Abstract

In this paper, multiplicative version of degree distance of a graph is defined and tight upper bounds of the graph operations have been found.

Subject Areas

Discrete Mathematics

Keywords

Join, Disjunction, Composition, Symmetric Difference, Multiplicative Degree Distance, Zagreb Indices and Coindices

1. Introduction

A topological index of a graph is a numerical quantity which is structural invariant, i.e. it is fixed under graph automorphism. The simplest topological indices are the number of vertices and edges of a graph. In this paper, we define and study a new topological index called multiplicative degree distance. All graphs considered are simple and connected graphs.

We denote the vertex and the edge set of a graph $G$ by $V(G)$ and $E(G)$, respectively. $d_G(v)$ denotes the degree of a vertex $v$ in $G$. The number of elements in the vertex set of a graph $G$ is called the order of $G$ and is denoted by $v(G)$. The number of elements in the edge set of a graph $G$ is called the size of $G$ and is denoted by $e(G)$. A graph with order $n$ and size $m$ edges is called a $(n,m)$-graph. For any $u,v \in V(G)$, the distance between $u$ and $v$ in $G$, denoted by $d_G(u,v)$, is the length of a shortest $(u,v)$-path in $G$. The edge connective the vertices $u$ and $v$ will be denoted by $uv$. The complement $\overline{G}$ of the graph $G$ is the graph with vertex set $V(G)$, in which two vertices in $\overline{G}$ are adjacent if...
and only if they are not adjacent in \( G \).

The join of graphs \( G_1 \) and \( G_2 \) is denoted by \( G_1 + G_2 \), and it is the graph with vertex set \( V(G_1) \cup V(G_2) \) and the edge set \( E(G_1) \cup E(G_2) \cup \{u_1u_2 \mid u_1 \in V(G_1), u_2 \in V(G_2) \} \). The composition of graphs \( G_1 \) and \( G_2 \) is denoted by \( G_1 \times G_2 \), and it is the graph with vertex set \( V(G_1) \times V(G_2) \), and two vertices \( u = (u_1, u_2) \) and \( v = (v_1, v_2) \) are adjacent if \( u_1 \) is adjacent to \( v_1 \) or \( u_2 \) and \( v_2 \) are adjacent. The disjunction of graphs \( G_1 \) and \( G_2 \) is denoted by \( G_1 \vee G_2 \), and it is the graph with vertex set \( V(G_1) \times V(G_2) \) and edge set \( E(G_1 \vee G_2) = \{(u_1, u_2)(v_1, v_2) \mid u_1v_1 \in E(G_1) \text{ or } u_2v_2 \in E(G_2) \} \). The symmetric difference of graphs \( G_1 \) and \( G_2 \) is denoted by \( G_1 \oplus G_2 \), and it is the graph with vertex set \( V(G_1) \times V(G_2) \) and edge set \( E(G_1 \oplus G_2) = \{(u_1, u_2)(v_1, v_2) \mid u_1v_1 \in E(G_1) \text{ or } u_2v_2 \in E(G_2) \text{ but not both} \} \).

Let \( G \) be a connected graph. The Wiener index \( W(G) \) of a graph \( G \) is defined as

\[
W(G) = \frac{1}{2} \sum_{u, v \in V(G)} d_G(u, v),
\]

where \( d_G(u, v) \) is the shortest path distance between vertices \( u \) and \( v \) in \( G \).

Dobrynin and Kochetova [1] and Gutman [2] independently proposed a vertex-degree-Weighted version of Wiener index called degree distance or Schultz molecular topological index, which is defined for a connected graph \( G \) as

\[
DD(G) = \frac{1}{2} \sum_{u, v \in V(G)} d_G(u, v)[d_G(u) + d_G(v)],
\]

where \( d_G(u) \) and \( d_G(v) \) are the degrees of vertices \( u \) and \( v \), respectively.

The Zagreb indices have been introduced more than thirty years ago by Gutman and Trianjestic [3]. The first Zagreb index \( M_1(G) \) of a graph \( G \) is defined as

\[
M_1(G) = \sum_{uv \in E(G)} [d_G(u) + d_G(v)].
\]

The second Zagreb index \( M_2(G) \) of a graph \( G \) is defined as

\[
M_2(G) = \sum_{u \in V(G)} d_G(u)d_G(v),
\]

respectively.

In [8], Hamzeh, Iranmanesh Hossein-Zadeh and M.V. Diudea recently...
introduced the generalized degree distance of graphs. Asma Hamzeh, Ali Iranmanesh and Samaneh Hossein-Zadeh, Cartesian product, composition, join, disjunction and symmetric difference of graphs and introduce generalized and modified generalized degree distance Polynomials of graphs, such that their first derivatives at $x = 1$, see [9].

In this paper, we define a new graph invariant named multiplicative version of degree distance of a graph denoted by $DD'(G)$ and defined by

$$
[DD'(G)]^2 = \prod_{u,v \in E(G)} d_G(u) + d_G(v).
$$

Therefore the study of this new topological index is important and we have obtained Sharp upper bounds for the graph operations such as join, disjunction, composition, symmetric difference of graphs.

2. The Multiplicative Degree Distance of Graph Operations

Lemma 2.1. [10] [11], Let $G_1$ and $G_2$ be two simple connected graphs. The number of vertices and edges of graph $G_i$ is denoted by $n_i$ and $e_i$ respectively for $i = 1, 2$. Then we have

1. $d_{G_1\oplus G_2}(u,v) = \begin{cases} 1, & uv \in E(G_1) \text{ or } uv \in E(G_2) \text{ or } (u \in V(G_1) \text{ and } v \in V(G_2)) \\ 2, & \text{otherwise} \end{cases}$

For a vertex $u$ of $G_1$, $d_{G_1\oplus G_2}(u) = d_{G_1}(u) + n_2$, and for a vertex $v$ of $G_2$, $d_{G_1\oplus G_2}(v) = d_{G_2}(v) + n_1$.

2. $d_{G_1 \times G_2}((u_1, v_1), (u_2, v_2)) = \begin{cases} d_{G_1}(u_1, u_2), & \text{otherwise} \\ 1, & u_1 \neq u_2 \\ 2, & u_1 = u_2, v_1, v_2 \in E(G_2) \end{cases}$

$\therefore d_{G_1 \times G_2}((u, v)) = n_2 d_{G_2}(u) + d_{G_2}(v)$.

3. $d_{G_1 \vee G_2}((u_1, v_1), (u_2, v_2)) = \begin{cases} d_{G_1}(u_1, u_2), & \text{otherwise} \\ 1, & u_1 \neq u_2 \\ 2, & u_1 = v_1, v_2 \in E(G_2) \end{cases}$

$\therefore d_{G_1 \vee G_2}((u, v)) = n_2 d_{G_2}(u) + d_{G_2}(v)$.

4. $d_{G_1 \oplus G_2}((u_1, v_1), (u_2, v_2)) = \begin{cases} d_{G_1}(u_1, u_2), & \text{otherwise} \\ 1, & u_1 \neq u_2 \\ 2, & u_1 = v_1, v_2 \in E(G_2) \end{cases}$

$\therefore d_{G_1 \oplus G_2}((u, v)) = n_2 d_{G_2}(u) + d_{G_2}(v)$.

Lemma 2.2. (Arithmetic Geometric inequality)

Let $x_1, x_2, \cdots, x_n$ be non-negative numbers. Then

$$
\frac{x_1 + x_2 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 x_2 \cdots x_n}
$$

Remark 2.3. For a graph $G$, let $A(G) = \{(x, y) \in V(G) \times V(G) \mid x \text{ and } y \text{ are adjacent in } G\}$ and let $B(G) = \{(x, y) \in V(G) \times V(G) \mid x \text{ and } y \text{ are not adjacent in } G\}$. For each $x \in V(G), (x, x) \in B(G)$. Clearly,
\( A(G) \cup B(G) = V(G) \times V(G) \). Let \( C(G) = \{(x, x) \mid x \in V(G)\} \) and \( D(G) = B(G) - C(G) \). Clearly \( B(G) = C(G) \cup D(G) \), \( C(G) \cap D(G) = \emptyset \).

The summation \( \sum_{(x, y) \in A(G)} \) runs over the ordered pairs of \( A(G) \). For simplicity, we write the summation \( \sum_{(x, y) \in B(G)} \) as \( \sum_{xy} \). Similarly, we write the summation \( \sum_{(x, y) \in C(G)} \) as \( \sum_{xy} \). Also the summation \( \sum_{xy \in E(G)} \) runs over the edges of \( G \). We denote the summation \( \sum_{x, y \in V(G)} \) by \( \sum_{xy} \) and similarly \( \sum_{x, y \in V(G)} \) by \( \prod_{xy} \). The summation \( \sum_{(x, y) \in D(G)} \) equivalent to \( \sum_{xy} \) and similarly the summation \( \sum_{(x, y) \in E(G)} \) equivalent to \( \sum_{xy} \).

