Algorithm of Iterative Process for Some Mappings and Iterative Solution of Some Diffusion Equation

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Abstract—In Hilbert spaces, through improving some corresponding conditions in some literature and extending some recent relevant results, a strong convergence theorem of some implicit iteration process for pseudocontraction mappings and explicit iteration process for nonexpansive mappings were established. And by using the result, some iterative solution for some equation of response diffusion were obtained.

Keywords—pseudocontraction mappings; nonexpansive mappings; implicit iteration process; explicit iteration process; diffusion equation.

1. Introduction

Let $E$ be Banach space, and $k$ be a nonempty closed convex subset of $E$. Suppose that $T$ is a mapping from $K$ to $K$, and $F(T)$ is a set of fixed point of $T$ with $F(T) \neq \emptyset$.

Assume that $J : E \rightarrow 2^E$ is regular dual mapping on $E$, and $J(x) = \{ f \in E^* | \langle x, f \rangle = \| f \|_1, x \in E \}$. 

As $E = H$ is Hilbert space, the internal product of $H$ is donate by the symbol $\langle \cdot, \cdot \rangle$, and the norm of $H$ is designated by symbol $\| \cdot \|$.

**Definition 1**: Mapping $T : K \rightarrow K$ is said to be pseudo contraction if for arbitrary $x, y \in K$, there exists $j(x-y) \in J(x-y)$ such that $\langle Tx - Ty, j(x-y) \rangle \leq \| x-y \|^2$.

$T$ is said to be strong pseudo contraction if there is $k \in (0, 1)$ such that $\langle Tx - Ty, j(x-y) \rangle \leq k \| x-y \|^2$ for arbitrary $x, y \in K$.

**Definition 2**: Mapping $T : K \rightarrow K$ is said to be nonexpansive if for arbitrary $x, y \in K$, there is $\| Tx - Ty \| \leq \| x-y \|$.

As we all know, that $T$ is pseudo contraction is equivalent to that for every $s > 0$ and every $x, y \in K$, there is $\| x-y \| \leq \| x-y + s(I-T)x - (I-T)y \|$ (1)

When $E = H$ is Hilbert space, $J : E \rightarrow 2^E$ is single value, and for arbitrary $x, y \in K$, there is $\langle x-y, j(x-y) \rangle = \| x-y \|^2$.

Obviously, nonexpansive mapping is pseudo contraction.

2. Lemmas and Methods

**Lemma 1**\[^{[1,2]}\] Let $E$ be a real Banach space, and $K$ be nonempty closed convex subset of $E$. Assume that $T : K \rightarrow K$ is continuous strong pseudo contraction mapping. Then $T$ is unique fixed point in $K$.

**Lemma 2**: Let $E$ be a real reflexive Banach space satisfying Opial condition, and $K$ be a nonempty closed convex subset of $E$. Suppose that $T : K \rightarrow K$ is continuous strong pseudo contraction mapping. Then for arbitrary $\{x_n\} \subset E$,

$x_n$ weakly converge to $x$, and $\|x_n - Tx_n\| \rightarrow 0$. So there is $(I-T)x = 0$.

**Lemma 3**\[^{[4]}\] Let $p > 1, r > 0$ be two certain real number, then Banach space is $(I-T)x = 0$ if and only if there is an increasing continuous function $g : [0, +\infty) \rightarrow [0, +\infty)$. $g(0) = 0$, such that

$\|x - \lambda y\| \leq \|x - y\| + (1-\lambda)g(\|x - y\|) + \lambda g(\|x - y\|)$

for all $x, y \in B_r$, where $\lambda \in [0,1]$, and $B_r$ is a closed spheroid which center is origin and radius is $r$, and $W_{fr}(\lambda) = \lambda^r(1-\lambda) + \lambda(1-\lambda)^r$.

**Lemma 4**: Let nonnegative real sequence $\{a_n\}$ satisfy the inequality:

$\| x_n - y_n \| \leq \| x_n - y_n \| + \delta_n$, $n \geq 0$, where $\gamma_n \in [0,1]$, $\gamma_n = \infty, \lim_{n \rightarrow +\infty} \gamma_n = 0$ or $\sum_{n=1}^{\infty} |\delta_n| < +\infty$, then $\lim_{n \rightarrow +\infty} a_n = 0$. 

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In Hilbert space, Moudaf\cite{6} has get strong convergence theorem of implicit iteration process of nonexpansive mapping, and $\chi\alpha\alpha\beta\alpha\alpha\beta\alpha\alpha\beta\alpha\alpha\beta\alpha$ has improved and extended some relative results in Reference \cite{7}.

