Statistical Theory of Turbulence by the Late Lamented Dr. shunichi Tsugé

Case Study on Flow through a Grid in Wind Tunnel

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***Abstract*—This paper is concerned with statistical theory of turbulence by the late lamented Dr. Shunichi Tsugé. The theory has been applied to the primary flow through a grid fixed vertically with respect to the horizontal axis of the wind tunnel. The first analytical solution has been obtained and explained the well-known "the inverse-linear decay law" of the turbulent intensity. It is believed that the present result is the first exact solution in the theory of turbulence.**

Keywords-grid-produced turbulence;exact solution;turbulent intensity;statistical theory of turbulence;applied mathematics

1. Introduction

Contrary to the firm kinetic theoretical basis of the Navier-Stokes equation for laminar flows, which verification dates back to Chapman[1], Enskog[2] and Grad[3], the history of its turbulent counterpart starts in late 1960s: It is known that two pioneering workers attempt to two-particle version of the Euler equation(Zhigulev[4]) and of the Navier-Stokes equation(Tsugé[5]), which are governing turbulent correlations for inviscid and viscid gases, respectively. It is notable that these two papers have proposed a rather striking thesis, completely contradicting to the conventional belief that "the kinetic theory is useless for turbulence, because it is merely concerned with molecular fluctuations having order of n(the mean number density), and thus are negligibly small compared with macroscopic turbulent fluctuations of order of n^2". In fact, human sensors are unable to perceive molecular fluctuations, consisting of thermal agitation such as molecular stress and heat-flux fluctuations, as discussed by Landau & Lifshitz[6], together with fluctuations due to real-gas effects.

An independent innovative hypothesis is proposed by Tsugé[7] and Grad[8] has made us possible to incorporate macroscopic turbulent fluctuations into the regime of the kinetic theory. That is, the two-particle molecular chaos due to Boltzman is replaced with a less stringent tertiary molecular chaos. This milder hypothesis leads us to a new finding that in a shear flow, turbulent correlations are survived over thousand mean free paths, or a macroscopic fluid-dynamic length, being detectable with any flow device used currently.

In 1974, it is shown by Tsugé [7]that the equations governing two-point correlations in an incompressible shear flow are separable into two Orr-Sommerfeld type equations at the respective points. It is, however, realized that physical meaning s of the variables in the equations are much different from those in the Orr-Sommerfeld equation. In the same paper, Tsugé[7] has proved that the fluid moments obtained from the one-particle kinetic equation are equivalent with the Navier-Stokes equation(Nakagawa [9]), and the two-particle version, the equations governing two-point correlations, reduces to the Kārman-Howarth equation.

The main purpose of the present paper is to obtain an exact solution for the flow through a grid in the wind tunnel based on the statistical theory of turbulence by Tsugé[7].

2. Equations Governing Boltzmann Function f and Double Correlation function g

In order for Boltzmann function f and double correlation function g to be identified by using variables in the BBGKY-hierarchy theory, after Bogoliubov, Born, Green, Kirkwood and Yvon, the following condition is required, for the averaging time τ must be longer than the time τ_g for satisfying the ergodicity;

$$\tau > \tau_g \qquad (1)$$

With the assumption (1), the dependent variables (f,g) may be described by the general framework of the hierarchy equations(Grad [10]):

$$(\partial/\partial t+u\cdot\partial/\partial y)e=J(a|\bar{a})[e\bar{e}+g(a,\bar{a})], \tag{2}$$

$$(\partial/\partial t+u\cdot\partial/\partial y+\bar{u}\cdot\partial/\partial\bar{y})g(a,\bar{a})=$$
$$J(a|\hat{a})[eg(\hat{a},\bar{a})+\hat{e}g(a,\bar{a})]+J(\bar{a}|\hat{a})[\bar{e}g(\hat{e},e)+\hat{e}g(\bar{e},e)], \tag{3}$$

where tertiary molecular chaos,

$$h(e,\bar{e},\hat{e})\equiv\langle\triangle e\triangle\bar{e}\triangle\hat{e}\rangle=0, \tag{4}$$

has been adopted to truncate the hierarchy system. It may be worth noting here that if one put $g(a,\bar{a})=\langle\triangle e\triangle\bar{e}\rangle=0$, binary molecular chaos, the above hierarchy system reduces to the Boltzman equation:

$$(\partial/\partial t+u\cdot\partial/\partial y)e=J(a|\bar{a})[e\bar{e}].$$

3. Flow Through a Grid in Wind Tunnel

It may be evident that in the flow through a grid fixed normal to the main flow direction , turbulence is generated and then it decays with increasing the distance from the grid, by experiencing the diffusion as well as viscous dissipation mainly. This turbulence is the topic to obtain the exact analytical solution.