**Lemma 2.4.** Let \( G \) be a graph. Then

\[
\sum_{xy} = 2e(G)
\]

**Proof:**

\[
\sum_{xy} = 2 \sum_{xy} = 2e(G)
\]

**Lemma 2.5.**

\[
\sum_{xy} = M_1(G)
\]

**Proof:** Let \( x \in V(G) \) and \( t = d_G(x) \). Let \( y_1, y_2, \ldots, y_t \) be the neighbours of \( x \). Each ordered pair \( (x, y_i), 1 \leq i \leq t \), contributes \( d_G(x) \) to the sum. Thus these ordered pairs contribute \( d_G^2(x) \) to the sum. Hence

\[
\sum_{xy} = \sum_{xy} = M_1(G)
\]

**Lemma 2.6.**

\[
\sum_{xy} = 2M_2(G)
\]

**Proof:** Clearly,

\[
\sum_{xy} = 2 \sum_{xy} = 2M_2(G)
\]

**Lemma 2.7.**

\[
\sum_{xy} = 2e(\overline{G}) + v(G)
\]

**Proof:**

\[
\sum_{xy} = \sum_{xy} + \sum_{xy} = 2e(\overline{G}) + v(G)
\]

**Lemma 2.8.**

\[
\sum_{xy} = 2e(\overline{G})(v(G)-1) + 2e(G) - M_1(\overline{G})
\]
Proof.

\[\sum_{xy \in G} d_G(x) = \sum_{(x,y) \in \partial(G)} d_G(x) + \sum_{(x,y) \in \overline{G}} d_G(x)\]

\[= \sum_{(x,y) \in \partial(G)} \{v(G) - 1 - d_G(x)\} + \sum_{(x,y) \in \overline{G}} d_G(x)\]

\[= (v(G) - 1) \sum_{(x,y) \in \partial(G)} 1 - \sum_{(x,y) \in \overline{G}} d_G(x) + 2e(G)\]

\[= (v(G) - 1) 2e(\overline{G}) - \sum_{xy \in \overline{G}} d_G(x) + 2e(G)\]

\[= 2e(\overline{G})(v(G) - 1) + 2e(G) - M_1(\overline{G})\text{ by Lemma 2.5}\]

Lemma 2.9.

\[\sum_{xy \in G} d_G(x)d_G(y) = 2\overline{M}_2(G) + M_1(G)\]

Proof:

\[\sum_{xy \in G} d_G(x)d_G(y) = \sum_{(x,y) \in \partial(G)} d_G(x)d_G(y) + \sum_{(x,y) \in \overline{G}} d_G(x)d_G(y)\]

\[= 2 \sum_{xy \in \overline{G}} d_G(x)d_G(y) + \sum_{xy \in \overline{G}} d_G(x)\]

\[= 2\overline{M}_2(G) + M_1(G)\]

Lemma 2.10.

\[\sum_{xy \in G} [d_G(x) + d_G(y)] = 2\overline{M}_1(G) + 4e(G)\]

Proof:

\[\sum_{xy \in G} [d_G(x) + d_G(y)] = \sum_{(x,y) \in \partial(G)} [d_G(x) + d_G(y)] + \sum_{(x,y) \in \overline{G}} [d_G(x) + d_G(y)]\]

\[= \sum_{xy \in \overline{G}} 2d_G(x) + 2 \sum_{xy \in \overline{G}} [d_G(x) + d_G(y)]\]

\[= 4e(G) + 2\overline{M}_1(G)\]

3. The Multiplicative Degree Distance of Composition of Graph

Theorem 3.1. Let \(G_i, i = 1, 2,\) be a \((n_i, m_i)\)-graph. Then

\[
\left[DD^*(G_i [G_j])\right]^2 \\
\leq \left\{\frac{1}{n_1n_2(n_1n_2-1)} \left[4M_1(G_i)W(G_i) + 4n_2m_2DD(G_i) + 4n_1\overline{M}_1(G_i)\right.\right. \\
+ 8n_2^2m_1(n_2-1) + 2n_1M_1(G_2) + 8m_1n_2m_2 + 4W(G_i)\overline{M}_1(G_j) \\
\left.\left.+ 2n_2DD(G_1)(2\overline{M}_2 + n_2)\right]\right\}^{n_1n_2(n_1n_2-1)}
\]

Proof:
\[
\left[ DD^* (G_1[G_2]) \right]^2
= \prod_{x_1, y_1 \in G_1} \prod_{x_2, y_2 \in G_2} d_{G_1[G_2]}((x, u), (y, v)) \left( d_{G_1[G_2]}(x, u) + d_{G_1[G_2]}(y, v) \right) \\
\leq \left\{ \frac{1}{n_1 n_2 (n_1 n_2 - 1)} \sum_{x_1, y_1 \in G_1} \sum_{x_2, y_2 \in G_2} d_{G_1[G_2]}((x, u), (y, v)) \left( d_{G_1[G_2]}(x, u) + d_{G_1[G_2]}(y, v) \right) \right\}^{n_1 n_2 (n_1 n_2 - 1)}
\]

\[
= \left\{ \frac{1}{n_1 n_2 (n_1 n_2 - 1)} \sum_{x_1, y_1 \in G_1} \sum_{x_2, y_2 \in G_2} d_{G_1[G_2]}((x, u), (y, v)) \left( d_{G_1[G_2]}(x, u) + d_{G_1[G_2]}(y, v) \right) \right\}^{n_1 n_2 (n_1 n_2 - 1)}
\]

\[
= \left\{ \frac{1}{n_1 n_2 (n_1 n_2 - 1)} \sum_{x_1, y_1 \in G_1} \sum_{x_2, y_2 \in G_2} d_{G_1[G_2]}((x, u), (y, v)) \left( d_{G_1[G_2]}(x, u) + d_{G_1[G_2]}(y, v) \right) \right\}^{n_1 n_2 (n_1 n_2 - 1)}
\]

\[
\left[ DD^* G_1[G_2] \right]^2 \leq \left\{ \frac{1}{n_1 n_2 (n_1 n_2 - 1)} \left( S_1 + S_2 + S_3 + S_4 \right) \right\}^{n_1 n_2 (n_1 n_2 - 1)}
\]

where \( S_1, S_2, S_3, S_4 \) are terms of the above sums taken in order.

Next we calculate \( S_1, S_2, S_3 \) and \( S_4 \) separately.

\[
S_1 = \sum_{x_1, y_1 \in G_1} \sum_{x_2, y_2 \in G_2} d_{G_1[G_2]}((x, u), (y, v)) \left( d_{G_1[G_2]}(x, u) + d_{G_1[G_2]}(y, v) \right)
\]

\[
= \sum_{x_1, y_1 \in G_1} \sum_{x_2, y_2 \in G_2} d_{G_1}(x, y) \left[ d_{G_1}(u) + n_2 d_{G_1}(x) + d_{G_1}(v) + n_2 d_{G_1}(y) \right] \text{ by Lemma 2.1}
\]

\[
= n_2 \sum_{x_1, y_1 \in G_1} \sum_{x_2, y_2 \in G_2} d_{G_1}(x, y) \left[ d_{G_1}(u) + n_2 d_{G_1}(x) + d_{G_1}(v) + n_2 d_{G_1}(y) \right]
\]

\[
= 4n_2 m_2 DD(G_1) + 4M_1(G_2) W(G_1)
\]

\[
S_2 = \sum_{x_1, y_1 \in G_1} \sum_{x_2, y_2 \in G_2} d_{G_1[G_2]}((x, u), (y, v)) \left( d_{G_1[G_2]}(x, u) + d_{G_1[G_2]}(y, v) \right)
\]

\[
= \sum_{x_1, y_1 \in G_1} \sum_{x_2, y_2 \in G_2} d_{G_1[G_2]}((x, u), (y, v)) \left( d_{G_1[G_2]}(x, u) + d_{G_1[G_2]}(y, v) \right)
\]

\[
= \sum_{x_1, y_1 \in G_1} \sum_{x_2, y_2 \in G_2} d_{G_1[G_2]}((x, u), (y, v)) \left( d_{G_1[G_2]}(x, u) + d_{G_1[G_2]}(y, v) \right)
\]

