LMI Approach to Suboptimal Guaranteed Cost Control for 2-D Discrete Uncertain Systems

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ABSTRACT

This paper studies the problem of the guaranteed cost control via static-state feedback controllers for a class of two-dimensional (2-D) discrete systems described by the Fornasini-Marchesini second local state-space (FMSLSS) model with norm bounded uncertainties. A convex optimization problem with linear matrix inequality (LMI) constraints is formulated to design the suboptimal guaranteed cost controller which ensures the quadratic stability of the closed-loop system and minimizes the associated closed-loop cost function. Application of the proposed controller design method is illustrated with the help of one example.

Keywords: Linear Matrix Inequality, Lyapunov Methods, Robust Stability, 2-D Discrete Systems, Uncertain Systems, Fornasini-Marchesini Second Local State-Space Model

1. Introduction

In the past few years, due to the rapid increase of a wide variety of applications of two-dimensional (2-D) discrete systems in many practical application domains such as digital filtering, image and video processing, seismographic data processing, thermal processes, gas absorption, water stream heating, control systems etc. [1-10], there has emerged a continuously growing interest in the system theoretic problems of 2-D discrete systems. Many authors have proposed and analyzed linear state-variable models for 2-D discrete systems [11-14]. The more popular models are Roesser model [11], Fornasini-Marchesini first model [13] and Fornasini-Marchesini second local state-space (FMSLSS) model [14]. Many publications relating to 2-D Lyapunov equation with constant coefficients for the Roesser model [11] have appeared [15-22]. The stability properties of 2-D discrete systems described by the FM first model [13] and Fornasini-Marchesini second local state-space model have been investigated extensively [23-29]. The stability analysis of 2-D discrete systems described by the FMSLSS model [14] has attracted a great deal of interest and many significant results have been obtained [22,30-44].

Due to assumptions in the modeling process and/or the changing operating conditions of a real world system, it is usually impossible for a mathematical model to describe the real world system exactly. The problem of designing robust controllers for 2-D uncertain systems has drawn the attention of several researchers in recent years [39,40]. When controlling a system subject to parameter uncertainty, it is also desirable to design a control system which is not only stable but also guarantees an adequate level of performance. One approach to this problem is the so-called guaranteed cost control approach [45]. This approach has the advantage of providing an upper bound on a given performance index and thus the system performance degradation incurred by the uncertainties is guaranteed to be less than this bound. Based on this idea, many significant results have been proposed [42-51]. In [42-44], the guaranteed cost control problem for 2-D discrete uncertain systems in FMSLSS setting has been considered and a robust controller design method has been established. The approach of [42] does not provide a true linear matrix inequality (LMI) based result which is not beneficial in terms of numerical complexity. Subsequently, in [43], an LMI based criterion for the existence of robust guaranteed cost controller has been formulated. Robust suboptimal guaranteed cost control for 2-D discrete uncertain systems in FMSLSS setting is an important problem.

In recent years, LMI has emerged as a powerful tool in control design problems [52-58]. The introduction of LMI in control theory has given a new direction in the area of robust control problems. A widely accepted method for solving robust control problems now is to simply
reduce them to LMI problems. Since solving LMIs is a convex optimization problem, such formulations offer a numerically efficient means of attacking problems that are difficult to solve analytically. These LMIs can be solved effectively by employing the recently developed Matlab LMI toolbox \[53\].

This paper, therefore, deals with the suboptimal guaranteed cost control problem for 2-D discrete uncertain systems described by FMSLSS model with norm-bounded uncertainties. The paper is organized as follows. In Section 2, we formulate the problem of robust guaranteed cost control for the uncertain 2-D discrete system described by the FMSLSS model and recall some useful results. An LMI based approach for the design of suboptimal guaranteed cost controller design method is given. Finally, some concluding remarks are given in Section 5.