The grid-produced turbulence is neither homogeneous nor isotropic, but an isotropic, for there exists a specific vector of the main flow direction(Fig.1).

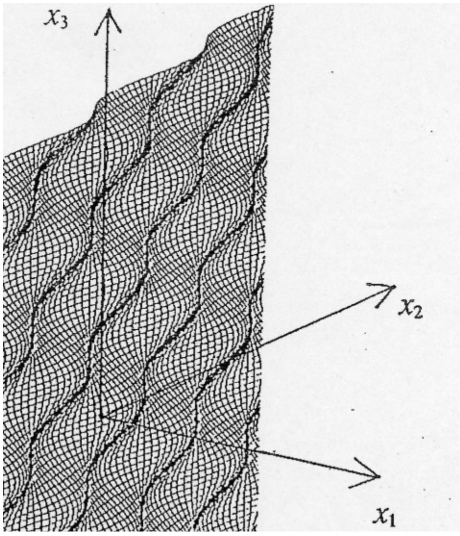

Fig.1 Cross section of two-dimensional plane waves in (x_2,x_3)-plane. x_1: primary flow direction.

Let us assume the two-point correlation is separable in the form,

$$R_{\alpha\beta}(y,\bar{y},t)=\check{R}[\int o_\alpha(y,t:\omega)\bar{o}_{\check{R}}(\bar{y},t:\omega)d\omega], \tag{5}$$

with

$$\bar{o}=o^*(*;\text{conjugate complex}), \tag{6}$$

where ω is the constant separating the variables, $\check{R}$[ ]denotes taking real part.

Then, equations governing o_α reduces to a set of integro-differential equations in the separated 3-dimensional space as

$$\partial o_r/\partial y_r=0, \tag{7}$$

$$(-i\omega+\partial/\partial t+u_r\cdot\partial/\partial y_r-\nu\cdot\partial^2/\partial y_r^2)o_j+\partial u_j/\partial y_r\cdot o_r+1/\rho\cdot\partial o_4/\partial y_j+\partial/\partial y_r\int_{-\infty}^{\infty}o_r(Q)o_j(\Omega-Q)dQ=0,$$
$$(j,r=1,2,3). \tag{8}$$

These (7) and (8) have been solved for the grid-produced turbulence in the wind tunnel flow $u=(U,0,0)$ with initial fluctuations given at the plane; $x_1=0$.

Instead of solving the complete boundary value problem, the first analytical solution associated with the present theory is sought to explain the existing experimental finding, viz., "the inverse-linear decay law" of the turbulent intensity.

A. Formulation of the Problem

Let the grid-produced turbulence be composed of a plane non-dispersive wave in the form,

$$o_\alpha=Q_\alpha(x_1,\Omega)\exp[i\Omega(\beta_2x_2+\beta_3x_3)],\ (\alpha=1,2,3,4) \tag{9}$$

with

$$\beta_2=k\cdot\cos\theta,\quad \beta_3=k\cdot\sin\theta,$$

where θ is the azimuth angle of the oblique wave plane normal to the mean flow direction(Fig.2).

