

Conventional and Enhanced Canonical Quantizations, Application to Some Simple Manifolds

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ABSTRACT

It is well known that the representations over an arbitrary configuration space related to a physical system of the Heisenberg algebra allow to distinguish the simply and non simply-connected manifolds [arXiv:quant-ph/9908.014, arXiv:hep-th/0608.023]. In the light of this classification, the dynamics of a quantum particle on the line is studied in the framework of the conventional quantization scheme as well as that of the enhanced quantization recently introduced by J. R. Klauder [arXiv:quant-ph/1204.2870]. The quantum action functional restricted to the phase space coherent states is obtained from the enhanced quantization procedure, showing the coexistence of classical and quantum theories, a fundamental advantage offered by this new approach. The example of the one dimensional harmonic oscillator is given. Next, the spectrum of a free particle on the two-sphere is recognized from the covariant diffeomorphic representations of the momentum operator in the configuration space. Our results based on simple models also point out the already-known link between interaction and topology at quantum level.

Keywords: Heisenberg Algebra; Conventional Quantization; Enhanced Quantization; Non Simply-Connected Manifolds; Interaction; Topology

1. Introduction

Our understanding of the nature through the physics has significantly developed since the advent of the quantum theory and its quantization techniques known as conventional quantization methods, which are essentially the canonical and the path integral quantizations. However, we are far from the end of the story, since numerous questions that are usually called “difficult problems” still remain to be solved.

It's well known that without geometrical and topological considerations, it is difficult to consistently quantize the physical models [1-3]. It's thus indispensable to pay a particular attention to the geometry when one studies physical systems. Moreover, the relationship between interaction and topology is more and more clear [4,5]. Then, in the search of innovative methods of quantization, J. R. Klauder has proposed, by taking into account the topological and geometrical concepts, a new approach of quantization based on a subclass of quantum states, the coherent states [6,7]. These quantum states express themselves by means of classical phase space variables and

thus constitute suitable mediators to realize the link between classical and quantum theories. Indeed, the canonical quantization rules associate to a given classical phase space variable, an operator belonging to a suitable Hilbert space, which must be at least hermitian or symmetric in mathematical parlance. The self-adjoint character is crucial for the observables and von Neumann was the first to put this forward, by proposing a formalism for systematically constructing self-adjoint operators by extensions [8]. However, after such conventional quantization, the quantum action which leads to the Schrödinger equation has nothing to do with the classical action which provides the Euler-Lagrange equations of movement and one thinks that it's one of the causes of inadequacy of conventional quantization procedures.

The enhanced quantization procedure provides a new interpretation of the very process of quantization that encompasses the well-know conventional quantization formalism and offers additional features as well. Obviously, this enhanced quantization formalism aims at improving quantization techniques by means of simple mathemati-

cal tools associated to well-established physical concepts like the principle action at quantum level. It is to be noted that usually, this principle of the action is used in classical theory to determine the Euler-Lagrange equations of motion. Then, these classical variables are promoted to the rank of operators acting on appropriate Hilbert spaces. In the enhanced quantum process, once the domains of self-adjoint operators are well set, one constructs the quantum action that is restricted to the state spaces that are at reach of the real world observer.

In this paper, taking into account the canonical quantization procedure as well as the new method of quantization introduced in Ref. [6] and in the framework of the most general representations of the Heisenberg algebra, we solve two simple models: the particle on the segment and the particle on the two-sphere, with the intention of putting in evidence of the implications of these approaches. As far as the particle on the line is concerned, we first study the case of the free movement and then we consider the presence of a delta-point interaction located at the middle of the segment, making non simply-connected the associated manifold. Concerning the two-sphere model, only the conventional procedure is given by considering the topological (which is trivial here) and the geometrical implications of the Heisenberg algebra. Enhanced quantization on the two-sphere requires the construction of the canonical coherent states on this manifold and is postponed for later.

The paper is organized as follows. In the next section, we point out the topological classification stemming from the representations of the Heisenberg algebra on an arbitrary manifold. In Section 3, we briefly give the importance of representations of the Heisenberg algebra in quantization procedures. In Section 4, we develop the procedures of conventional and enhanced quantizations on the finite non-punctured and punctured lines. The spectrum of the free particle on the two-sphere is given in Section 5. Section 6 encloses the work with some discussions and outlooks.