In this paper, by applying a new implicit iteration sequence $x_{n}=\alpha_{n}f(x_{n})+\beta_{n}x_{n}+\gamma_{n}T_{n}$, and explicit iterative sequence $y_{n+1}=\alpha_{n}f(y_{n})+\beta_{n}y_{n}+\gamma_{n}T_{n}$, we shall consider the problem involving the fixed point of strong pseudo construction and nonexpansive mapping on closed convex set $K$. When exact conditions are satisfied, $\{x_{n}\}$ and $\{y_{n}\}$ all strongly converge to the fixed point of $T$. When the conditions for $\{\alpha_{n}\}$ and $\{\beta_{n}\}$ in Reference \cite{6},\cite{7} are widened, and as $\beta_{n}=0$, we can obtain the iterative sequence in Reference \cite{6},\cite{7}, and then we improve and extend some relative results and obtain some equation of diffusion by applying the above results.

Let $T:K \rightarrow K$ be continuous pseudo construction mapping, and $f:K \rightarrow K$ be continuous strong pseudo construction mapping with constant $\alpha$ $(0<\alpha<1)$, and suppose that $\alpha_{n}+\beta_{n}+\gamma_{n}=1$ for $\alpha_{n},\beta_{n},\gamma_{n} \in (0,1)$, and we stucture mapping $S_{n}:K \rightarrow K$, $S_{n}x=\alpha_{n}f(x)+\beta_{n}x+\gamma_{n}T_{n}$. Then $S_{n}$ is continuous strong pseudo construction mapping. By virtue of Lemma 1, $S_{n}$ has unique fixed point $x_{n}$, then we have

$$x_{n}=S_{n}x_{n}=\alpha_{n}f(x_{n})+\beta_{n}x_{n}+\gamma_{n}T_{n}$$

(2)

3. Main Results

**Theorem 1** Let $E$ be a Hilbert space, and $K$ be a nonempty closed convex subset of $E$. Assume that $f:K \rightarrow K$ is continuous strong pseudo construction mapping with constant $\alpha$ $(0<\alpha<1)$, and $f$ is bounded on bounded set, and $T:K \rightarrow K$ is continuous pseudo construction mapping. Then

(a) If $\sigma_{n}=\frac{\alpha_{n}}{1-\beta_{n}} \rightarrow 0$ or $\limsup_{n\rightarrow \infty} \sigma_{n}<1$, and there is $p \in F(T)$ such that $\|f(x_{n})-p\|^{2}-\|x_{n}-p\|^{2} \rightarrow 0$, then implicit iterative sequence (2) strongly converges to the point of $F(T)$.

(b) If $T$ is nonexpansive mapping and $f$ is construction mapping with constant $\alpha$, as $\frac{\|x_{n}-x_{n}\|}{\sigma_{n}} \rightarrow 0$ and $\sum_{n=1}^{\infty} \sigma_{n}=\infty$, the explicit iterative sequence $y_{n+1}=\alpha_{n}f(y_{n})+\beta_{n}y_{n}+\gamma_{n}T_{n}$ strongly converges to the point of $F(T)$.

**Proof.**

(a) Because $\forall p \in F(T)$,

$$\|x_{n}-p\|^{2}$$

$$=\langle \alpha_{n}(f(x_{n})-p)+\beta_{n}(x_{n}-p)+\gamma_{n}(T_{n}-p),x_{n}-p \rangle$$

$$\leq \alpha_{n}\|f(x_{n})-p\|^{2}+\beta_{n}\|x_{n}-p\|^{2}+\gamma_{n}\|T_{n}-p\|^{2}+\gamma_{n}\|x_{n}-p\|^{2}$$

we have

$$\|x_{n}-p\|^{2} \leq \frac{\alpha_{n}}{1-\alpha_{n}-\beta_{n}-\gamma_{n}}\|f(x_{n})-p\|^{2}+\frac{\alpha_{n}}{1-\alpha_{n}}\|f(x_{n})-p\|^{2}. $$

Hence $\{x_{n}\}, \{f(x_{n})\}, \{T_{n}\}$ are bounded.