\[
= 0 + \sum_{x_1, y_1 \in G_1} \sum_{x_2, y_2 \in G_2} d_{G_1[G_2]}((x, u), (y, v)) \left( d_{G_1[G_2]}(x, u) + d_{G_1[G_2]}(y, v) \right) \text{ by Lemma 2.1}
\]

\[
= 2 \sum_{x_1, y_1 \in G_1} \sum_{x_2, y_2 \in G_2} d_{G_1}(x, y) \left[ d_{G_1}(u) + d_{G_1}(v) \right]
\]

\[
= 4n_1 M_1(G_2) + 8n_2 m_2 (n_2 - 1)
\]
\[ S_3 = \sum_{x,y \in G_2} d_{G_2}(x,y) \left[ d_{G_1}(x) + d_{G_1}(y) \right] \]

\[ = \sum_{x,y \in G_2} d_{G_2}(x,y) \left[ d_{G_2}(u) + d_{G_2}(v) \right] \]

by Lemma 2.1

\[ = 2nM_1(G_2) + 8nM_2 \]

\[ S_4 = \sum_{x,y \in G_2} d_{G_2}(x,y) \left[ d_{G_1}(x) + d_{G_1}(y) \right] \]

by Lemma 2.1

\[ = 4W(G_2)M_1(G_2) + 2nDD(G_2)(2m_2 + n_2) \]

Lemma 3.2.

\[ DD^* K_n[G_n] = \left[ 2(nr - 1) \right]^{\frac{nr(n-1)}{2}} \]

Proof: Clearly the graph \( K_n[G_n] \) is the complete graph \( K_{nr} \).

\[ : DD^* (K_n[G_n]) = DD^* (K_{nr}) = \left[ 2(nr - 1) \right]^{\frac{nr(n-1)}{2}} \] (1)

Remark 3.3. Let \( G_1 = K_n \) and \( G_2 = K_r \). We get,

\[ DD(G_1) = 2(n-1)^2, M_1(G_2) = 0, n_1 = n, M_2(G_2) = r(r-1)^2, \]

\[ n_2 = r, m_1 = \frac{n(n-1)}{2}, M_2(G_2) = 0, n_1 = n, n_2 = r, m_2 = \frac{r(r-1)}{2} \]

\[ : \text{In Theorem 3.1, put } G_1 = K_n \text{ and } G_2 = K_r, \text{ we get} \]

\[ DD^* K_n[K_n] \leq \left[ 2(nr - 1) \right]^{\frac{nr(n-1)}{2}} \] (2)

From (1) and (2) our bound is tight

4. The Multiplicative Degree Distance of Join of Graph

Theorem 4.1. Let \( G_i, i = 1, 2, \) be a \( (n_i, m_i) \)-graph and let \( \bar{m}_i = e(G_i) \). Then
\[
\left[ DD^* (G_1 + G_2) \right]^2 \leq \frac{1}{(n_1 + n_2)(n_1 + n_2 - 1)} \left\{ 2M_1 (G_1) + 4n_2 m_1 + 4\overline{M}_1 (G_1) + 8n_2 \overline{m}_1 \\
+ 2m_2 + 2m_1 + n_1 n_2 (n_1 + n_2) + 2M_1 (G_2) \\
+ 4n_2 m_1 + 4\overline{M}_1 (G_2) + 8n_2 \overline{m}_2 \right\}^{(n_1 + n_2)(n_1 + n_2 - 1)}
\]

Proof:

\[
\left[ DD^* (G_1 + G_2) \right]^2 = \prod_{x, y \in \mathcal{X}(G_1 + G_2)} d_{(G_1 + G_2)}(x, y) \left[ d_{(G_1 + G_2)}(x) + d_{(G_1 + G_2)}(y) \right] \\
\leq \left[ \frac{1}{(n_1 + n_2)(n_1 + n_2 - 1)} \sum_{x, y \in \mathcal{X}(G_1 + G_2)} d_{(G_1 + G_2)}(x, y) \left[ d_{(G_1 + G_2)}(x) + d_{(G_1 + G_2)}(y) \right] \right]^{(n_1 + n_2)(n_1 + n_2 - 1)} \\
= \left[ \frac{1}{(n_1 + n_2)(n_1 + n_2 - 1)} \sum_{x \in \mathcal{X}(G_1), y \in \mathcal{X}(G_2)} \sum_{y \in \mathcal{X}(G_1)} d_{(G_1 + G_2)}(x, y) \left[ d_{(G_1 + G_2)}(x) + d_{(G_1 + G_2)}(y) \right] \right]^{(n_1 + n_2)(n_1 + n_2 - 1)} \\
= \left[ \frac{1}{(n_1 + n_2)(n_1 + n_2 - 1)} \sum_{x \in \mathcal{X}(G_1), y \in \mathcal{X}(G_2)} \sum_{y \in \mathcal{X}(G_1)} d_{(G_1 + G_2)}(x, y) \left[ d_{(G_1 + G_2)}(x) + d_{(G_1 + G_2)}(y) \right] \right]^{(n_1 + n_2)(n_1 + n_2 - 1)} \\
+ \sum_{y \in \mathcal{X}(G_2)} \sum_{x \in \mathcal{X}(G_1)} d_{(G_1 + G_2)}(x, y) \left[ d_{(G_1 + G_2)}(x) + d_{(G_1 + G_2)}(y) \right]^{(n_1 + n_2)(n_1 + n_2 - 1)} \\
= \left[ \frac{1}{(n_1 + n_2)(n_1 + n_2 - 1)} \sum_{x \in \mathcal{X}(G_1), y \in \mathcal{X}(G_2)} \sum_{y \in \mathcal{X}(G_1)} d_{(G_1 + G_2)}(x, y) \left[ d_{(G_1 + G_2)}(x) + d_{(G_1 + G_2)}(y) \right] \right]^{(n_1 + n_2)(n_1 + n_2 - 1)} \\
+ \sum_{y \in \mathcal{X}(G_2)} \sum_{x \in \mathcal{X}(G_1)} d_{(G_1 + G_2)}(x, y) \left[ d_{(G_1 + G_2)}(x) + d_{(G_1 + G_2)}(y) \right]^{(n_1 + n_2)(n_1 + n_2 - 1)} \\
+ \sum_{y \in \mathcal{X}(G_2)} \sum_{x \in \mathcal{X}(G_1)} d_{(G_1 + G_2)}(x, y) \left[ d_{(G_1 + G_2)}(x) + d_{(G_1 + G_2)}(y) \right]^{(n_1 + n_2)(n_1 + n_2 - 1)} \\
+ \sum_{y \in \mathcal{X}(G_2)} \sum_{x \in \mathcal{X}(G_1)} d_{(G_1 + G_2)}(x, y) \left[ d_{(G_1 + G_2)}(x) + d_{(G_1 + G_2)}(y) \right]^{(n_1 + n_2)(n_1 + n_2 - 1)} \\
+ \sum_{y \in \mathcal{X}(G_2)} \sum_{x \in \mathcal{X}(G_1)} d_{(G_1 + G_2)}(x, y) \left[ d_{(G_1 + G_2)}(x) + d_{(G_1 + G_2)}(y) \right]^{(n_1 + n_2)(n_1 + n_2 - 1)} \\
+ \sum_{y \in \mathcal{X}(G_2)} \sum_{x \in \mathcal{X}(G_1)} d_{(G_1 + G_2)}(x, y) \left[ d_{(G_1 + G_2)}(x) + d_{(G_1 + G_2)}(y) \right]^{(n_1 + n_2)(n_1 + n_2 - 1)} \\
+ \sum_{y \in \mathcal{X}(G_2)} \sum_{x \in \mathcal{X}(G_1)} d_{(G_1 + G_2)}(x, y) \left[ d_{(G_1 + G_2)}(x) + d_{(G_1 + G_2)}(y) \right]^{(n_1 + n_2)(n_1 + n_2 - 1)} \\
+ \sum_{y \in \mathcal{X}(G_2)} \sum_{x \in \mathcal{X}(G_1)} d_{(G_1 + G_2)}(x, y) \left[ d_{(G_1 + G_2)}(x) + d_{(G_1 + G_2)}(y) \right]^{(n_1 + n_2)(n_1 + n_2 - 1)} \\
+ \sum_{y \in \mathcal{X}(G_2)} \sum_{x \in \mathcal{X}(G_1)} d_{(G_1 + G_2)}(x, y) \left[ d_{(G_1 + G_2)}(x) + d_{(G_1 + G_2)}(y) \right]^{(n_1 + n_2)(n_1 + n_2 - 1)}
\[
\left[ DD^* (G_1 + G_2) \right]^2 \leq \frac{1}{(n_1 + n_2)(n_1 + n_2 - 1)} (J_1 + 2J_2 + J_3)
\]

where \( J_1, J_2, J_3 \) are terms of the above sums taken in order.