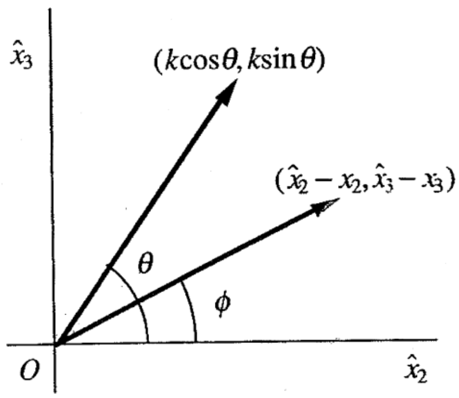

Fig.2 Definition sketch of angles θ and ϕ

Let, then, Q_α make a Fourier transform into F_α in order to eliminate the nonlinear convolution integral of (8),

$$Q_\alpha(x_1, \Omega)=1/(2\pi) \int^{\infty}_{-\infty} F_\alpha(x_1, s)\exp(-i\Omega s))ds. \quad (10)$$

Note inverse Fourier transform $F_\alpha(x_1, s)$ is defined as

$$F_\alpha(x_1, s)= \int^{\infty}_{-\infty} Q_\alpha(x_1, \Omega)\exp(is\Omega))d\Omega,$$

where $Q_\alpha(x_1, \Omega)$ is an infinitely differentiable function of bounded support.

Substituting (9), together with (10) into (7) and (8), we have

$$\partial F_1/\partial x_1+ \cdot\partial F_2/\partial s+\beta_3\cdot\partial F_3/\partial s=0, \quad (11)$$

$$L(F_1)+1/\rho\cdot\partial F_4/\partial x_1+NL(F_1)=0, \quad (12)$$

$$L(F_2)+1/\rho\cdot\beta_2\partial F_4/\partial s+NL(F_2)=0, \quad (13)$$

$$L(F_3)+1/\rho\cdot\beta_3\partial F_4/\partial s+NL(F_3)=0, \quad (14)$$

with

$$L=U\partial/\partial x_1 - \nu\Delta, \quad (15)$$

$$\Delta=\partial^2/\partial x_1^2+(\beta_2^2+\beta_3^2)\cdot \partial^2/\partial s^2, \quad (16)$$

$$NL= F_1\cdot\partial/\partial x_1+(\beta_2F_2+\beta_3F_3)\cdot \partial/\partial s. \quad (17)$$

Then, the following non-dimensional expressions are introduced into (11)-14),

$$\xi = x_1/M, \ \eta =s/M, \ f_1= F_1/U, \ f_2= F_2/U, \ f_3= F_3/U, \ f_4= F_4/(\rho U^2), \ R_e=UM/\nu, \quad (18)$$

we obtain

$$\partial f_1/\partial \xi+\partial f/\partial \eta=0, \quad (19)$$
$$J(f_1)+ \partial f_4/\partial \xi+H(f_1)=0, \quad (20)$$

$$J(f)+ \partial f_4/\partial \eta+H(f)=0, \quad (21)$$

$$J(g)+H(g)=0, \quad (22)$$

$$f=\beta_2f_2+\beta_3f_3, \quad (23)$$

$$g=\beta_3f_2-\beta_2f_3, \quad (24)$$

or, inversely

$$f_2=(f\cdot\cos\theta+g\cdot\sin\theta)/k, \quad (25)$$

$$f_3=(f\cdot\sin\theta-g\cdot\cos\theta)/k, \quad (26)$$

$$J=\partial/\partial\xi-1/R_e\cdot\Delta, \quad (27)$$

$$\Delta=\partial^2/\partial\xi^2+k^2\cdot \partial^2/\partial\eta^2, \quad (28)$$

$$H= f_1\cdot\partial/\partial\xi+f\cdot\partial/\partial\eta. \quad (29)$$

The turbulent correlations defined by (5) may be also expressed in terms of the Fourier transformed dependent variables as follows,

$$R_{\alpha\beta}(y, \bar{y}, t)= U^2/(2\pi)^2\int_0^{2\pi}\left[\int_{-\infty}^{\infty}f_jf_id\eta\right]d\theta, (j, I;1, 2, 3). \quad (30)$$