If $\sigma_{n}=\frac{\alpha_{n}}{1-\beta_{n}} \rightarrow 0$, then using formula (2), we can write $x_{n}=\sigma_{n}f(x_{n})+(1-\sigma_{n})T_{n}$, and then we obtain

$$\|x_{n}-T_{n}\| = \sigma_{n}\|f(x_{n})-T_{n}\| \rightarrow 0.$$ (3)

If $\limsup_{n\rightarrow \infty} \sigma_{n}<1$ and there is $p \in F(T)$ such that

$$\|f(x_{n})-p\|^{2}-\|x_{n}-p\|^{2} \rightarrow 0,$$

then by virtue of formula (1) and Lemma 3, we obtain

$$\|x_{n}-p\|^{2}$$

$$\leq \|x_{n}-p+\frac{1-\sigma_{n}}{2\sigma_{n}}(x_{n}-T_{n})\|^{2}$$

$$=\|x_{n}-p+\frac{1-\sigma_{n}}{2\sigma_{n}}(f(x_{n})-T_{n})\|^{2}$$

$$=\|\frac{1}{2}(f(x_{n})-p)+\frac{1}{2}(x_{n}-p)\|^{2}$$

$$\leq \frac{1}{2}\|f(x_{n})-p\|^{2}+\frac{1}{2}\|x_{n}-p\|^{2}-\frac{1}{4}g(\|x_{n}-f(x_{n})\|),$$

and then

$$\frac{1}{2}g(\|x_{n}-f(x_{n})\|) \leq \|f(x_{n})-p\|^{2}-\|x_{n}-p\|^{2} \rightarrow 0.$$ (4)

So we have $\|x_{n}-f(x_{n})\| \rightarrow 0$.

Whereas

$$\|x_{n}-T_{n}\| = \sigma_{n}\|f(x_{n})-T_{n}\| = \frac{\sigma_{n}}{1-\sigma_{n}}\|x_{n}-f(x_{n})\| \rightarrow 0.$$ (4)

Because $\{x_{n}\}$ is bounded, and $E$ is Hilbert space, we have that $x_{n}$ weakly converge to $q \in K$. By virtue of formula...
(3) or (4) and Lemma 2, we have \( q \in F(T) \).

Because
\[
\|x_n - q\|^2 = <\sigma_n(f(x_n) - q) + (1 - \sigma_n)(Tx_n - q), x_n - q > \\
\leq \alpha \sigma_n \|x_n - q\|^2 + \sigma_n <f(q) - q, x_n - q > + (1 - \sigma_n) \|x_n - q\|^2,
\]
we obtain \( \|x_n - q\|^2 \leq \frac{1}{1 - \alpha} <f(q) - q, x_n - q >. \) Since \( x_n \) weakly converges to \( q \), \( x_n \) strongly converges to \( q \in F(T) \).

(b) Because
\[
\|x_n - x_{n+1}\| = \|\sigma_n f(x_n) - \sigma_n f(x_{n+1}) + (1 - \sigma_n)Tx_n - (1 - \sigma_n)Tx_{n+1}\|
\leq \alpha \sigma_n \|x_n - x_{n+1}\| + \|f(x_n) - f(x_{n+1})\| + (1 - \sigma_n) \|x_n - x_{n+1}\|
\leq \alpha \sigma_n \|x_n - x_{n+1}\| + \|f(x_n) - f(x_{n+1})\| + (1 - \alpha \sigma_n) \|x_n - x_{n+1}\|\]
we obtain
\[
\|x_n - x_{n+1}\| \leq \frac{\alpha \sigma_n}{1 - \alpha \sigma_n} \|x_n - x_{n+1}\| \leq \frac{M}{2}, \text{ where } \|f(x_{n+1})\| \leq \frac{M}{2}, \text{ and } \|Tx_n\| \leq \frac{M}{2}.
\]

Hence we have \( \|y_{n+1} - q\| \leq \|y_n - q\| + \|x_n - q\| \rightarrow 0 \), which means that \( \{y_n\} \) strongly converges to \( q \in F(T) \).

Note.

Theorem 1 improves and extends some relative results in Reference [6] and [7].

As follows, we will discuss iterative solution of some response diffusion equation.

Let \( E = L^2(I) = \{ x(t, s) \mid (t, s) \in I, x(t, s) \text{ is Lebesgue intergrable on } I \}, I = [a, b] \times [c, d] \), and \( \forall x, y \in E \), we define \( \langle x, y > = \int_I x(t, s) y(t, s) dt ds \). Then \( E \) is Hilbert space, and \( \|x\|^2 = \langle x, x > = \int_I x^2(t, s) dt ds \), \( \langle y, f(x) > = \langle y, x >, \forall x, y \in E \).