Next we calculate \( J_1, J_2 \) and \( J_3 \) separately one by one. Now,

\[
J_1 = \sum_{x \in F(G_1)} \sum_{y \in F(G_2)} d_{(G_1+G_2)}(x,y) \left[ d_{(G_1+G_2)}(x) + d_{(G_1+G_2)}(y) \right]
= \sum_{x \in F(G_1)} d_{(G_1+G_2)}(x,y) \left[ d_{(G_1+G_2)}(x) + d_{(G_1+G_2)}(y) \right]
= \sum_{y \in F(G_2)} d_{(G_1+G_2)}(x,y) \left[ d_{(G_1+G_2)}(x) + d_{(G_1+G_2)}(y) \right] + \sum_{x \in F(G_1)} d_{(G_1+G_2)}(x,y) \left[ d_{(G_1+G_2)}(x) + d_{(G_1+G_2)}(y) \right]
+ \sum_{y \in F(G_2)} d_{(G_1+G_2)}(x,y) \left[ d_{(G_1+G_2)}(x) + d_{(G_1+G_2)}(y) \right]
= 1 \cdot \sum_{x \in F(G_1)} d_{(G_1+G_2)}(x) + d_{(G_1+G_2)}(y) + 0
+ 2 \cdot \sum_{y \in F(G_2)} d_{(G_1+G_2)}(x) + d_{(G_1+G_2)}(y) + 0
= \sum_{x \in F(G_1)} \left[ d_{G_1}(x) + n_2 + d_{G_2}(y) + n_2 \right]
+ \sum_{y \in F(G_2)} \left[ d_{G_1}(x) + n_2 + d_{G_2}(y) + n_2 \right] \text{ by Lemma 2.1}
+ 2 \sum_{y \in F(G_2)} \left[ d_{G_1}(x) + d_{G_2}(y) \right] + 2n_2 \sum_{y \in F(G_2)} 1
+ 2 \left[ \sum_{y \in F(G_2)} \left[ d_{G_1}(x) + d_{G_2}(y) \right] + 2n_2 \sum_{y \in F(G_2)} 1 \right]
= 2M(G_1) + 4n_2m_2 + 4 \bar{M}_1(G_1) + 8n_2 \bar{m}_1
\]

\[
J_2 = \sum_{x \in F(G_1)} \sum_{y \in F(G_2)} d_{(G_1+G_2)}(x,y) \left[ d_{(G_1+G_2)}(x) + d_{(G_1+G_2)}(y) \right]
= 1 \sum_{x \in F(G_1)} \sum_{y \in F(G_2)} \left[ d_{(G_1+G_2)}(x) + d_{(G_1+G_2)}(y) \right]
= \sum_{x \in F(G_1)} \sum_{y \in F(G_2)} \left[ d_{G_1}(x) + n_2 + d_{G_2}(y) + n_2 \right] \text{ by Lemma 2.1}
= \sum_{x \in F(G_1)} d_{G_1}(x) + \sum_{y \in F(G_2)} 1 + \sum_{x \in F(G_1)} \sum_{y \in F(G_2)} 1 + \sum_{x \in F(G_1)} \sum_{y \in F(G_2)} 1
= 2m_1n_2 + 2m_2n_1 + (n_1 + n_2) n_1 n_2
\]

\[
J_3 = \sum_{x \in F(G_1)} \sum_{y \in F(G_2)} d_{(G_1+G_2)}(x,y) \left[ d_{(G_1+G_2)}(x) + d_{(G_1+G_2)}(y) \right]
= \sum_{x, y \in F(G_2)} d_{(G_1+G_2)}(x,y) \left[ d_{(G_1+G_2)}(x) + d_{(G_1+G_2)}(y) \right]
= \sum_{y \in F(G_2)} d_{(G_1+G_2)}(x,y) \left[ d_{(G_1+G_2)}(x) + d_{(G_1+G_2)}(y) \right] + \sum_{x \in F(G_1)} d_{(G_1+G_2)}(x,y) \left[ d_{(G_1+G_2)}(x) + d_{(G_1+G_2)}(y) \right]
= \sum_{x \in F(G_1)} \sum_{y \in F(G_2)} d_{(G_1+G_2)}(x,y) \left[ d_{(G_1+G_2)}(x) + d_{(G_1+G_2)}(y) \right]
\]
\[= 1 \sum_{xy \in G_2} \left[ d_{(G_1 + G_2)}(x) + d_{(G_1 + G_2)}(y) \right] + 2 \sum_{xy \in G_2, x \neq y} \left[ d_{(G_1 + G_2)}(x) + d_{(G_1 + G_2)}(y) \right] + 0 \]
\[= \sum_{xy \in G_2} \left[ d_{G_2}(x) + n_i + d_{G_2}(y) + n_i \right] + 2 \sum_{xy \in G_2, x \neq y} \left[ d_{G_2}(x) + n_i + d_{G_2}(y) + n_i \right] \text{ by Lemma 2.1} \]
\[= \sum_{xy \in G_2} \left[ d_{G_2}(x) + d_{G_2}(y) \right] + 2n_i \sum_{xy \in G_2} 1 + 2\left\{ \sum_{xy \in G_2, x \neq y} \left[ d_{G_2}(x) + d_{G_2}(y) \right] + 2n_i \sum_{xy \in G_2} 1 \right\} \]
\[= 2M_1(G_2) + 4n_i m_2 + 4\overline{M}_1(G_2) + 8n_i \overline{m}_2 \]

\[\left[ DD^*(G_1 + G_2) \right]^2 \leq \frac{1}{(n_1 + n_2)(n_1 + n_2 + 1)} \left[ 2M_1(G_1) + 4n_i m_1 + 4\overline{M}_1(G_1) + 8n_i \overline{m}_1 \right] + 2m_1 n_2 + 2m_1 n_1 + n_1 n_2 (n_1 + n_2) + 2M_1(G_2) + 4n_i m_2 + 4\overline{M}_1(G_2) + 8n_i \overline{m}_2 \right\} \]

**Lemma 4.2.**

\[DD^*[K_n + K_r] = \left[2(n + r - 1)\right]^{(n + r)(n + r - 1)} \]

**Proof:** Clearly the graph \(K_n + K_r\) is the complete graph \(K_{n+r}\),

\[DD^*[K_n + K_r] = DD^*[K_{n+r}] = \left[2(n + r - 1)\right]^{(n + r)(n + r - 1)} \quad (3)\]

**Remark 4.3.** Let \(G_1 = K_n\) and \(G_2 = K_r\). We get,

\[M_1(G_1) = n(n-1)^2,\]
\[m_1 = \frac{n(n-1)}{2},\]
\[M_1(G_2) = r(r-1)^2,\]
\[\overline{M}_1(G_2) = 0,\]
\[m_2 = \frac{r(r-1)}{2},\]
\[\overline{M}_1(G_2) = 0, n_i = n, n_2 = r, m_i = 0, \overline{m}_2 = 0.\]

\[\therefore \text{In Theorem 4.1, put } G_1 = K_n, G_2 = K_r, \text{ we get }\]

\[DD^*[K_n + K_r] \leq \left[2(n + r - 1)\right]^{(n + r)(n + r - 1)} \quad (4)\]

From (3) and (4) our bound is tight.