2. Problem Formulation and Preliminaries

The following notations are used throughout the paper:

- \( R^n \) real vector space of dimension \( n \)
- \( R^{m \times n} \) set of \( n \times m \) real matrices
- \( 0 \) null matrix or null vector of appropriate dimension
- \( I \) identity matrix of appropriate dimension
- \( G^T \) transpose of matrix \( G \)
- \( G > 0 \) matrix \( G \) is positive definite symmetric
- \( G < 0 \) matrix \( G \) is negative definite symmetric
- \( \text{det}(G) \) determinant of matrix \( G \)
- \( \lambda_{\text{max}}(G) \) maximum eigenvalue of matrix \( G \).

In this paper, we are concerned with the problem of guaranteed cost control for 2-D discrete uncertain systems described by FMSLSS model \[14\]. The system under consideration is given by

\[
x(i+1, j+1) = (A_i + \Delta A_i)x(i+1, j) + (B_i + \Delta B_i)u(i+1, j),
\]

\[
A = \begin{bmatrix} A_i & A_j \end{bmatrix},
\]

\[
B = \begin{bmatrix} B_i & B_j \end{bmatrix},
\]

\[
C = \begin{bmatrix} C_i & C_j \end{bmatrix},
\]

where \( x(i, j) \in R^n \) and \( u(i, j) \in R^m \) are the state and control input, respectively. The matrices \( A_i \in R^{m \times n} \) and \( B_i \in R^{m \times n} \) \( (k = 1, 2) \) are known constant matrices representing the nominal plant, \( \Delta A_i \) and \( \Delta B_i \) \( (k = 1, 2) \) are real valued matrix functions representing parameter uncertainties in the system model. The parameter uncertainties under consideration are assumed to be norm-bounded and of the form

\[
[\Delta A \Delta B] = LF(i, j)[M_1 \quad M_2],
\]

where

\[
\Delta A = \begin{bmatrix} \Delta A_1 & \Delta A_2 \end{bmatrix}, \quad \Delta B = \begin{bmatrix} \Delta B_1 & \Delta B_2 \end{bmatrix},
\]

\[
M_1 = [M_{11} \quad M_{12}], \quad M_2 = [M_{21} \quad M_{22}].
\]

In the above, \( L, M_1 \) and \( M_2 \) can be regarded as known structural matrices of uncertainty and \( F(i, j) \) is an unknown matrix representing parameter uncertainty which satisfies

\[
\|F(i, j)\| \leq 1.
\]

It may be mentioned that the uncertainty of (1c) satisfying (1f) has been widely adopted in robust control literature \[38,39,42-44,59-62\]. The matrices \( L \) and \( M_1 \) \( (M_2) \) specify how the elements of the nominal matrices \( A \) \( (B) \) are affected by the uncertain parameters in \( F(i, j) \). Note that \( F(i, j) \) can always be restricted as (1f) by appropriately selecting \( L, M_1 \) \( (M_2) \). Therefore, there is no loss of generality in choosing \( F(i, j) \) as in (1f).

It is assumed that the system (1a) has a finite set of initial conditions \[22,34,36,38,43,44\] i.e., there exist two positive integers \( p \) and \( q \) such that

\[
x(i, 0) = 0, \quad i \geq p; \quad x(0, j) = 0, \quad j \geq q,
\]

and the initial conditions are arbitrary, but belong to the set \[42-44\]

\[
S = \left\{ x(i, 0), x(0, j) \in R^n : x(i, 0) = MN, \quad x(0, j) = MN, \quad N_i^kN_i^k < 1 \quad (k = 1, 2) \right\},
\]

where \( M \) is a given matrix.

Associated with the uncertain system (1) is the cost function \[43,44\]:

\[
J = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \left[ u^T(i+1, j)R_iu(i+1, j) + u^T(i, j+1)R_ju(i, j+1) \right]
\]

\[
+ \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \bar{W}_i \bar{v}_j,
\]

where

\[
\bar{v}_j = \begin{bmatrix} x(i+1, j) \\ x(i, j+1) \end{bmatrix},
\]

\[
0 < R_i \in R^{m \times m} \quad (k = 1, 2),
\]

\[
W_i = \begin{bmatrix} Q_i & 0 \\ 0 & Q_i \end{bmatrix},
\]

\[
0 < Q_i \in R^{n \times n} \quad (k = 1, 2).
\]

Suppose the system state is available for feedback, the objective of this paper is to develop a procedure to de-
A control law (3) is said to be an optimal quadratic guaranteed cost control if it ensures the quadratic stability of the closed-loop system (4) and minimizes the closed-loop cost function (5).