Consider the problem involving solution of some first order diffusion equation:

\[
\frac{\partial x}{\partial t} = -u_0 \frac{\partial x}{\partial s} - Gx - hx, \quad x(s, 0) = x_n(s), x(0, t) = x_1(t),
\]

where \( G \) is continuous mapping on \( E \), and \( u_0 \geq 0 \) is constant, and \( h = h(t, s) \geq 0 \).

This problem is equivalent to the integral equation as follows:
\[
\int_0^s x(t, s) dt + \int_0^t x(t, s) h(t, s) dt ds + \int_0^t x(t, s) Gx dt ds
\]

where \( G \) is continuous mapping on \( E \), and then \( K \) is nonempty closed convex subset of \( E \).

Let \( H : K \rightarrow K \).

If \( G \) satisfies (A): \( \forall x, y \in K, xGx \leq yGy \), then let
\[
T : K \rightarrow K, Tx = -hx + x.
\]

If \( G \) satisfies (B): there is \( L_1 > 0 \) such that \( \|xGx - yGy\| \leq L_1 \|x - y\| \) for arbitrary \( x, y \in K \). Then \( H \) is Lipschitz mapping on \( K \), and then we have \( L \) such that \( \forall x, y \in K, \|Hx - Hy\| \leq L \|x - y\| \).

Let \( H_1 = \frac{2}{L}, T_1 : K \rightarrow K, T_1 x = -H_1 x + x \).

Theorem 2 Let integral equation (7) has solution, then

(i) If \( G \) satisfies (A), when \( \alpha_n = \frac{\alpha_n}{1 - \beta_n} \rightarrow 0 \) or \( \limsup_{n \rightarrow \infty} \sigma_n < 1 \), and there is \( \rho \in F(T) \) such that \( \|f(x_n) - p\|^2 \rightarrow 0 \). Implicit iterative sequence \( x_n = \alpha_n f(x_n) + \beta_n x_n + \gamma_n Tx_n \) strongly converges to the fixed point of \( T \) which is solution of equation (7).

(ii) If \( G \) satisfies (B), when \( \sum_{n=0}^{\infty} \alpha_n = \frac{\sigma_n - \sigma_n}{\alpha_n} \rightarrow 0 \), explicit iterative sequence \( y_{n+1} = \alpha_n f(y_n) + \beta_n y_n + \gamma_n Ty_n \) strongly converges to the fixed point of \( T \) which is solution of equation (7).

Proof.

(i) Now, \( \forall x, y \in K, (Hx - Hy)(x - y) \) is nonnegative on
Then we have \( <Hx-Hy,x-y> \geq 0 \), that is said that \( T \) is pseudo contraction mapping on \( K \). Using Theorem 1, we obtain the result.

(ii) Now, \( \forall x,y \in K, \|H_1x-H_1y\| \leq 2\|x-y\| \) (8)

\[
(T_nx-T_ny)^2 \\
= [-H_x(x+(-H_y+y))]^2 \\
= (H_x+y-H_x)x)^2 - 2\|x-y\|H_x+H_yx + (x-y)^2 \\
= (H_x+y-H_x)[H_x+y-H_xx - 2\|y-x\|] + (x-y)^2 \\
\]

If \( \|H_1y-H_1x\| \leq 2\|y-x\| \), then we obtain

\[
(T_nx-T_ny)^2 \leq (x-y)^2 \\
\]

If \( \|H_1y-H_1x\| \geq 2\|y-x\| \leq 0 \), then we obtain

\[
(T_nx-T_ny)^2 \leq (H_1y-H_1x)^2 - 2\|y-x\|H_1y-H_1x - 2\|y-x\| + (x-y)^2 \\
= (H_1y-H_1x)^2 - 4\|y-x\|^2 + (x-y)^2 \\
\]

Hence, by virtue of formula (8), we have

\[
\|T_nx-T_ny\|^2 \leq \|x-y\|^2 \\
\]

That is said that \( T_n \) is nonexpansive mapping on \( K \). Using Theorem 1, we obtain the result.

Therefore, through improving some corresponding conditions in literature [6],[7], and extending some recent relevant results, Theorem 1 was established. Theorem 1 is a strong convergence theorem of some implicit iteration process for pseudocontraction mappings and explicit iteration process for nonexpansive mappings. By applying Theorem 1, the iterative solution for some equation of response diffusion was obtained, Theorem 2 was established.

**REFERENCES**