**5. The Multiplicative Degree Distance of Disjunction of Graph**

**Theorem 5.1.** Let \(G_i, i = 1, 2\), be a \((n_i, m_i)\)-graph and let \(\overline{m}_i = e(G_i)\). Then
\[
\left( DD^* (G_1 \lor G_2) \right)^2 \\
\leq \left[ \frac{1}{n_1 n_2 (n_1 n_2 - 1)} \left\{ 2m_2 n_2 \left( 2\bar{M}_1 (G_1) + 4m_1 \right) + 2n_1 \left( 2\bar{m}_1 + n_1 \right) M_1 (G_2) \right. \right. \\
\left. \left. - 2M_1 (G_2) (2\bar{m}_1 (n_1 - 1) + 2m_1 - M_1 (G_1)) + 2n_1 M_1 (G_1) (2\bar{m}_1 + n_1) \right. \right. \\
\left. \left. + 2m_2 n_2 \left( 2\bar{M}_1 (G_2) + 4m_1 \right) - 2M_1 (G_1) \left( 2(n_1 - 1) \bar{m}_2 + 2m_2 - M_1 (G_1) \right) \right. \right. \\
\left. \left. + 4n_1 M_1 (G_2) m_2 + 4n_1 M_1 (G_2) - 2M_1 (G_1) M_1 (G_2) \right. \right. \\
\left. \left. + 2n_2 \left( 2\bar{M}_1 (G_1) + 4m_1 \right) (2\bar{m}_1 + n_1) + 2n_1 \left( 2\bar{M}_1 (G_1) + 4m_1 \right) (2\bar{m}_1 + n_1) \right. \right. \\
\left. \left. - 4 \left( 2\bar{m}_1 (n_1 - 1) + 2m_1 - M_1 (G_1) \right) \left( 2\bar{m}_1 (n_1 - 1) + 2m_1 - M_1 (G_1) \right) \right. \right. \\
\left. \left. - 8m_2 n_2^2 - 4m_1^2 + 16m_1 m_2 \right) \right]\right\}^{\eta_2 (\eta_2 - 1)}
\]

Proof:

\[
\left( DD^* (G_1 \lor G_2) \right)^2 \\
= \prod_{x,y \in G_1} \prod_{x,y \in G_2} d(G_1 \lor G_2) ((x,u),(y,v)) \left[ d(G_1 \lor G_2) (x,u) + d(G_1 \lor G_2) (y,v) \right]
\]

\[
\leq \left\{ \frac{1}{n_1 n_2 (n_1 n_2 - 1)} \sum_{x,y \in G_1} \sum_{x,y \in G_2} d(G_1 \lor G_2) ((x,u),(y,v)) \left[ d(G_1 \lor G_2) (x,u) + d(G_1 \lor G_2) (y,v) \right] \right\}^{\eta_2 (\eta_2 - 1)}
\]

\[
= \left\{ \frac{1}{n_1 n_2 (n_1 n_2 - 1)} \sum_{x,y \in G_1} \sum_{x,y \in G_2} d(G_1 \lor G_2) ((x,u),(y,v)) \left[ d(G_1 \lor G_2) (x,u) + d(G_1 \lor G_2) (y,v) \right] \right\}^{\eta_2 (\eta_2 - 1)}
\]

\[
+ \sum_{u\in G_2} d(G_1 \lor G_2) ((x,u),(y,v)) \left[ d(G_1 \lor G_2) (x,u) + d(G_1 \lor G_2) (y,v) \right] \right\}^{\eta_2 (\eta_2 - 1)}
\]

\[
= \left\{ \frac{1}{n_1 n_2 (n_1 n_2 - 1)} \sum_{x,y \in G_1} \sum_{x,y \in G_2} d(G_1 \lor G_2) ((x,u),(y,v)) \left[ d(G_1 \lor G_2) (x,u) + d(G_1 \lor G_2) (y,v) \right] \right\}^{\eta_2 (\eta_2 - 1)}
\]

where \( A_1, A_2, A_3, A_4 \) are terms of the above sums taken in order.

Next we calculate \( A_1, A_2, A_3 \) and \( A_4 \) separately one by one. Now,
\[ A_i = \sum_{xy \in E_G} \sum_{uv \in E_G} d_{(G_i \cup G_j)}((x,u),(y,v)) \left[ d_{(G_i \cup G_j)}(x,u) + d_{(G_i \cup G_j)}(y,v) \right] \]

\[ = \sum_{xy \in E_G} \sum_{uv \in E_G} 1 \left[ n_d G_i (x) + n_d G_i (u) - d_{G_i}(x)d_{G_i}(u) + n_d G_i (y) \right] + n_d G_i (v) - d_{G_i}(x)d_{G_i}(v) \]

by Lemma 2.1

\[ = n_2 \sum_{xy \in E_G} \sum_{uv \in E_G} \left[ d_{G_i}(x) + d_{G_i}(y) \right] + n_1 \sum_{xy \in E_G} \sum_{uv \in E_G} \left[ d_{G_i}(u) + d_{G_i}(v) \right] \]

\[ - \sum_{xy \in E_G} \sum_{uv \in E_G} d_{G_i}(x)d_{G_i}(u) - \sum_{xy \in E_G} \sum_{uv \in E_G} d_{G_i}(y)d_{G_i}(v) \]

\[ = n_2 \left[ \sum_{xy \in E_G} \left[ d_{G_i}(x) + d_{G_i}(y) \right] \right] \left[ \sum_{xy \in E_G} 1 \right] + n_1 \left[ \sum_{xy \in E_G} \left[ d_{G_i}(u) + d_{G_i}(v) \right] \right] \]

\[ - \left[ \sum_{xy \in E_G} \left[ d_{G_i}(x) \right] \sum_{uv \in E_G} \left[ d_{G_i}(u) \right] - \left[ \sum_{xy \in E_G} \left[ d_{G_i}(y) \right] \sum_{uv \in E_G} \left[ d_{G_i}(v) \right] \right] \]

\[ = 2M_1(G_i)(2\overline{m}_1 + n_2) + 2n_1M_1(G_j) + 2M_1(G_j) \]

\[ A_2 = \sum_{xy \in E_G} \sum_{uv \in E_G} d_{(G_i \cup G_j)}((x,u),(y,v)) \left[ d_{(G_i \cup G_j)}(x,u) + d_{(G_i \cup G_j)}(y,v) \right] \]

\[ = \sum_{xy \in E_G} \sum_{uv \in E_G} \left[ n_d G_i (x) + n_d G_i (u) - d_{G_i}(x)d_{G_i}(u) + n_d G_i (y) \right] \]

\[ + n_d G_i (v) - d_{G_i}(x)d_{G_i}(v) \]

by Lemma 2.1

\[ = n_2 \sum_{xy \in E_G} \sum_{uv \in E_G} \left[ d_{G_i}(x) + d_{G_i}(y) \right] + n_1 \sum_{xy \in E_G} \sum_{uv \in E_G} \left[ d_{G_i}(u) + d_{G_i}(v) \right] \]

\[ - \sum_{xy \in E_G} \sum_{uv \in E_G} d_{G_i}(x)d_{G_i}(u) - \sum_{xy \in E_G} \sum_{uv \in E_G} d_{G_i}(y)d_{G_i}(v) \]

\[ = n_2 \left[ \sum_{xy \in E_G} \left[ d_{G_i}(x) + d_{G_i}(y) \right] \right] \left[ \sum_{xy \in E_G} 1 \right] + n_1 \left[ \sum_{xy \in E_G} \left[ d_{G_i}(u) + d_{G_i}(v) \right] \right] \]

\[ - \left[ \sum_{xy \in E_G} \left[ d_{G_i}(x) \right] \sum_{uv \in E_G} \left[ d_{G_i}(u) \right] - \left[ \sum_{xy \in E_G} \left[ d_{G_i}(y) \right] \sum_{uv \in E_G} \left[ d_{G_i}(v) \right] \right] \]

\[ = 2n_2M_1(G_i)(2\overline{m}_2 + n_2) + 2n_1M_1(G_j) + 2M_1(G_j) \]

\[ A_3 = \sum_{xy \in E_G} \sum_{uv \in E_G} d_{(G_i \cup G_j)}((x,u),(y,v)) \left[ d_{(G_i \cup G_j)}(x,u) + d_{(G_i \cup G_j)}(y,v) \right] \]

\[ = \sum_{xy \in E_G} \sum_{uv \in E_G} \left[ n_d G_i (x) + n_d G_i (u) - d_{G_i}(x)d_{G_i}(u) + n_d G_i (y) \right] \]

\[ + n_d G_i (v) - d_{G_i}(x)d_{G_i}(v) \]

by Lemma 2.1

\[ = n_2 \sum_{xy \in E_G} \sum_{uv \in E_G} \left[ d_{G_i}(x) + d_{G_i}(y) \right] + n_1 \sum_{xy \in E_G} \sum_{uv \in E_G} \left[ d_{G_i}(u) + d_{G_i}(v) \right] \]

\[ - \sum_{xy \in E_G} \sum_{uv \in E_G} d_{G_i}(x)d_{G_i}(u) - \sum_{xy \in E_G} \sum_{uv \in E_G} d_{G_i}(y)d_{G_i}(v) \]

\[ = n_2 \left[ \sum_{xy \in E_G} \left[ d_{G_i}(x) + d_{G_i}(y) \right] \right] \left[ \sum_{xy \in E_G} 1 \right] + n_1 \left[ \sum_{xy \in E_G} \left[ d_{G_i}(u) + d_{G_i}(v) \right] \right] \]

