Statistical Theory of Turbulence  
by the Late Lamented Dr. shunichi Tsugé  

Case Study on Flow through a Grid in Wind Tunnel  

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Abstract—This paper is concerned with statistical theory of turbulence by the late lamented Dr. Shunichi Tsugé. The theory has been applied to the primary flow through a grid fixed vertically with respect to the horizontal axis of the wind tunnel. The first analytical solution has been obtained and explained the well-known “the inverse-linear decay law” of the turbulent intensity. It is believed that the present result is the first exact solution in the theory of turbulence.

Keywords—grid-produced turbulence;exact solution;turbulent intensity;statistical theory of turbulence;applied mathematics  

1. Introduction  
Contrary to the firm kinetic theoretical basis of the Navier-Stokes equation for laminar flows, which verification dates back to Chapman[1], Enskog[2] and Grad[3], the history of its turbulent counterpart starts in late 1960s: It is known that two pioneering workers attempt to two-particle version of the Euler equation(Zhigulev[4]) and of the Navier-Stokes equation(Tsugé[5]), which are governing turbulent correlations for inviscid and viscous gases, respectively. It is notable that these two papers have proposed a rather striking thesis, completely contradicting to the conventional belief that “the kinetic theory is useless for turbulence, because it is merely concerned with molecular fluctuations having order of n(the mean number density), and thus are negligibly small compared with macroscopic turbulent fluctuations of order of n²”. In fact, human sensors are unable to perceive molecular fluctuations, consisting of thermal agitation such as molecular stress and heat-flux fluctuations, as discussed by Landau & Lifshitz[6], together with fluctuations due to real-gas effects.

An independent innovative hypothesis is proposed by Tsugé[7] and Grad[8] has made us possible to incorporate macroscopic turbulent fluctuations into the regime of the kinetic theory. That is, the two-particle molecular chaos due to Boltzmann is replaced with a less stringent tertiary molecular chaos. This milder hypothesis leads us to a new finding that in a shear flow, turbulent correlations are survived over thousand mean free paths, or a macroscopic fluid-dynamic length, being detectable with any flow device used currently.

In 1974, it is shown by Tsugé[7] that the equations governing two-point correlations in an incompressible shear flow are separable into two Orr-Sommerfeld type equations at the respective points. It is, however, realized that physical meaning of the variables in the equations are much different from those in the Orr-Sommerfeld equation. In the same paper, Tsugé[7] has proved that the fluid moments obtained from the one-particle kinetic equation are equivalent with the Navier-Stokes equation(Nakagawa[9]), and the two-particle version, the equations governing two-point correlations, reduces to the Kármán-Howarth equation.

The main purpose of the present paper is to obtain an exact solution for the flow through a grid in the wind tunnel based on the statistical theory of turbulence by Tsugé[7].

2. Equations Governing Boltzmann Function f and Double Correlation function g  

In order for Boltzmann function f and double correlation function g to be identified by using variables in the BBGKY-hierarchy theory, after Bogoliubov, Born, Green, Kirkwood and Yvon, the following condition is required, for the averaging time τ must be longer than the time τ_g for satisfying the ergodicity:

\[ \tau > \tau_g \]  

(1)
With the assumption (1), the dependent variables \( f, g \) may be described by the general framework of the hierarchy equations \[ \text{Grad} [10]: \]

\[
\left( \frac{\partial}{\partial t} + u \cdot \frac{\partial}{\partial y} \right) e = J(a | \bar{a}) \left[ e \bar{e} + g(a, \bar{a}) \right], \tag{2}
\]

\[
\left( \frac{\partial}{\partial t} + u \cdot \frac{\partial}{\partial y} + \bar{u} \cdot \frac{\partial}{\partial \bar{y}} \right) g(a, \bar{a}) = J(a \mid \bar{a}) \left[ e g(e, \bar{e}) + e g(e, \bar{e}) \right] + J(\bar{a} \mid \bar{a}) \left[ e g(e, \bar{e}) + e g(e, \bar{e}) \right], \tag{3}
\]

where tertiary molecular chaos,

\[
h(e, \bar{e}, \bar{e}) = \frac{\Delta e \Delta \bar{e} \Delta \bar{e} > 0}{0}, \tag{4}
\]

has been adopted to truncate the hierarchy system. It may be worth noting here that if one put \( g(a, \bar{a}) = \Delta e \Delta \bar{e} \Delta \bar{e} = 0 \), binary molecular chaos, the above hierarchy system reduces to the Boltzmann equation:

\[
\left( \frac{\partial}{\partial t} + u \cdot \frac{\partial}{\partial y} \right) e = J(a \mid \bar{a}) \left[ e \bar{e} \right].
\]

3. Flow Through a Grid in Wind Tunnel

It may be evident that in the flow through a grid fixed normal to the main flow direction, turbulence is generated and then it decays with increasing the distance from the grid, by experiencing the diffusion as well as viscous dissipation mainly. This turbulence is the topic to obtain the exact analytical solution.