It may be justified that the grid-produced turbulence is axis-symmetric with respect to the mean flow direction, namely, homogeneous in any plane normal to its direction. Such a turbulent flow is, therefore, described by superimposing the plane waves considered here, and by averaging over the angle θ within (x_2, x_3);

$$R_{\alpha\beta}(y, \bar{y}, t)=U^2/(2\pi)^2\int_0^{2\pi} \left[\!\left[\int_{-\infty}^{\infty}f_j(\xi, \eta)f_i\left[\bar{\xi}, \ \eta+kr\cdot\cos(\theta-\phi)\right]d\eta\right]\!\right]d\theta,$$

with

$$r=\left[(\bar{x}_2-x_2)^2+(\bar{x}_3-x_3)^2\right]^{1/2},$$

where θ and ϕ are angles defined in Fig.2

B. The Exact Solution of Grid-produced Turbulence

It may be straight forward that (19)-(22) are two-dimensional, so that it may be possible to introduce a stream function Ψ in the form,

$$f_1=\partial\Psi/\partial\eta, f=-\partial\Psi/\partial\xi.$$

Then, combining (20), and (21) in order to eliminate f_4, we have

$$(J+\Psi_\eta\cdot\partial/\partial\xi-\Psi_\xi\cdot\partial/\partial\eta)\Delta\Psi=0. \quad (31)$$

This suggests that any harmonic function for Ψ, namely, solution of the Laplace differential equation, $\Delta\Psi=0$, turns out to be an exact solution of the above full-nonlinear equation (31). A particular solution, which is no more than a version of the general solutions, the relevant integral constants being specified by the boundary conditions at $\eta = \mp\infty$, and $\xi = \infty$, and whose components(Ψ_ξ, Ψ_η) exhibiting the decay law, may be expressed by

$$\Psi = \Lambda \arctan(\eta / k\xi). \tag{32}$$

It is easy to verify that by substituting (32) into the Laplace differential equation

$$\Delta\Psi = (\partial^2/\partial\xi^2 + k^2 \cdot \partial^2/\partial\eta^2)\Psi = 0,$$

Ψ is the solution. Moreover, substitution of (32) into (31) results in the following relations,

$$f_1 = \Psi_\eta = \Lambda/(k\xi)\,[1+(\eta/k\xi)^2]^{-1}, \tag{33}$$

and

$$f = -\Psi_\xi = \Lambda\eta/(k\xi^2)\,[1+(\eta/k\xi)^2]^{-1}. \tag{34}$$

The turbulent intensity in the ξ-direction, which is the non-dimensional longitudinal coordinate of x_1/M, can be calculated by substituting f_1 in (33) into (30), and integrating it with respect to η and θ, and results in

$$\langle(\Delta u_1)^2\rangle/U^2 = \Lambda^2/(4k\xi), \tag{35}$$,

or

$$U^2/\langle(\Delta u_1)^2\rangle = 4k\xi/\Lambda^2. \tag{36}$$

4. Conculsion

The present result (36) shows that the inverse of the mean squared fluctuation of the turbulent velocity component in the x_1 mean flow direction is proportional to the normalized coordinate of ξ. In Fig.3 are compared the predicted inverse decay rate of the turbulent velocity in the mean flow direction with the classical data taken by Batchelor & Townsend[11], who have confirmed experimentally the turbulent energy decay maintains similarity irrespective of the difference in the Reynolds number $R_e = UM/\nu$.

It is believed that (36) is the first exact solution in statistical theory of turbulence, so it has a permanent value.

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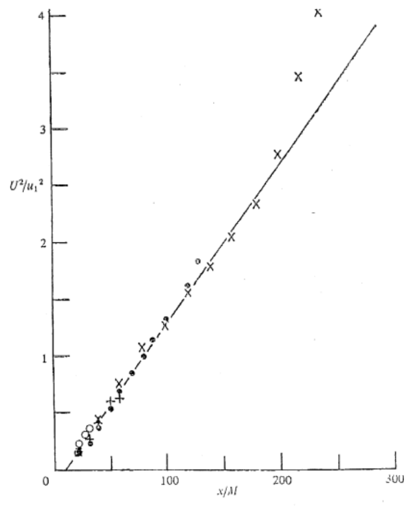

Fig.3 Similarity of energy decay at different Reynolds numbers (after Batchelor & Townsend[11]).

X:M=0.635, ●:M=1.27cm, +:M=2.54cm, ○:M=5.08cm
- : present theory,
U=longitudinal velocity=1286cm/s,
u_1= longitudinal velocity fluctuation,
x=longitudinal coordinate,
M=grid mesh size.