\[ - \left[ \sum_{xy \in E_G} \left[ d_{G_i}(x) \right] \sum_{uv \in E_G} \left[ d_{G_i}(u) \right] - \left[ \sum_{xy \in E_G} \left[ d_{G_i}(y) \right] \sum_{uv \in E_G} \left[ d_{G_i}(v) \right] \right] \]

\[ = 4n_2m_2M_1(G_i) + 4n_1m_1M_1(G_j) - 2M_1(G_i)M_1(G_j) \]
\[ A_4 = \sum_{x \in G_1 \ \text{and} \ v \in G_2} d_{(G_1, G_2)}((x, u), (y, v)) \left[ d_{(G_1, G_2)}(x, u) + d_{(G_1, G_2)}(y, v) \right] \]

\[ = 2 \sum_{x \in G_1 \ \text{and} \ u \in G_2} \left[ d_{(G_1, G_2)}(x, u) + d_{(G_1, G_2)}(y, v) \right] \]

\[ - 2 \sum_{x \in G_1 \ \text{and} \ y \in G_2} \sum_{u \in G_1 \ \text{and} \ v \in G_2} \left[ d_{(G_1, G_2)}(x, u) + d_{(G_1, G_2)}(y, v) \right] \]

\[ A_4 = 2A_3 - 2A_6, \quad \text{where} \quad A_3 \quad \text{and} \quad A_6 \quad \text{are terms of the above sums taken in order.} \]

Now,

\[ A_4 = \sum_{x \in G_1 \ \text{and} \ u \in G_2} \sum_{y \in G_2} d_{(G_1, G_2)}(x, u) + d_{(G_1, G_2)}(y, v) \]

\[ = \sum_{x \in G_1 \ \text{and} \ u \in G_2} \sum_{y \in G_2} \left[ n_2d_{G_1}(x) + n_1d_{G_2}(u) - d_{G_1}(x)d_{G_2}(u) + n_2d_{G_2}(y) \right] \]

\[ + n_1d_{G_1}(v) - d_{G_1}(x)d_{G_2}(v) \] \quad \text{by Lemma 2.1}

\[ = n_2\left( \sum_{y \in G_2} \left[ d_{G_1}(x) + d_{G_2}(y) \right] \right) \left( \sum_{u \in G_2} 1 \right) + n_1\left( \sum_{y \in G_2} 1 \right) \left( \sum_{u \in G_2} \left[ d_{G_2}(u) + d_{G_2}(v) \right] \right) \]

\[ - \left( \sum_{y \in G_2} \left[ d_{G_2}(y) \right] \right) \left( \sum_{u \in G_2} \left[ d_{G_2}(u) \right] \right) - \left( \sum_{y \in G_2} \left[ d_{G_2}(y) \right] \right) \left( \sum_{u \in G_2} \left[ d_{G_2}(v) \right] \right) \]

\[ = n_2 \left( 2\bar{M}_1(x) + n_1 \right) \left( 2\bar{M}_2 + n_2 \right) + n_1 \left( 2\bar{M}_1(y) + 4m_1 \right) \left( 2\bar{M}_2 + n_1 \right) \]

\[ - 2 \left( 2\bar{M}_1(n_1 - 1) + 2m_1 - M_1(\bar{G}_1) \right) \left( 2\bar{M}_2(n_2 - 1) + 2m_2 - M_1(\bar{G}_2) \right) \]

\[ A_6 = \sum_{x \in G_1 \ \text{and} \ y \in G_2} \sum_{u \in G_2 \ \text{and} \ v \in G_2} \left[ d_{(G_1, G_2)}(x, u) + d_{(G_1, G_2)}(y, v) \right] \]

\[ = \sum_{x \in G_1 \ \text{and} \ y \in G_2} \sum_{u \in G_2 \ \text{and} \ v \in G_2} \left[ n_2d_{G_1}(x) + n_1d_{G_2}(u) - d_{G_1}(x)d_{G_2}(u) + n_2d_{G_2}(y) \right] \]

\[ + n_1d_{G_1}(v) - d_{G_1}(x)d_{G_2}(v) \] \quad \text{by Lemma 2.1}

\[ = n_2\left( \sum_{y \in G_2} \left[ d_{G_1}(x) + d_{G_2}(y) \right] \right) + n_1\left( \sum_{y \in G_2} 1 \right) \left( \sum_{u \in G_2} \left[ d_{G_2}(u) + d_{G_2}(v) \right] \right) \]

\[ - \sum_{y \in G_2} \left( \sum_{u \in G_2} \left[ d_{G_2}(y) \right] \right) \left( \sum_{u \in G_2} \left[ d_{G_2}(u) \right] \right) - \sum_{y \in G_2} \left( \sum_{u \in G_2} \left[ d_{G_2}(y) \right] \right) \left( \sum_{u \in G_2} \left[ d_{G_2}(v) \right] \right) \]

\[ = n_2 \left( \sum_{y \in G_2} \left[ d_{G_1}(x) + d_{G_2}(y) \right] \right) \left( \sum_{u \in G_2} 1 \right) + n_1 \left( \sum_{y \in G_2} 1 \right) \left( \sum_{u \in G_2} \left[ d_{G_2}(u) + d_{G_2}(v) \right] \right) \]

\[ - \left( \sum_{y \in G_2} \left[ d_{G_2}(y) \right] \right) \left( \sum_{u \in G_2} \left[ d_{G_2}(u) \right] \right) - \left( \sum_{y \in G_2} \left[ d_{G_2}(y) \right] \right) \left( \sum_{u \in G_2} \left[ d_{G_2}(v) \right] \right) \]

\[ = 4m_1n_2 + 4m_2n_1 - 8m_1m_2 \]
\[
\left[ DD'(G_i \vee G_j) \right]^2 \\
\leq \left[ \frac{1}{n_i n_j (n_i n_j - 1)} \left\{ 2m_2 n_2 \left( 2\overline{M}_1 (G_i) + 4m_1 \right) + 2n_i (2\overline{m}_1 + n_i) M_1 (G_j) \\
- 2M_1 (G_i) \left( 2\overline{m}_1 (n_i - 1) + 2m_i - M_1 (\overline{G}_i) \right) + 2n_i M_1 (G_i) (2\overline{m}_i + n_i) \right\} \\
+ 2n_i m_i (2\overline{M}_1 (G_j) + 4m_1) - 2M_1 (G_j) \left( 2(n_j - 1) \overline{m}_2 + 2m_2 - M_1 (\overline{G}_j) \right) \\
+ 4n_i M_1 (G_j) m_2 + 4n_i m_i M_1 (G_j) - 2M_1 (G_i) M_1 (G_j) \\
+ 2n_j \left( 2\overline{M}_1 (G_i) + 4m_1 \right) \left( 2\overline{m}_2 + 2n_2 \right) + 2n_i \left( 2\overline{M}_1 (G_j) + 4m_1 \right) (2\overline{m}_i + n_i) \\
- 4 \left( 2\overline{m}_1 (n_i - 1) + 2m_i - M_1 (\overline{G}_i) \right) \left( 2\overline{m}_i (n_j - 1) + 2m_2 - M_1 (\overline{G}_j) \right) \right\} \\
- 8mn_2^2 - 4n_2^2 m_2 + 16m_2 m_2 \right]\]

\textbf{Lemma 5.2.}

\[ DD' [K_m \vee K_n] = (2mn - 2) \frac{mn(mn-1)}{2} \]

\textbf{Proof:} Clearly the graph \( K_m \vee K_n \) is the complete graph \( K_{mn} \).

\[ DD' (K_m \vee K_n) = DD' (K_{mn}) = (2mn - 2) \frac{mn(mn-1)}{2} \quad (5) \]

\textbf{Remark 5.3.} Let \( G_1 = K_m \) and \( G_2 = K_n \). We get

\[ n_i = m_i = m(m-1)/2, m_j = n(n-1)/2, \overline{m}_i = 0, \overline{m}_j = 0 \]

\[ M_1 (G_i) = M_1 (K_m) = m(m-1)/2, M_1 (G_j) = M_1 (K_n) = n(n-1)/2 \]

\[ M_1 (\overline{G}_i) = M_1 (\overline{K}_m) = 0, M_1 (\overline{G}_j) = M_1 (\overline{K}_n) = 0, M_1 (G_i) = \overline{M}_1 (K_m) = 0 \]

\textbf{Remark 5.3.} In Theorem 5.1, put \( G_1 = K_m \) and \( G_2 = K_n \), we get

\[ DD' [K_m \vee K_n] \leq (2mn - 2) \frac{mn(mn-1)}{2} \quad (6) \]

From (5) and (6) our bound is tight.