2. Basics and Consequences of Representations of the Heisenberg Algebra

Let us sketch the results and then recall the main information stemming from the representations of the Heisenberg algebra over an arbitrary manifold: the *topological classification* of these representations. The results given hereafter are obtained from a purely algebraic analysis which is absolutely independent of the specific details and dynamics of a given physical system as defined through its classical Lagrangian (see Ref. [1] for a detailed development); it solely relies on the existence of canonically conjugate pairs of phase space degrees of freedom.

The space of quantum states is assumed to be equipped with an hermitian inner product $\langle \cdot | \cdot \rangle$ and spanned by position eigenstates $|q\rangle$ of the position operators \hat{q}^α ($\alpha = 1, 2, \dots, d$) for all the possible values of q^α as local coordinates over the manifold M so that $\hat{q}^\alpha |q\rangle = q^\alpha |q\rangle$. The general configuration space covariant representation of the Heisenberg algebra determined by the following relations,

$$\begin{aligned} [\hat{q}^\alpha, \hat{p}_\beta] &= i\hbar \delta_\beta^\alpha, [\hat{q}^\alpha, \hat{q}^\beta] = 0 = [\hat{p}_\alpha, \hat{p}_\beta], \\ (\hat{q}^\alpha)^\dagger &= \hat{q}^\alpha, (\hat{p}_\alpha)^\dagger = \hat{p}_\alpha, \end{aligned} \quad (1)$$

is constructed in terms of two structures defined over M , namely its metric structure $g_{\alpha\beta}(q)$ as well as a flat $U(1)$ gauge connection $A_\alpha(q)$.

The metric structure is required to specify the diffeomorphic covariant normalization of the position eigenstates,

$$\begin{aligned} \langle q_2 | q_1 \rangle &= \frac{1}{\sqrt{g(q_2)}} \delta^{(d)}(q_2 - q_1), \\ \mathbf{1} &= \int_M d^d q \sqrt{g(q)} |q\rangle \langle q|, \\ \langle \psi | \phi \rangle &= \int_M d^d q \sqrt{g(q)} \psi^*(q) \phi(q), \end{aligned} \quad (2)$$

with

$$g(q) = \det g_{\alpha\beta}(q), \quad (3)$$

$\psi(q) = \langle q | \psi \rangle$ and $\phi(q) = \langle q | \phi \rangle$ being the configuration space wave functions of the states $|\psi\rangle$ and $|\phi\rangle$. This parametrization of the inner product of the position eigenstate basis $|q\rangle$ comes about, since the consideration of the position matrix elements $\langle q_2 | \hat{q}^\alpha | q_1 \rangle$ of the coordinate operators \hat{q}^α as well as their hermiticity property $(\hat{q}^\alpha)^\dagger = \hat{q}^\alpha$ implies the equations

$(q_2^\alpha - q_1^\alpha) \langle q_2 | q_1 \rangle = 0$, while the choice of absolute normalization in terms of $g(q)$ is made in so that configuration space wave functions $\psi(q) = \langle q | \psi \rangle$ of states transform as scalar fields under diffeomorphisms in M .

The $U(1)$ gauge connection structure $A_\alpha(q)$ is related to the arbitrariness in the choice of phase for the position eigenstates $|q\rangle$, which is not yet fixed by the choice of normalization of their inner products $\langle q_2 | q_1 \rangle$. In fact, a local change in this phase implies a local $U(1)$ gauge transformation of $A_\alpha(q)$,

$$\begin{aligned} |q\rangle &\rightarrow |q\rangle_{(2)} = e^{i\chi(q)/\hbar} |q\rangle, \\ A_\alpha(q) &\rightarrow A_\alpha^{(2)}(q) = A_\alpha(q) + \frac{\partial \chi(q)}{\partial q^\alpha}. \end{aligned} \quad (4)$$