The grid-produced turbulence is neither homogeneous nor isotropic, but an isotropic, for there exists a specific vector of the main flow direction (Fig.1).

Let us assume the two-point correlation is separable in the form,

\[
R_{x_2, x_3}(y, \bar{y}, t) = \mathcal{R} \left[ \int o_{x_2}(y, t ; \omega) o_{x_3}(\bar{y}, t ; \omega) d\omega \right], \tag{5}
\]

with

\[
o = \sigma^* (\sigma; \text{conjugate complex}), \tag{6}
\]

where \( \sigma \) is the constant separating the variables, \( \mathcal{R} \) denotes taking real part.

Then, equations governing \( o_{x_2} \) reduces to a set of integro-differential equations in the separated 3-dimensional space as

\[
\frac{\partial o_{x_2}}{\partial y} = 0, \tag{7}
\]

\[
(-i \omega + \frac{\partial}{\partial y} + u_j \frac{\partial}{\partial x_j} + \frac{\partial}{\partial x_j} \frac{\partial}{\partial y} + \frac{\partial^2}{\partial x_j^2} o_j + \frac{\partial}{\partial y} \frac{\partial}{\partial y} - \frac{\partial^2}{\partial x_j^2}) o_j + \frac{\partial}{\partial y} \int_0^\infty o_j(Q) dQ = 0, \tag{8}
\]

These (7) and (8) have been solved for the grid-produced turbulence in the wind tunnel flow \( u = (0, 0, 0) \) with initial fluctuations given at the plane \( x_1 = 0 \).

Instead of solving the complete boundary value problem, the first analytical solution associated with the present theory is sought to explain the existing experimental finding, viz., “the inverse-linear decay law” of the turbulent intensity.

A. Formulation of the Problem

Let the grid-produced turbulence be composed of a plane non-dispersive wave in the form,

\[
o_{x_2} = Q_{x_2}(x_1, \Omega) \exp \left[ i \Omega (\beta_2 x_2 + \beta_3 x_3) \right] , (\alpha = 1, 2, 3, 4)
\]

with

\[
\beta_2 = k \cdot \cos \theta, \quad \beta_3 = k \cdot \sin \theta,
\]

where \( \theta \) is the azimuth angle of the oblique wave plane normal to the mean flow direction (Fig.2).
Let, then, $Q_a$ make a Fourier transform into $F_a$ in order to eliminate the nonlinear convolution integral of (8),

$$Q_a(x_1, \Omega) = 1/(2\pi) \int_{-\infty}^\infty F_a(x_1, s) \exp(-is\Omega) \, ds. \quad (10)$$

Note inverse Fourier transform $F_a(x_1, s)$ is defined as

$$F_a(x_1, s) = \int_{-\infty}^\infty Q_a(x_1, \Omega) \exp(is\Omega) \, d\Omega,$$

where $Q_a(x_1, \Omega)$ is an infinitely differentiable function of bounded support.

Substituting (9), together with (10) into (7) and (8), we have

$$\partial F_i/\partial x_1 + \partial F_i/\partial s + \beta_2 \cdot \partial F_i/\partial s = 0, \quad (11)$$

$$L(F_i) + 1/\rho \cdot \partial F_i/\partial x_1 + NL(F_i) = 0, \quad (12)$$

$$L(F_i) + 1/\rho \cdot \beta_2 \cdot \partial F_i/\partial s + NL(F_i) = 0, \quad (13)$$

$$L(F_i) + 1/\rho \cdot \beta_3 \cdot \partial F_i/\partial s + NL(F_i) = 0, \quad (14)$$

with

$$L = U \partial / \partial x_1 - v \Delta, \quad (15)$$

$$\Delta = \partial^2 / \partial x_1^2 + (\beta_2^2 + \beta_3^2) \cdot \partial^2 / \partial s^2, \quad (16)$$

$$NL = F_1 \cdot \partial / \partial x_1 + (\beta_2 F_2 + \beta_3 F_3) \cdot \partial / \partial s. \quad (17)$$

Then, the following non-dimensional expressions are introduced into (11)-14,

$$\xi = x_1/M, \eta = x_3/M, f_1 = F_1/U, f_2 = F_2/U, f_3 = F_3/U, f_4 = f_4/\rho \Omega^2, R_s = UM/\nu, \quad (18)$$

we obtain

$$\partial f_i/\partial \xi + \partial f_i/\partial \eta = 0, \quad (19)$$

$$J(f_i) + \partial f_i/\partial \xi + H(f_i) = 0, \quad (20)$$

$$J(f) + \partial f/\partial \eta + H(f) = 0, \quad (21)$$

$$J(g) + H(g) = 0, \quad (22)$$

$$f = \beta_2 f_2 + \beta_3 f_3, \quad (23)$$

$$g = \beta_3 f_2 - \beta_2 f_3. \quad (24)$$

or, inversely

$$f_5 = (f \cdot \cos \theta + g \cdot \sin \theta) / k, \quad (25)$$

$$f_7 = (f \cdot \sin \theta - g \cdot \cos \theta) / k, \quad (26)$$

$$J = \partial / \partial \xi - 1/R_s \cdot \Delta, \quad (27)$$

$$\Delta = \partial^2 / \partial \xi^2 + k^2 \cdot \partial^2 / \partial \eta^2, \quad (28)$$

$$H = f_1 \cdot \partial / \partial \xi + f_4 \cdot \partial / \partial \eta. \quad (29)$$