\textbf{6. The Multiplicative Degree Distance of Symmetric difference of Graph}

\textbf{Theorem 6.1.}

\[ \left[ DD^* (G_i \oplus G_j) \right]^2 \]

\[ \leq \left[ \frac{1}{n_i n_j (n_i n_j - 1)} \left\{ 2m_2 n_2 \left( 2\overline{M}_1 (G_i) + 4m_1 \right) + 2n_i (2\overline{m}_1 + n_i) M_1 (G_j) (2\overline{m}_i + n_i) \right\} \\
- 2M_1 (G_i) \left( 2\overline{m}_1 (n_i - 1) + 2m_i - M_1 (\overline{G}_i) \right) + 2n_i M_1 (G_i) (2\overline{m}_i + n_i) \right\} \\
+ 2n_i m_i (2\overline{M}_1 (G_j) + 4m_1) - 2M_1 (G_j) \left( 2(n_j - 1) \overline{m}_2 + 2m_2 - M_1 (\overline{G}_j) \right) \\
+ 4n_i M_1 (G_j) m_2 + 4n_i m_i M_1 (G_j) - 2M_1 (G_i) M_1 (G_j) \\
+ 2n_j \left( 2\overline{M}_1 (G_i) + 4m_1 \right) \left( 2\overline{m}_2 + 2n_2 \right) + 2n_i \left( 2\overline{M}_1 (G_j) + 4m_1 \right) (2\overline{m}_i + n_i) \\
- 4 \left( 2\overline{m}_1 (n_i - 1) + 2m_i - M_1 (\overline{G}_i) \right) \left( 2\overline{m}_i (n_j - 1) + 2m_2 - M_1 (\overline{G}_j) \right) \right\} \\
- 2 \left( 4n_2^2 m_i + 4n_2^2 m_2 - 16m_2 m_2 \right) \right]^{n_i n_j (n_i n_j - 1)} \]
Proof:

\[
[DD^* (G_1 \oplus G_2)]^2 \\
= \prod_{x \in G_1} \prod_{u \in G_2, x \neq u} \prod_{y \in G_2} \prod_{v \neq y} d_{(G_1 \oplus G_2)}((x, u), (y, v)) [d_{(G_1 \oplus G_2)}(x, u) + d_{(G_1 \oplus G_2)}(y, v)] \\
\leq \frac{1}{n_1 n_2 (n_1 n_2 - 1)} \left[ \sum_{x \in G_1} \sum_{u \in G_2, x \neq u} \sum_{y \in G_2} \sum_{v \neq y} d_{(G_1 \oplus G_2)}((x, u), (y, v)) [d_{(G_1 \oplus G_2)}(x, u) + d_{(G_1 \oplus G_2)}(y, v)] \right]^{n_1 n_2 (n_1 n_2 - 1)} \\
= \frac{1}{n_1 n_2 (n_1 n_2 - 1)} \left[ \sum_{x \in G_1} \sum_{u \in G_2, x \neq u} \sum_{y \in G_2} \sum_{v \neq y} d_{(G_1 \oplus G_2)}((x, u), (y, v)) [d_{(G_1 \oplus G_2)}(x, u) + d_{(G_1 \oplus G_2)}(y, v)] \right]^{n_1 n_2 (n_1 n_2 - 1)} \\
+ \sum_{x \in G_1} \sum_{u \in G_2} d_{(G_1 \oplus G_2)}((x, u), (y, v)) [d_{(G_1 \oplus G_2)}(x, u) + d_{(G_1 \oplus G_2)}(y, v)]^{n_1 n_2 (n_1 n_2 - 1)} \\
= \frac{1}{n_1 n_2 (n_1 n_2 - 1)} \left[ \sum_{x \in G_1} \sum_{u \in G_2, x \neq u} \sum_{y \in G_2} \sum_{v \neq y} d_{(G_1 \oplus G_2)}((x, u), (y, v)) [d_{(G_1 \oplus G_2)}(x, u) + d_{(G_1 \oplus G_2)}(y, v)] \right]^{n_1 n_2 (n_1 n_2 - 1)} \\
\leq \frac{1}{n_1 n_2 (n_1 n_2 - 1)} (C_1 + C_2 + C_3 + C_4) \\
\]

where \( C_1, C_2, C_3, C_4 \) are terms of the above sums taken in order.

Next we calculate \( C_1, C_2, C_3, C_4 \) separately.

\( C_1 = \sum_{x \in G_1} \sum_{u \in G_2} d_{(G_1 \oplus G_2)}((x, u), (y, v)) [d_{(G_1 \oplus G_2)}(x, u) + d_{(G_1 \oplus G_2)}(y, v)] \)

\[
= \sum_{x \in G_1} \sum_{u \in G_2} \left[ n_1 d_{G_1}(x) + n_2 d_{G_2}(u) - 2d_{G_1}(x)d_{G_2}(u) + n_2 d_{G_2}(y) \\ + n_1 d_{G_2}(v) - 2d_{G_1}(y)d_{G_2}(v) \right] \text{ by Lemma 2.1} \\
= n_1 \sum_{x \in G_1} d_{G_1}(x) + n_2 \sum_{u \in G_2} d_{G_2}(u) + n_1 \sum_{x \in G_1} d_{G_2}(u) + n_2 \sum_{x \in G_1} d_{G_2}(v) \\
- 2 \sum_{x \in G_1} d_{G_1}(x)d_{G_2}(u) - 2 \sum_{x \in G_1} d_{G_1}(y)d_{G_2}(v) \\
= n_1 \left[ \sum_{x \in G_1} d_{G_1}(x) + d_{G_1}(y) \right] + n_2 \left[ \sum_{u \in G_2} d_{G_2}(u) + d_{G_2}(v) \right] \\
- 2 \left[ \sum_{x \in G_1} d_{G_1}(x)d_{G_2}(u) \right] - 2 \left[ \sum_{x \in G_1} d_{G_1}(y)d_{G_2}(v) \right] \\
= 2n_1 n_2 (2\bar{M}_1(G_1) + 4m_1 + 2n_1 M_1(G_2)(2\bar{m}_1 + n_1) \\
- 4M_1(G_2)(2\bar{m}_1 (n_1 - 1) + 2m_1 - M_1(\bar{G}_1)) \right] \\
\]

\( C_2 = \sum_{y \in G_1} \sum_{v \in G_2} d_{(G_1 \oplus G_2)}((x, u), (y, v)) [d_{(G_1 \oplus G_2)}(x, u) + d_{(G_1 \oplus G_2)}(y, v)] \)
\[ C_2 = \sum_{x,y \in \mathcal{G}_1} \sum_{u,v \in \mathcal{G}_2} d_{(G_1 \ast \mathcal{G}_2)}((x,u),(y,v)) \left[ d_{(G_1 \ast \mathcal{G}_2)}(x,u) + d_{(G_1 \ast \mathcal{G}_2)}(y,v) \right] \]

\[ = \sum_{x,y \in \mathcal{G}_1} \sum_{u,v \in \mathcal{G}_2} \left[ n_x d_{x_{G_1}}(x) + n_y d_{y_{G_2}}(y) - 2d_{x_{G_1}}(x)d_{y_{G_2}}(y) \right] \]

\[ + n_x d_{x_{G_1}}(x) + n_y d_{y_{G_2}}(y) - 2d_{x_{G_1}}(x)d_{y_{G_2}}(y) \] by Lemma 2.1

\[ = n_x \sum_{x,y \in \mathcal{G}_1} \sum_{u,v \in \mathcal{G}_2} \left[ d_{x_{G_1}}(x) + d_{y_{G_2}}(y) \right] + n_y \sum_{x,y \in \mathcal{G}_1} \sum_{u,v \in \mathcal{G}_2} \left[ d_{x_{G_1}}(u) + d_{y_{G_2}}(v) \right] \]

\[ - 2 \sum_{x,y \in \mathcal{G}_1} \sum_{u,v \in \mathcal{G}_2} d_{x_{G_1}}(x)d_{y_{G_2}}(y) - 2 \sum_{x,y \in \mathcal{G}_1} \sum_{u,v \in \mathcal{G}_2} d_{x_{G_1}}(x)d_{y_{G_2}}(y) \]

\[ = 2n_x M_1(G_1)(\bar{m}_2 + n_y) + 2n_y m_1(2\bar{M}_1(G_2) + 4m) \]

\[ - 4M_1(G_1)(2(n_x - 1)\bar{m}_2 + 2m - M_1(\bar{G}_2)) \]