The further restriction for the $U(1)$ connection to be flat, namely $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha = 0$, stems from the condition that all momentum operators \hat{p}_α commute with one another, namely $[\hat{p}_\alpha, \hat{p}_\beta] = 0$. Indeed, the

configuration space representation of these operators is constructed with the parametrization of their position matrix elements as

$$\begin{aligned} \langle q_2 | \hat{p}_\alpha | q_1 \rangle = & -\frac{i\hbar}{g^{1/4}(q_2)} \frac{\partial}{\partial q_2^\alpha} \left(\frac{1}{g^{1/4}(q_2)} \delta^{(d)}(q_2 - q_1) \right) \\ & + \frac{1}{g^{1/4}(q_2)} A_\alpha(q_2) \delta^{(d)}(q_2 - q_1), \end{aligned} \tag{5}$$

leading to the following representation in terms of configuration space wave functions of states and thus involving the $U(1)$ gauge covariant derivative,

$$\langle q | \hat{p}_\alpha | \psi \rangle = -\frac{i\hbar}{g^{1/4}(q)} \left[\frac{\partial}{\partial q^\alpha} + \frac{i}{\hbar} A_\alpha(q) \right] g^{1/4}(q) \psi(q). \tag{6}$$

Likewise, the configuration space representation of the position operators \hat{q}_α is issued by

$$\langle q | \hat{q}_\alpha | \psi \rangle = q^\alpha \psi(q). \tag{7}$$

Consequently, by considering how the vector field $A_\alpha(q)$ transforms under any local change (0.4) of phase in the position eigenstates $|q\rangle$, one finds that this field is actually a $U(1)$ gauge connection related to arbitrariness, whose field strength must be identically vanishing ($F_{\alpha\beta} = 0$) so that the last set of commutation relations $[\hat{p}_\alpha, \hat{p}_\beta] = 0$ defining the Heisenberg algebra will also be obeyed.

In conclusion, all inequivalent unitary representations of the Heisenberg algebra are labeled by all possible gauge equivalence classes of flat $U(1)$ connections over the configuration manifold M . These inequivalent representations are thus in one-to-one correspondence with the $U(1)$ holonomies for all non contractible cycles in M . In the case of a simply-connected manifold, none of these cycles exist, and the Heisenberg algebra thus admits only a single covariant representation over M , which is associated to the trivial choice for a flat connection, $A_\alpha(q) = 0$. This result generalizes to these kinds of curved manifolds, the well-known fact that over a flat Euclidean space, there only exists the usual von Neumann representation of the Heisenberg algebra. However, when the configuration space M is non simply-connected, including examples even as simple as a circle or a torus of arbitrary dimension, there exists an infinite number of inequivalent representations of the Heisenberg algebra, labeled by the $U(1)$ holonomies for all non contractible cycles in M . This includes of course the possibility of vanishing holonomies for all cycles, a situation which then corresponds to the representation with $A_\alpha(q) = 0$ up to the local arbitrariness in the phase of the position eigenstates $|q\rangle$. When non vanishing, such $U(1)$ holono-

mies may be viewed as being associated to an Aharonov-Bohm flux line threading the corresponding cycles in M . Clearly, even though the Heisenberg algebra is local as far as its characteristics are concerned, these results show that a consistent representation depends on the global topology structure of the configuration space manifold M as measured through its $U(1)$ holonomies.

3. The Representations of the Heisenberg Algebra in Quantization Procedures

3.1. Conventional Quantization

Conventional quantization is the one to have been experimented since the very beginning of the modern quantization procedure. It links to classical phase space variables $X(q, p)$, quantum quantities which are required to be at least symmetric or straight out self-adjoint, denoted $X(\hat{q}, \hat{p})$ and belonging to specific Hilbert spaces. Dynamics in the classical space is expressed by means of the basic Poisson bracket¹ $\{q_\alpha, p_\beta\} = \delta_{\alpha\beta}$ while at quantum level, the corresponding commutator is given by $[\hat{q}_\alpha, \hat{p}_\beta] = i\hbar \delta_{\alpha\beta}$.

The kinematical characterization of the quantized system given in the previous section still need to be completed with the specification of a quantum dynamics, namely a quantum Hamiltonian \hat{H} in correspondence with the classical one, given by

$$H(p, q) = \frac{1}{2m} p_\alpha g^{\alpha\beta}(q) p_\beta, \tag{8}$$

p_α being the phase space degrees of freedom conjugate to the local coordinates q^α through the usual canonical Poisson brackets. m is the mass of the particle. The only one possible choice for a diffeomorphic scalar quantum Hamiltonian is given by,

$$\begin{aligned} \hat{H} = & \frac{1}{2m} g^{-1/4}(\hat{q}) [\hat{p}_\alpha + B_\alpha(\hat{q})] g^{1/2} \\ & \times g^{\alpha\beta}(\hat{q}) [\hat{p}_\beta + B_\beta(\hat{q})] g^{-1/4}(\hat{q}), \end{aligned} \tag{9}$$

where the vector field $B_\alpha(\hat{q})$ is given by the following representation

$$B_\alpha(\hat{q}) : -\frac{1}{2} \hbar \sqrt{g(q)} \frac{\partial}{\partial q^\alpha} \left(\frac{1}{\sqrt{g(q)}} \right). \tag{10}$$