As an extension of the result for the global asymptotic stability condition of 2-D discrete FMSLSS model given in [14,30-33], one can easily arrive at the following lemma.

**Lemma 2.1** [44] The 2-D discrete uncertain system (1) is globally asymptotically stable if and only if

\[
\det \left\{ I - z_1 A_1 + L F M_{11} \right\} \neq 0
\]

for all \((z_1, z_2, F) \in \mathbb{U}^2\),

where \(\mathbb{U}^2 = \{(z_1, z_2, F) : |z_1| \leq 1, |z_2| \leq 1, \|F\| \leq 1\} \).

**Definition 2.2** [42-44] Consider the uncertain system (1) and cost function (2), then the static-state feedback controller \(u(i, j) = Kx(i, j)\) is said to define a quadratic guaranteed cost control for system (1) if and only if

\[
J = \sum_{i=1}^{n} \sum_{j=1}^{n} z_{ij}^T W_2 z_{ij} \rightarrow \min,
\]

where

\[
J = \sum_{i=1}^{n} \sum_{j=1}^{n} z_{ij}^T W_2 z_{ij},
\]

and

\[
W_2 = \begin{bmatrix}
Q_1 + K^T R K & 0 \\
0 & Q_2 + K^T R_2 K
\end{bmatrix}.
\]

**Lemma 2.3** [52, 63] For real matrices \(M, L, Q\) of appropriate dimensions, where \(M = M^T\) and \(Q = Q^T > 0\), then \(M + L^T Q L < 0\) if and only if

\[
\begin{bmatrix}
M & L^T \\
L & -Q^{-1}
\end{bmatrix} < 0.
\]

or equivalently

\[
\begin{bmatrix}
-Q^{-1} & L \\
L^T & M
\end{bmatrix} < 0.
\]

**Lemma 2.4** [44] Suppose there exists a quadratic guaranteed cost matrix \(0 < P = P^T \in R^{n \times n}\) if there exist a \(2n \times 2n\) positive definite symmetric matrix \(W_2\) given by (5b) and an \(n \times n\) positive definite symmetric matrix \(P_1\) such that

\[
\Gamma_{cl} + W_2 < 0,
\]

where

\[
\Gamma_{cl} = \begin{bmatrix}
A_{11} + B_{11} K & A_{12} + B_{12} K \\
A_{21} + B_{21} K & A_{22} + B_{22} K
\end{bmatrix}^T P
\]

\[
\times \begin{bmatrix}
P_1 & 0 \\
0 & P_1 - P
\end{bmatrix},
\]

and

\[
A_{11} = A_1 + \Delta A_1 = A_1 + L F M_{11},
\]

\[
A_{12} = A_2 + \Delta A_2 = A_2 + L F M_{12},
\]

\[
A_{21} = A_1 + \Delta A_1 = A_1 + L F M_{21},
\]

\[
B_{12} = B_2 + \Delta B_2 = B_2 + L F M_{22}.
\]

The following lemmas are needed in the proof of our main result.

**Lemma 2.2** [42,44,51] Let \(A \in R^{n \times n}, H \in R^{m \times k}, E \in R^{k \times n}\) and \(Q = Q^T \in R^{n \times n}\) be given matrices. Then there exists a positive definite matrix \(P\) such that

\[
\]

for all \(F\) satisfying \(F^T F \leq I\), if and only if there exists a scalar \(\varepsilon > 0\) such that

\[
\begin{bmatrix}
-P^{1} + \varepsilon H H^T & A \\
A^T & \varepsilon^{1} E^T E - Q
\end{bmatrix} < 0.
\]

**3. Main Result**

In this section, we establish that the problem of determining quadratic guaranteed cost control for system (1) and cost function (2) can be recast to a convex optimization problem. The main result may be stated as follows.