The turbulent correlations defined by (5) may be also expressed in terms of the Fourier transformed dependent variables as follows,

$$R_{\alpha \beta}(\gamma, \delta, t) = \frac{U^2}{(2\pi)^2} \int_{-\infty}^{\infty} \text{f}_\gamma \text{f}_\delta \, d\theta, (\text{j}, \text{i}: 1, 2, 3). \quad (30)$$

It may be justified that the grid-produced turbulence is axis-symmetric with respect to the mean flow direction, namely, homogeneous in any plane normal to its direction. Such a turbulent flow is, therefore, described by superimposing the plane waves considered here, and by averaging over the angle $\theta$ within $(x_3, x_3)$;

$$R_{\alpha \beta}(\gamma, \delta, t) = \frac{U^2}{(2\pi)^2} \int_{0}^{2\pi} \text{f}_j \text{f}_j \, d\theta, (\text{j}, \text{i}: 1, 2, 3). \quad (31)$$

$$r = [(x_3 - x_3)^2 + (x_3 - x_3)^2]^{1/2},$$

where $\alpha$ and $\phi$ are angles defined in Fig.2.

**B. The Exact Solution of Grid-produced Turbulence**

It may be straightforward that (19)-(22) are two-dimensional, so that it may be possible to introduce a stream function $\Psi$ in the form,

$$f_i = \partial \Psi / \partial \eta, f_4 = \partial \Psi / \partial \xi.$$

Then, combining (20), and (21) in order to eliminate $f_4$, we have

$$(J + \Psi \cdot \partial / \partial \xi - \Psi \cdot \partial / \partial \eta) \Delta \Psi = 0. \quad (31)$$

This suggests that any harmonic function for $\Psi$, namely, solution of the Laplace differential equation, $\Delta \Psi = 0$, turns out to be an exact solution
of the above full-nonlinear equation (31). A particular solution, which is no more than a version of the general solutions, the relevant integral constants being specified by the boundary conditions at \( \eta = \mp \infty \), and \( \xi = \infty \), and whose components \( (\Psi_\xi, \Psi_\eta) \) exhibiting the decay law, may be expressed by

\[ \Psi = \arctan \left( \frac{\eta}{k \xi} \right). \]  

(32)

It is easy to verify that by substituting (32) into the Laplace differential equation

\[ \Delta \Psi = (\partial^2 / \partial \xi^2 + k^2 \cdot \partial^2 / \partial \eta^2) \Psi = 0, \]

\( \Psi \) is the solution. Moreover, substitution of (32) into (31) results in the following relations,

\[ f_1 = \Psi_\xi = A \left( -\frac{\eta}{k \xi} \right) \left( 1 + \left( \frac{\eta}{k \xi} \right)^2 \right)^{-1}, \]  

(33)

and

\[ f = -\Psi_\eta = A \eta \left( \frac{1}{k \xi^2} \right) \left( 1 + \left( \frac{\eta}{k \xi} \right)^2 \right)^{-1}. \]  

(34)

The turbulent intensity in the \( \xi \)-direction, which is the non-dimensional longitudinal coordinate of \( x_1/M \), can be calculated by substituting \( f_1 \) in (33) into (30), and integrating it with respect to \( \eta \) and \( \theta \), and results in

\[ \langle (\Delta u_1)^2 \rangle / U^2 = \frac{\Lambda}{(4k \xi)}, \]  

(35)

or

\[ U^2 / \langle (\Delta u_1)^2 \rangle = 4k \xi / \Lambda^2. \]  

(36)

4. Conclussion

The present result (36) shows that the inverse of the mean squared fluctuation of the turbulent velocity component in the \( x_1 \) mean flow direction is proportional to the normalized coordinate of \( \xi \). In Fig.3 are compared the predicted inverse decay rate of the turbulent velocity in the mean flow direction with the classical data taken by Batchelor & Townsend[11], who have confirmed experimentally the turbulent energy decay maintains similarity irrespective of the difference in the Reynolds number \( R_e = UM/\nu \).

It is believed that (36) is the first exact solution in statistical theory of turbulence, so it has a permanent value.

REFERENCES


Fig.3 Similarity of energy decay at different Reynolds numbers (after Batchelor & Townsend[11]).
$X: M=0.635$, ● $M=1.27$ cm, $+: M=2.54$ cm, ○ $M=5.08$ cm

− : present theory,
$U=$ longitudinal velocity $=1286$ cm/s,
$u_1 =$ longitudinal velocity fluctuation,
$x =$ longitudinal coordinate,
$M =$ grid mesh size.