Consequently, it's important, taking into account the above topological classification, to pay a required attention to the character of the underlying manifolds to configuration spaces, theatre of physical phenomena. Finally, the quantum Hamiltonian which is the generator a the dynamics of the system admits the following configura-

¹This convention corresponds to bosonic systems. Fermionic systems with Grassmann odd degrees of freedom use their own bracket.

tion space representation, where we omit the hat over the quantum quantities,

$$H : -\frac{\hbar^2}{2m} \frac{1}{\sqrt{g(q)}} \left[\partial_\alpha + \frac{i}{\hbar} A_\alpha(q) \right] \sqrt{g(q)} \times g^{\alpha\beta}(q) \left[\partial_\beta + \frac{i}{\hbar} A_\beta(q) \right], \quad (11)$$

where the ordinary derivatives are replaced by $U(1)$ covariant derivatives. Clearly, the presence of the topological quantum variable A_α shows that one must pay attention to the topological features of the manifold associated to the configuration space, for a given system.

3.2. Enhanced Quantization

Let's now turn ourselves to the recently introduced enhanced canonical quantization² procedure. We do not have the intention to give an exhaustive account on this specific and new method. We only want to sketch the ideas that span this quantization procedure and point out the place of the representations of the Heisenberg algebra in that quantization technique. For more details, the reader could consult the papers [6,7].

It aims at constructing a quantum action which will allow to consistently describe the system in the appropriate quantum space—a specific Hilbert space—as well as in the classical phase space. The ideal quantum space for this procedure is found to be the one spanned by the phase space coherent states which express themselves by means of conjugate quantum variables corresponding to the classical conjugate phase space degrees of freedom (q, p) . Note that these variables are directly at the reach of a classical experimenter and then the corresponding quantum states are restricted to those that can be macroscopically created.

Taking into account the above, one also need to pay attention to the geometrical as well as the topological features of the considered manifold, since the general configuration space covariant representation of the momentum operator, given in the relation (6), is constructed in terms of a flat $U(1)$ gauge connection $A_\alpha(q)$ —a purely quantum degree of freedom—as well as the metric structure $g_{\alpha\beta}(q)$ over this manifold.

Finally, even if the topological degrees of freedom $A_\alpha(q)$ are absent in the differential representations of the momenta and the Hamiltonians commonly encountered in the literature, this corresponds to the particular case of the trivial holonomy and is then included in the general case described above.

²Another alternative to the conventional quantization method has been introduced by the author, namely the *affine quantization* which is intended to play a role in Quantum Gravity.

4. Conventional and Enhanced Canonical Quantizations on the Line

4.1. Conventional Quantization on the Line

Let us concentrate our analysis on the case of a particle moving on a finite line represented by the interval (a_1, a_2) . From the self-adjoint extensions point of view, it's the more general case; semi finite or infinite range domains being straightforward generalizations. Moreover, we shall include the situation in which the finite line is punctured at a spot standing for its middle. The physical interpretation of such model is the following: consider a quantum particle confined in a one-dimensional box submitted to a singular delta-point interaction localized at the center $x_0 : \delta_{x_0} = 0$ if $x \in (a_1, a_2) \setminus \{x_0\}$ and $\delta_{x_0} \rightarrow \infty$ if $x = x_0$. The manifold linked to such a punctured line is no more simply-connected at the opposite of the non punctured line which is simply-connected. Consequently, we must consider the most general representations of operators \hat{p} and \hat{q} given by the relations (6) and (7).

By exploiting the following property of the gauge field $A_\alpha(q)$ on the line, where λ is an a priori real parameter,

$$A dx = \hbar \lambda dx, \quad (12)$$

and taking into account the fact that $g(q) = 1$ here, we get the following representation of the quantum momentum,

$$\hat{p}_\alpha : -i\hbar \left(\frac{\partial}{\partial x} + i\lambda \right), \quad (13)$$

while the spectrum of $\hat{q}_\alpha \equiv X$ is confined in the segment (a_1, a_2) .