**Theorem 3.1** Consider system (1) and cost function (2), then there exists a suboptimal static-state feedback controller \(u(i, j) = Kx(i, j)\) that solves the addressed robust guaranteed cost control problem if the following optimization problem

minimize \((\varepsilon + \lambda)\)

subject to

\[
\begin{cases}
(i). \, (13), \\
(ii). \, -\lambda I M^T < 0
\end{cases}
\]

has a feasible solution \(\varepsilon > 0, U \in R^{n \times k}\), \(0 < S = S^T \in R^{m \times m}\) and \(0 < Y = Y^T \in R^{m \times m}\). The constraint (13) is given by

\[
\begin{bmatrix}
I & M \\
0 & -S
\end{bmatrix} < 0.
\]
where

\[
\begin{align*}
\bar{A}_1 &= A_1 + B_1 U, \\
\bar{A}_2 &= A_2 + B_2 U, \\
\bar{M}_{11} &= M_{11}^T + U^T M_{21}^T, \\
\bar{M}_{12} &= M_{12}^T + U^T M_{22}^T.
\end{align*}
\]

In this situation, a suboptimal control law is

\[
K = U S^{-1}
\]

which ensures the minimization of the upper bound of (2) for the closed-loop uncertain system.

Proof: Using (5b) and (7b), matrix Inequality (7a) can be expressed as

\[
\begin{bmatrix}
A_{11} + B_1 K & A_{12} + B_2 K \\
A_{21} + B_1 K & A_{22} + B_2 K
\end{bmatrix}^T P
\]

which, in view of (7c)-(7f), takes the form

\[
\begin{bmatrix}
P_1 & 0 \\
0 & P - P_1
\end{bmatrix} < 0.
\]

Applying Lemma 2.2, (17) can be rearranged as

\[
\begin{bmatrix}
-\bar{P} + Q_2 + K^T R_2 K & 0 \\
0 & -(P - P_1) + Q_2 + K^T R_2 K
\end{bmatrix} < 0.
\]

Premultiplying and postmultiplying (18) by the matrix

\[
\begin{bmatrix}
-e^{-1/2} P^{-1} & 0 \\
0 & e^{-1/2} P^{-1}
\end{bmatrix}
\]

one obtains

\[
\begin{align*}
-\varepsilon P^{-1} + LL^T &= \varepsilon (A_1 + B_1 K)^T P^{-1} \\
\varepsilon P^{-1} (A_1 + B_1 K)^T &= \varepsilon P^{-1} \left\{ -P_1 + Q_2 + K^T R_2 K + \varepsilon \left( M_{11} + M_{21} K \right)^T \left( M_{11} + M_{21} K \right) \right\} P^{-1} \\
\varepsilon P^{-1} (A_2 + B_2 K)^T &= \varepsilon P^{-1} \left( M_{12} + M_{22} K \right)^T \left( M_{11} + M_{21} K \right) P^{-1} \\
\varepsilon P^{-1} \left( M_{11} + M_{21} K \right)^T \left( M_{12} + M_{22} K \right) &= \varepsilon P^{-1} \left( M_{12} + M_{22} K \right)^T \left( M_{12} + M_{22} K \right) P^{-1}
\end{align*}
\]

< 0,
which can be rewritten as
\[
\begin{bmatrix}
-S + LL^T & \bar{A}_1 \\
\bar{A}_1^T & -Y + e^{-1}SQ \quad S + e^{-1}U^T R U + \bar{M}_{11} \bar{M}_{11}^T \\
\bar{A}_2 & 0 & -(S - Y)
\end{bmatrix}< 0
\]
where
\[
S = \varepsilon P^{-1},
\]
\[
Y = e^{-1}SP_1S,
\]
and \(\bar{A}_1, \bar{A}_2, \bar{M}_{11}\) and \(\bar{M}_{12}\) are defined in (15).

Equation (19) can be expressed as Equation (22).

The equivalence of (22) and (13) follows from the convexity of the objective function and of the constraints.

This completes the proof of Theorem 3.1.

Remark 3.1 It should be pointed out that the optimization problem given by (14) is an LMI eigenvalue problem [52,53], which provides a procedure to design suboptimal guaranteed cost controller.

