(s),{}^cD_{0+}^qv(s))|ds\\ &\le\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}[|f_1(s,u(s),v(s),{}^cD_{0+}^pu(s),{}^cD_{0+}^qv(s))\\ &\qquad-f_1(s,0,0,0,0)|+|f_1(s,0,0,0,0)|]ds\\ &+\frac{|\lambda_1-t|}{\Lambda\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}[|f_1(s,u(s),v(s),{}^cD_{0+}^pu(s),\\ &\qquad{}^cD_{0+}^qv(s))-f_1(s,0,0,0,0)|+|f_1(s,0,0,0,0)|]ds\\ &+\frac{\lambda_2|\lambda_1-t|}{\Lambda\Gamma(\alpha-p)}\int_0^1(1-s)^{\alpha-p-1}[|f_1(s,u(s),v(s),{}^cD_{0+}^pu(s),\\ &\qquad{}^cD_{0+}^qv(s))-f_1(s,0,0,0,0)|+|f_1(s,0,0,0,0)|]ds\\ &\le[(n_1+n_2+n_3\frac{1}{\Gamma(2-p)}+n_4\frac{1}{\Gamma(2-q)})r+N_1]\\ &\times(\frac{(\Lambda+\lambda_1)}{\Lambda\Gamma(\alpha+1)}+\frac{\lambda_1\lambda_2}{\Lambda\Gamma(\alpha-p+1)}),\end{aligned}$$

$$\begin{aligned}&|(T_1(u,v))'(t)|\\ &\le\frac{1}{\Gamma(\alpha-1)}\int_0^t(t-s)^{\alpha-2}|f_1(s,u(s),v(s),{}^cD_{0+}^pu(s),\\ &\qquad{}^cD_{0+}^qv(s))|ds\\ &+\frac{1}{\Lambda\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}|f_1(s,u(s),v(s),{}^cD_{0+}^pu(s),\\ &\qquad{}^cD_{0+}^qv(s))|ds\\ &+\frac{\lambda_2}{\Lambda\Gamma(\alpha-p)}\int_0^1(1-s)^{\alpha-p-1}|f_1(s,u(s),v(s),\\ &\qquad{}^cD_{0+}^pu(s),{}^cD_{0+}^qv(s))|ds\\ &\le\frac{1}{\Gamma(\alpha-1)}\int_0^t(t-s)^{\alpha-2}[|f_1(s,u(s),v(s),{}^cD_{0+}^pu(s),\\ &\qquad{}^cD_{0+}^qv(s))-f_1(s,0,0,0,0)|+|f_1(s,0,0,0,0)|]ds\\ &+\frac{1}{\Lambda\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}[|f_1(s,u(s),v(s),{}^cD_{0+}^pu(s),\\ &\qquad{}^cD_{0+}^qv(s))-f_1(s,0,0,0,0)|+|f_1(s,0,0,0,0)|]ds\\ &+\frac{\lambda_2}{\Lambda\Gamma(\alpha-p)}\int_0^1(1-s)^{\alpha-p-1}[|f_1(s,u(s),v(s),\\ &\qquad{}^cD_{0+}^pu(s),{}^cD_{0+}^qv(s))-f_1(s,0,0,0,0)|\end{aligned}$$

$$+|f_1(s,0,0,0,0)|]ds$$
$$\le [(n_1+n_2+n_3\frac{1}{\Gamma(2-p)}+n_4\frac{1}{\Gamma(2-q)})r+N_1]$$
$$\times(\frac{(\Lambda\alpha+1)}{\Lambda\Gamma(\alpha+1)}+\frac{\lambda_2}{\Lambda\Gamma(\alpha-p+1)}).$$

Hence

$$\|T_1(u,v)(t)\|_X$$
$$\le [(n_1+n_2+n_3\frac{1}{\Gamma(2-p)}+n_4\frac{1}{\Gamma(2-q)})r+N_1]M_1\le\frac{r}{2}.$$

In the same way, we can obtain that

$$\|T_2(u,v)(t)\|_X$$
$$\le [(n_1^*+n_2^*+n_3^*\frac{1}{\Gamma(2-p)}+n_4^*\frac{1}{\Gamma(2-q)})r+N_2]M_2\le\frac{r}{2}.$$