The property which is expressed by the relation (13) is interesting. It makes non explicit the presence of the gauge field $A_\alpha(q)$ in the momentum and the hamiltonian operators \hat{p} and \hat{H} , respectively. This gauge potential is represented by the useful quantum variable λ .

Let us continue our analysis by choosing the segment to be centered at the origin of the configuration space *i.e.* $(-a, a)$, $a > 0$, with $x_0 = 0$. Let's consider at quantum level the operators X and P linked up to the classical phase space variables x and p respectively, satisfying the commutation relation $[X, P] = i\hbar$. Moreover, these operators are required to be self-adjoint on the finite interval including its borders and the only possibility which is known to be safe to satisfy this requirement, is to extend these by the well-known means of the von Neumann self-adjoint extensions formalism. Consequently, let's extend the momentum operator

$P \equiv \hat{p}_\alpha$, $\mathcal{D}(P) = L^2(dx, (-a, a))$. It amounts firstly to find a domain $\mathcal{D}_0(P)$ in which the operator $(P, \mathcal{D}_0(P))$ is symmetric and closed; secondly to de-

termine its adjoint $(P^\dagger, \mathcal{D}(P^\dagger))$ with the intention of calculating the deficiency indices of P ; and finally to characterize, if they do exist, the self-adjoint extensions of P .

P is densely defined, since the closure of $\mathcal{D}(P)$ reads $\overline{\mathcal{D}(P)} = L^2(\mathbb{R})$, and is symmetric in the following domain

$$\mathcal{D}_o(P) = \{\phi \in \mathcal{D}(P) / \phi(a) = \phi(-a) = \phi(0) = 0\}. \quad (14)$$

The adjoint of this momentum operator is formally given by $P^\dagger = P$, $\mathcal{D}(P^\dagger) = L^2(dx, (-a, a))$. To determine the deficiency indice numbers, we must count the number of linearly independent solutions, belonging to $L^2(dx, (-a, a))$, of the following equation

$$P^\dagger \phi_\pm(x) = \pm \frac{i\hbar}{d} \phi_\pm(x), \quad P^\dagger : -i\hbar \left(\frac{\partial}{\partial x} + i\lambda \right), \quad (15)$$

where the quantity d is introduced for dimensional regularization. The solutions are readily found to be

$$\phi_\pm(x) = C_\pm e^{\left(\frac{1}{\mp} - i\lambda\right)x}, \quad \text{leading to } \text{def}(P) = (1, 1).$$

Characterize the self-adjoint extensions boils down to determine the boundary conditions which make the operator $(P, \mathcal{D}(P))$ self-adjoint everywhere on the punctured segment, including its border. We know from the von Neumann's theory [8-10] that self-adjoint extensions of P are parametrized by a $U(1)$ matrix, *i.e.* by a phase factor $e^{i\theta}$, $\theta \in [-\pi, +\pi]$. We have to enlarge $\mathcal{D}_o(P)$ and decrease $\mathcal{D}(P^\dagger)$ so that they coincide. By denoting $P_\theta = (P, \mathcal{D}_\theta)$ the extended momentum operator, we get

$$\begin{aligned} \mathcal{D}_\theta &= \{\phi \in \mathcal{D}(P) = L^2(dx, (-a, a)), \\ &\phi(-a) = e^{i\theta} \phi(0), \phi(0) = e^{i\theta} \phi(a)\}. \end{aligned} \quad (16)$$

By considering the above, we readily find the spectrum of the momentum operator on the punctured segment where C is a non-vanishing constant:

$$\begin{aligned} P\phi(x) &= p_{\lambda,\nu} \phi(x), \quad \phi(x) = Ce^{i\left(\frac{p_{\lambda,\nu}}{\hbar} - \lambda\right)x}, \\ p_{\lambda,\nu} &= \hbar(\lambda - \nu), \quad \nu = \theta/a. \end{aligned} \quad (17)$$

Note that in the absence of the point interaction, the purely topological quantum variable disappears, since the manifold associated to the segment is simply-connected. Consequently, $\lambda = 0$ and the self-adjoint boundary condition reduces to $\phi(-a) = e^{i\alpha} \phi(a)$ leading to the P -spectrum issued by $\phi_o(x) = C \exp\left(\frac{i}{\hbar} \gamma x\right)$, $p_\gamma = \hbar \gamma$,

$$\gamma = \frac{\alpha}{2a}, \quad \alpha \in [-\pi, \pi].$$

Let us deal with the determination of the spectrum of the Hamiltonian. The deficiency indices of this operator

are finite and equal (2,2). All self-adjoint extensions of the hamiltonian are characterized by four parameters.