4. Application to the Guaranteed Cost Control of Dynamical Processes Described by the Darboux Equation

In this section, we shall demonstrate the application of our proposed method (Theorem 3.1) in robust guaranteed cost control of processes in the Darboux equation. It is known that some dynamical processes in gas absorption, water stream heating and air drying can be described by the Darboux equation [3,7,8]:
\[
\varepsilon^2 s(x,t) = a_x s(x,t) + a_x s(x,t) + a_0 s(x,t) + b f(x,t)
\]
with the initial conditions
\[
s(x,0) = p(x), \quad s(0,t) = q(t)
\]
where \(s(x,t)\) is an unknown function at space \(x \in [0,x_f]\) and time \(t \in [0,\infty]\), \(a_1, a_2, a_0\) and \(b\) are real constants and \(f(x,t)\) is the input function.

Let
\[
r(x,t) = \frac{\partial s(x,t)}{\partial t} - a_2 s(x,t)
\]
then (24) can be transformed into an equivalent system of first-order differential equation of the form:
\[
\begin{bmatrix}
\partial r(x,t) \\
\partial s(x,t)
\end{bmatrix} = \begin{bmatrix}
a_1 & a_2 & a_0 \\
1 & a_2 & 0
\end{bmatrix} \begin{bmatrix} r(x,t) \\
s(x,t) \end{bmatrix} + \begin{bmatrix} b \\
0
\end{bmatrix} f(x,t).
\]

It follows from (26) that
\[
r(0,t) = \frac{\partial s(x,t)}{\partial t} \bigg|_{x=0} - a_2 s(0,t) = \frac{dq(t)}{dt} - a_2 q(t) = z(t).
\]

Taking
\[
\begin{bmatrix}
-L & 0 & 0 & 0 & 0 & 0 \\
0 & -Y & 0 & 0 & 0 & 0 \\
0 & 0 & -\varepsilon I & 0 & 0 & 0 \\
0 & 0 & 0 & -\varepsilon I & 0 & 0 \\
0 & 0 & 0 & 0 & -\varepsilon I & 0 \\
0 & 0 & 0 & 0 & 0 & -\varepsilon I
\end{bmatrix}^{-1} \begin{bmatrix}
L & 0 & 0 & 0 & 0 & 0 \\
0 & \bar{M}_{11} & \bar{M}_{12} & 0 & 0 & 0 \\
0 & 0 & \bar{Q}_{1/2} S & 0 & 0 & 0 \\
0 & 0 & 0 & \bar{Q}_{1/2} S & 0 & 0 \\
0 & 0 & 0 & 0 & \bar{R}_{1/2} U & 0 \\
0 & 0 & 0 & 0 & 0 & \bar{R}_{1/2} U
\end{bmatrix} < 0.
\]
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\[ r(i, j) = r(i \Delta x, j \Delta t), \]  \( r(i, j) = s(i \Delta x, j \Delta t), \)  \( f(x, t) = u(i, j) \)

and applying the forward difference quotients for both derivatives in (27), it is easy to verify that (27) can be expressed in the following form:

\[
\begin{bmatrix}
  r(i, j) \\
  s(i, j)
\end{bmatrix} =
\begin{bmatrix}
  (1 + a_0 \Delta x) & (a_1 + a_0) \Delta x \\
  0 & 0
\end{bmatrix}
\begin{bmatrix}
  r(i-1, j) \\
  s(i-1, j)
\end{bmatrix}
\]

\[
+ \begin{bmatrix}
  0 & 0 \\
  \Delta t & (1 + a_2 \Delta t)
\end{bmatrix}
\begin{bmatrix}
  r(i-1, j-1) \\
  s(i-1, j-1)
\end{bmatrix}
\]

\[
+ \begin{bmatrix}
  0 \\
  0
\end{bmatrix}
\begin{bmatrix}
  h\Delta x \\
  0
\end{bmatrix}
\begin{bmatrix}
  u(i-1, j) \\
  u(i, j-1)
\end{bmatrix}
\]

with the initial conditions

\[ s(i, 0) = p(i \Delta x), \quad r(0, j) = z(j \Delta t). \]