Consequently, $\|T(u,v)(t)\|_{X\times X}\le r$. Now for (u_2,v_2), $(u_1,v_1)\in X\times X$ and for any $t\in[0,1]$, we get

$$|T_1(u_2,v_2)(t)-T_1(u_1,v_1)(t)|$$
$$\le\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}|f_1(s,u_2(s),v_2(s),{}^cD_{0+}^pu_2(s),$$
$${}^cD_{0+}^qv_2(s))-f_1(s,u_1(s),v_1(s),$$
$${}^cD_{0+}^pu_1(s),\ {}^cD_{0+}^qv_1(s))|ds$$
$$+\frac{|\lambda_1-t|}{\Lambda\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}|f_1(s,u_2(s),v_2(s),$$
$${}^cD_{0+}^pu_2(s),{}^cD_{0+}^qv_2(s))$$
$$-f_1(s,u_1(s),v_1(s),{}^cD_{0+}^pu_1(s),{}^cD_{0+}^qv_1(s))|ds$$
$$+\frac{\lambda_2|\lambda_1-t|}{\Lambda\Gamma(\alpha-p)}\int_0^1(1-s)^{\alpha-p-1}|f_1(s,u_2(s),v_2(s),$$
$${}^cD_{0+}^pu_2(s),{}^cD_{0+}^qv_2(s))$$
$$-f_1(s,u_1(s),v_1(s),{}^cD_{0+}^pu_1(s),{}^cD_{0+}^qv_1(s))|ds$$
$$\le(\frac{(\Lambda+\lambda_1)}{\Lambda\Gamma(\alpha+1)}+\frac{\lambda_1\lambda_2}{\Lambda\Gamma(\alpha-p+1)})(n_1+n_2+n_3\frac{1}{\Gamma(2-p)}$$
$$+n_4\frac{1}{\Gamma(2-q)})(\|u_2-u_1\|+\|v_2-v_1\|),$$

$$|(T_1(u_2,v_2))'(t)-(T_1(u_1,v_1))'(t)|$$
$$\le\frac{1}{\Gamma(\alpha-1)}\int_0^t(t-s)^{\alpha-2}|f_1(s,u_2(s),v_2(s),{}^cD_{0+}^pu_2(s),$$
$${}^cD_{0+}^qv_2(s))-f_1(s,u_1(s),v_1(s),$$
$${}^cD_{0+}^pu_1(s),{}^cD_{0+}^qv_1(s))|ds$$
$$+\frac{1}{\Lambda\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}|f_1(s,u_2(s),v_2(s),$$
$${}^cD_{0+}^pu_2(s),{}^cD_{0+}^qv_2(s))$$
$$-f_1(s,u_1(s),v_1(s),{}^cD_{0+}^pu_1(s),$$
$${}^cD_{0+}^qv_1(s))|ds$$
$$+\frac{\lambda_2}{\Lambda\Gamma(\alpha-p)}\int_0^1(1-s)^{\alpha-p-1}|f_1(s,u_2(s),v_2(s),$$
$${}^cD_{0+}^pu_2(s),{}^cD_{0+}^qv_2(s))$$
$$-f_1(s,u_1(s),v_1(s),{}^cD_{0+}^pu_1(s),{}^cD_{0+}^qv_1(s))|ds$$
$$\le(\frac{(\Lambda\alpha+1)}{\Lambda\Gamma(\alpha+1)}+\frac{\lambda_2}{\Lambda\Gamma(\alpha-p+1)})$$
$$\times(n_1+n_2+n_3\frac{1}{\Gamma(2-p)}+n_4\frac{1}{\Gamma(2-q)})$$
$$\times(\|u_2-u_1\|+\|v_2-v_1\|).$$

Hence

$$\|T_1(u_2,v_2)(t)-T_1(u_1,v_1)(t)\|_X$$
$$\le M_1(n_1+n_2+n_3\frac{1}{\Gamma(2-p)}+n_4\frac{1}{\Gamma(2-q)})$$
$$\times(\|u_2-u_1\|_X+\|v_2-v_1\|_X).$$

Similarly to the above discussion, we can obtain

$$\|T_2(u_2,v_2)(t)-T_2(u_1,v_1)(t)\|_X$$
$$\le M_2(n_1^*+n_2^*+n_3^*\frac{1}{\Gamma(2-p)}+n_4^*\frac{1}{\Gamma(2-q)})$$
$$\times(\|u_2-u_1\|_X+\|v_2-v_1\|_X).$$

As a result

$$\|T(u_2,v_2)(t)-T(u_1,v_1)(t)\|_{X\times X}$$
$$\le[M_1(n_1+n_2+n_3\frac{1}{\Gamma(2-p)}+n_4\frac{1}{\Gamma(2-q)})$$
$$+M_2(n_1^*+n_2^*+n_3^*\frac{1}{\Gamma(2-p)}+n_4^*\frac{1}{\Gamma(2-q)})]$$
$$\times(\|u_2-u_1\|_X+\|v_2-v_1\|_X).$$

Since

$$[M_1(n_1+n_2+n_3\tfrac{1}{\Gamma(2-p)}+n_4\tfrac{1}{\Gamma(2-q)})$$
$$+M_2(n_1^*+n_2^*+n_3^*\tfrac{1}{\Gamma(2-p)}+n_4^*\tfrac{1}{\Gamma(2-q)})]<1$$

therefore T is a contraction operator.

Thus the conclusion of the theorem holds by using the fixed point theorem of contraction mapping principle. The proof is completed.

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