We pay attention to the parity symmetry whose the singular point interaction localized at $x_o = 0$ must be respectful. This parity expresses as [11]: ${}^P \psi(u) = \psi(-u)$, where u stands for the space coordinate. It reduces the number of parameters of self-adjoint extensions. Consequently, the self-adjoint boundary conditions reduce to the following, with the condition $a^2 - bc = 0$;
 $a, b, c \in \mathbb{R}$.

$$\begin{pmatrix} \psi_I \\ \partial_x \psi_I \end{pmatrix} = \begin{pmatrix} a & b \\ c & a \end{pmatrix} \begin{pmatrix} \psi_{II} \\ \partial_x \psi_{II} \end{pmatrix}. \quad (18)$$

In this notation, the wave functions ψ_I and ψ_{II} defined in $(-a, 0)$ and $(0, +a)$ respectively, are given by the following expressions,

$$\psi_I(x) = \frac{1}{\sqrt{a}} e^{i\sqrt{\lambda}x}, \quad \psi_{II}(x) = \frac{1}{\sqrt{a}} e^{-i\sqrt{\lambda}x}. \quad (19)$$

The energy levels of the system are given by

$$E_{\lambda,\nu=0} = \frac{\hbar^2}{2m} \lambda^2. \quad (20)$$

Finally, if we didn't pay attention to the parity symmetry, the energy levels would be given by

$$E_{\lambda,\nu} = \frac{\hbar^2}{2m} (\lambda - \nu)^2$$

and the self-adjoint boundary conditions would be written as follows, with $ad - bc = 0$; $a, b, c, d \in \mathbb{R}$.

$$\begin{pmatrix} \psi_I \\ \partial_x \psi_I \end{pmatrix} = e^{i\theta} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \psi_{II} \\ \partial_x \psi_{II} \end{pmatrix}. \quad (21)$$

The resulting spectrum in the absence of the point interaction (which means that the particle moves freely from $-a$ to $+a$) is rather straightforward.

4.2. Enhanced Canonical Quantizations for the Free Particle on the Line

4.2.1. Coherent States-Restricted Quantum Action Functional

Let us define the eigenstates $|x\rangle$ and $|n_\gamma\rangle$ related to the operators X and P_γ respectively, satisfying the relations

$$\langle x|x'\rangle = \delta_{[-a,a]}(x - x'), \quad \langle n_\gamma | m_\gamma \rangle = \delta_{n,m}. \quad (22)$$

We have

$$\begin{aligned} X|x\rangle &= x|x\rangle, \quad P_\gamma |n_\gamma\rangle = p_\gamma |n_\gamma\rangle, \\ \langle x|P_\gamma |n_\gamma\rangle &= (-i\hbar) \partial_x \langle x|n_\gamma\rangle = p_\gamma \langle x|n_\gamma\rangle. \end{aligned} \quad (23)$$

The canonical coherent states are introduced by the means of the following unitary operators

$$e^{-\frac{ixP_\gamma}{\hbar}}, e^{\frac{ipX}{\hbar}}, (x, p) \in [-a, a] \times \mathbb{R}. \quad (24)$$

These coherent states are given by the following relation [12],

$$|p, x\rangle = e^{-\frac{i}{\hbar}xP_\gamma} e^{\frac{i}{\hbar}pX} |\eta_\gamma\rangle, \tag{25}$$

where $|\eta_\gamma\rangle$ is a ket standing for the fiducial state. A straightforward computation shows that the states $\{|p, x\rangle\}$ satisfy the following normalization and the unit resolution conditions

$$\langle p, x | p, x \rangle = \langle \eta_\gamma | e^{-\frac{i}{\hbar}pX} e^{\frac{i}{\hbar}xP_\gamma} e^{-\frac{i}{\hbar}xP_\gamma} e^{\frac{i}{\hbar}pX} |\eta_\gamma\rangle = \langle \eta_\gamma | \eta_\gamma \rangle = 1, \tag{26}$$

$$\int_{\mathbb{R}^2} |p, x\rangle \langle p, x| \frac{dp dx}{2\pi\hbar} = \mathbb{I}. \tag{27}$$

The set of states $\{|p, x\rangle\}$ provides a family of over-complete normalized states and stands for the set of phase space coherent states.