\[ C_3 = \sum_{x,y \in \mathcal{G}_1} \sum_{u,v \in \mathcal{G}_2} d_{(G_1 \ast \mathcal{G}_2)}((x,u),(y,v)) \left[ d_{(G_1 \ast \mathcal{G}_2)}(x,u) + d_{(G_1 \ast \mathcal{G}_2)}(y,v) \right] \]

\[ = \sum_{x,y \in \mathcal{G}_1} \sum_{u,v \in \mathcal{G}_2} \left[ n_x d_{x_{G_1}}(x) + n_y d_{y_{G_2}}(y) - 2d_{x_{G_1}}(x)d_{y_{G_2}}(y) \right] \]

\[ + n_x d_{x_{G_1}}(x) + n_y d_{y_{G_2}}(y) - 2d_{x_{G_1}}(x)d_{y_{G_2}}(y) \] by Lemma 2.1

\[ = n_x \sum_{x,y \in \mathcal{G}_1} \sum_{u,v \in \mathcal{G}_2} \left[ d_{x_{G_1}}(x) + d_{y_{G_2}}(y) \right] + n_y \sum_{x,y \in \mathcal{G}_1} \sum_{u,v \in \mathcal{G}_2} \left[ d_{x_{G_1}}(u) + d_{y_{G_2}}(v) \right] \]

\[ - 2 \sum_{x,y \in \mathcal{G}_1} \sum_{u,v \in \mathcal{G}_2} d_{x_{G_1}}(x)d_{y_{G_2}}(y) - 2 \sum_{x,y \in \mathcal{G}_1} \sum_{u,v \in \mathcal{G}_2} d_{x_{G_1}}(x)d_{y_{G_2}}(y) \]

\[ = 4n_x M_1(G_1)m_2 + 4n_y M_1(G_2) - 4M_1(G_1)m_1(G_2) \]

\[ C_4 = \sum_{x,y \in \mathcal{G}_1} \sum_{u,v \in \mathcal{G}_2} d_{(G_1 \ast \mathcal{G}_2)}((x,u),(y,v)) \left[ d_{(G_1 \ast \mathcal{G}_2)}(x,u) + d_{(G_1 \ast \mathcal{G}_2)}(y,v) \right] \]

\[ = 2 \sum_{x,y \in \mathcal{G}_1} \sum_{u,v \in \mathcal{G}_2} \left[ d_{(G_1 \ast \mathcal{G}_2)}(x,u) + d_{(G_1 \ast \mathcal{G}_2)}(y,v) \right] \]

\[ - 2 \sum_{x,y,u,v \in \mathcal{G}_1, x = y} \sum_{u,v \in \mathcal{G}_2} d_{(G_1 \ast \mathcal{G}_2)}(x,u) + d_{(G_1 \ast \mathcal{G}_2)}(y,v) \]

\[ C_4 = 2C_3 - 2C_6, \] where \( C_3 \) and \( C_6 \) denote the sums of the above terms in order.

Now,
\[ C_2 = \sum_{x,y \in G_1} \sum_{u,v \in G_2} \left[ d_{(G_1 \oplus G_2)}(x,u) + d_{(G_1 \oplus G_2)}(y,v) \right] \]
\[ = \sum_{x,y \in G_1} \sum_{u,v \in G_2} \left[ n_2 d_{G_1}(x) + n_2 d_{G_1}(u) - 2d_{G_1}(x) d_{G_2}(u) \right] \]
\[ + n_2 d_{G_2}(y) + n_2 d_{G_2}(v) - 2d_{G_2}(y) d_{G_2}(v) \right] \text{ by Lemma 2.1} \]
\[ = n_2 \sum_{x,y \in G_1} \sum_{u,v \in G_2} \left[ d_{G_1}(x) + d_{G_2}(y) \right] + n_2 \sum_{x,y \in G_1} \sum_{u,v \in G_2} \left[ d_{G_2}(u) + d_{G_2}(v) \right] \]
\[ - 2 n_2 \sum_{x,y \in G_1} \sum_{u,v \in G_2} d_{G_1}(x) d_{G_2}(u) - 2 \sum_{x,y \in G_1} \sum_{u,v \in G_2} d_{G_2}(y) d_{G_2}(v) \]
\[ = n_2 \left[ \sum_{x,y \in G_1} \left[ d_{G_1}(x) + d_{G_2}(y) \right] \right] \left( \sum_{u,v \in G_2} 1 \right) + n_2 \left( \sum_{x,y \in G_1} \sum_{u,v \in G_2} \left[ d_{G_2}(u) + d_{G_2}(v) \right] \right) \]
\[ - 2 \left( \sum_{x,y \in G_1} \sum_{u,v \in G_2} d_{G_1}(x) d_{G_2}(u) \right) - 2 \left( \sum_{x,y \in G_1} \sum_{u,v \in G_2} d_{G_2}(y) d_{G_2}(v) \right) \]
\[ = n_2 \left[ \sum_{x,y \in G_1} \left[ d_{G_1}(x) + d_{G_2}(y) \right] \right] \left( \sum_{u,v \in G_2} 1 \right) + n_2 \left( \sum_{x,y \in G_1} \sum_{u,v \in G_2} \left[ d_{G_2}(u) + d_{G_2}(v) \right] \right) \]
\[ - 2 \left( \sum_{x,y \in G_1} \sum_{u,v \in G_2} d_{G_1}(x) d_{G_2}(u) \right) - 2 \left( \sum_{x,y \in G_1} \sum_{u,v \in G_2} d_{G_2}(y) d_{G_2}(v) \right) \]
\[ = 4n_2^2 m_1 + 4n_2^2 m_2 - 16m_1 m_2 \]

\[ \left[ DD^*(G_1 \oplus G_2) \right]^2 \leq \left( \frac{1}{n_2} \right) \left[ \frac{2n_2 m_2 (2\tilde{M}_1(G_1) + 4m_1) + 2n_2 M_1(G_2) (2\tilde{m}_2 + n_2)}{n_2 (n_2 - 1)} \right] \]
\[ - 4M_1(G_2) (2\tilde{m}_1(n_2 - 1) + 2m_2 - M_1(\tilde{G}_1)) + 2n_2 M_1(G_2) (2\tilde{m}_2 + n_2) \]
\[ + 2n_2 m_1 (2\tilde{M}_1(G_1) + 4m_2) - 4M_1(G_1) (2(n_2 - 1)\tilde{m}_2 + 2m_2 - M_1(\tilde{G}_1)) \]
\[ + 4n_2 M_1(G_1) m_1 + 4n_2 m_1 M_1(G_2) - 4M_1(G_1) M_1(G_2) \]
\[ + 2 \left( n_2 (2\tilde{M}_1(G_1) + 4m_2) (2m_2 + n_2) + n_2 (2\tilde{M}_1(G_2) + 4m_2) (2m_2 + n_1) \right) \]
\[ - 4 (2\tilde{m}_1(n_2 - 1) + 2m_2 - M_1(\tilde{G}_1)) (2\tilde{m}_2(n_2 - 1) + 2m_2 - M_1(\tilde{G}_2)) \]
\[ - 2 \left( 4n_2^2 m_1 + 4n_2^2 m_2 - 16m_1 m_2 \right) \]
\[ n_2 (n_2 - 1) \]
Proof: Clearly the graph $K_m \oplus K_1$ is the complete graph $K_w$

$$DD^*[K_m \oplus K_1] = DD^*[K_w] = (2m - 2)\frac{n(m-1)}{2}$$  \hspace{1cm} (7)

Remark 6.3. Let $G_1 = K_m$ and $G_2 = K_1$. We get

$$n_1 = m, \quad n_2 = 1, \quad m_1 = \frac{m(m-1)}{2}, \quad m_2 = 0, \quad \overline{m}_1 = 0, \quad \overline{m}_2 = 0$$

$$M_1(G_1) = M_1(K_m) = m(m-1)^2, \quad M_1(G_2) = M_1(K_1) = 0$$

$$\overline{M}_1(G_1) = \overline{M}_1(K_m) = 0, \quad \overline{M}_1(G_2) = \overline{M}_1(K_1) = 0$$

$$\overline{M}_1(G_1) = \overline{M}_1(K_m) = 0, \quad \overline{M}_1(G_2) = \overline{M}_1(K_1) = 0$$

\[ \therefore \quad \text{In Theorem 6.1, put } G_1 = K_m \text{ and } G_2 = K_1, \text{ we get} \]

$$DD^*[K_m \oplus K_1] \leq (2m - 2)\frac{n(m-1)}{2}$$  \hspace{1cm} (8)

From (7) and (8) our bound is tight.

References


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