By setting

\[ x(i, j) = \begin{bmatrix}
  r(i, j) \\
  s(i, j)
\end{bmatrix}, \]

(30) can be converted into the following FMSLSS model:

\[ x(i+1, j+1) = \begin{bmatrix}
  0 & 0 \\
  \Delta t & (1 + a_2 \Delta t)
\end{bmatrix}
\begin{bmatrix}
  x(i+1, j) \\
  x(i, j+1)
\end{bmatrix}
\]

\[
+ \begin{bmatrix}
  0 \\
  0
\end{bmatrix}
\begin{bmatrix}
  h\Delta x \\
  0
\end{bmatrix}
\begin{bmatrix}
  u(i+1, j) \\
  u(i, j+1)
\end{bmatrix}
\]

with the initial conditions

\[ x(i, 0) = \begin{bmatrix}
  -a_2 p(i \Delta x) \\
  p(i \Delta x)
\end{bmatrix}, \quad x(0, j) = \begin{bmatrix}
  z(j \Delta t) \\
  q(j \Delta t)
\end{bmatrix} \]

Now, consider the problem of suboptimal guaranteed cost control of a system represented by (33) with

\[ a_0 = \frac{1}{15}, \]
\[ a_1 = -\frac{3}{5}, \]
\[ a_2 = -\frac{1}{3}, \]
\[ b = 2, \]
\[ \Delta x = 0.5, \]
\[ \Delta t = 0.9 \]

and the initial conditions (34) satisfy (1g) and (1h) with

\[ p = q = 2. \]

It is also assumed that the above system is subjected to parameter uncertainties of the form (1c)-(f) with

\[ L = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \]
\[ M_{11} = \begin{bmatrix} 0.0005 & 0 \end{bmatrix}, \]
\[ M_{12} = \begin{bmatrix} 0 & 0.005 \end{bmatrix}, \]
\[ M_{21} = 0, \]
\[ M_{22} = -0.007. \]

Associated with the uncertain system (33)-(37), the cost function is given by (2) with

\[ Q_1 = \begin{bmatrix} 0.09 & 0 \\ 0 & 0.09 \end{bmatrix}, \]
\[ Q_2 = \begin{bmatrix} 0.9 & 0 \\ 0 & 0.9 \end{bmatrix}, \]
\[ R_1 = R_2 = 0.0025. \]

Applying Lemma 2.1, it is easy to verify that the above system is unstable. We wish to construct a suitable guaranteed cost controller for this system, such that the corresponding cost bound is minimized. To this end, we apply our proposed method (Theorem 3.1) to find the suboptimal guaranteed cost controller. It is found using the LMI toolbox in Matlab [53] that the optimization problem (14) is feasible for the present example and the optimal solution is given by

\[ S = \begin{bmatrix} 5.0381 & -4.5348 \\ -4.5348 & 7.4153 \end{bmatrix}, \]
\[ Y = \begin{bmatrix} 1.2294 & 0.1296 \\ 0.1296 & 1.1950 \end{bmatrix}, \]
\[ U = \begin{bmatrix} 0.3528 & -1.3176 \end{bmatrix}, \]
\[ \varepsilon = 11.0117, \]
\[ \lambda = 0.00121. \]

By Theorem 3.1, the suboptimal guaranteed cost controller for this system is

\[ u(i, j) = \begin{bmatrix} -0.19999 & -0.29999 \end{bmatrix} x(i, j), \]

and the least upper bound of the corresponding closed-loop cost function is

\[ J = 0.02682. \]

5. Conclusions

In this paper, we have presented a method of designing a suboptimal guaranteed cost controller via static-state
feedback for a class of 2-D discrete systems described by the FMSLSS model with norm bounded uncertainties. A suboptimal guaranteed cost controller is obtained through a convex optimization problem which can be solved by using Matlab LMI Toolbox [53]. Application of presented controller design method is demonstrated through processes described by a Darboux equation [3,7,8]. The presented method can also be applied for the robust guaranteed cost controller design for metal rolling control problem [4,9,10].

REFERENCES