We are now able to discuss the dynamics in the space spanned by these coherent states. Let us introduce the quantum generator that constitutes the Hamiltonian $\mathcal{H}(P, e^{iX}, e^{-iX})$. Let's also consider the quantum action restricted to the coherent states $|p(t), x(t)\rangle$ instead of arbitrary states $|\psi(t)\rangle$, defined as follows

$$A_{Q(R)} = \int_0^T \langle p(t), x(t) | [i\hbar\partial_t - \mathcal{H}] | p(t), x(t) \rangle dt. \tag{28}$$

We chose a class of fiducial states satisfying $\langle \eta_\gamma | X | \eta_\gamma \rangle = 0$ and $\langle \eta_\gamma | P_\gamma | \eta_\gamma \rangle = \hbar\gamma$, so that we have

$$\langle p(t), x(t) | [i\hbar\partial_t] | p(t), x(t) \rangle = (\hbar\gamma + p) \dot{x}. \tag{29}$$

The restricted quantum action becomes

$$A_{Q(R)} = \int_0^T \left\{ [p\dot{x} - H_\gamma(p(t), x(t))] + \hbar\gamma\dot{x} \right\} dt, \tag{30}$$

where $H_\gamma(p(t), x(t)) = H_{c,\gamma} + O(\hbar, p(t), x(t))$. In this equation, $H_{c,\gamma}$ is the classical Hamiltonian. The quantum parameter γ induces a surface term $\hbar\gamma\dot{x}$ in $A_{Q(R)}$ which does not have any influence on the classical equations of motion. Hence we may write

$$A_{Q(R)} = A_c + O(\hbar). \tag{31}$$

The evaluation of the diagonal elements of the matrix representation of \mathcal{H} in the coherent states basis, setting $\gamma' = \hbar\gamma$, gives

$$\begin{aligned} & \langle p(t), x(t) | \mathcal{H}(P_\gamma, e^{iX}, e^{-iX}) | p(t), x(t) \rangle \\ &= \langle \eta_\gamma | \left((P_\gamma + p)^2 + V(e^{i(X+x)}, e^{-i(X+x)}) \right) | \eta_\gamma \rangle \\ &= (\gamma' + p)^2 - \gamma'^2 + \langle \eta_\gamma | P_\gamma^2 | \eta_\gamma \rangle + a_0 \\ & \quad + \sum_{n=1}^m [a_n \cos nx + b_n \sin nx] + O(\hbar) \\ &= (p + \gamma')^2 + V(e^{ix}, e^{-ix}) + O(\hbar), \end{aligned} \tag{32}$$

where the constants $\langle \eta_\gamma | P_\gamma^2 | \eta_\gamma \rangle$ and γ'^2 are included in $O(\hbar)$. We deduce that

$$A_{Q(R)} = \int_0^T \left\{ (p + \gamma') \dot{x} - \left[(p + \gamma')^2 + V(e^{ix}, e^{-ix}) \right] + O(\hbar) \right\} dt. \tag{33}$$

with the canonical following change of the momentum, $\tilde{p} = p + \gamma'$ we finally obtain

$$A_{Q(R)} = \int_0^T \left\{ \tilde{p} \dot{x} - \left[\tilde{p}^2 + V(e^{ix}, e^{-ix}) \right] \right\} dt + O(\hbar) = A_c + O(\hbar). \tag{34}$$

Let us give the concrete example of the one-dimensional harmonic oscillator of mass m . The restricted quantum action is the following, $|0\rangle$ being the Fock vacuum state,

$$\begin{aligned} A_{Q(R)} &= \int_0^T \langle q, p | [i\hbar\partial_t - \mathcal{H}] | q, p \rangle dt, |q, p\rangle \\ &= e^{-\frac{i}{\hbar}q\hat{p}} e^{\frac{i}{\hbar}p\hat{q}} |0\rangle, \\ \mathcal{H} &= \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{q}^2. \end{aligned} \tag{35}$$

By the means of the Baker-Campbel-Hausdorf formula and after some lines of computation, one obtains

$$\begin{aligned} A_{Q(R)} &= \int_0^T \left\{ [p\dot{q} - H(p, q)] - \frac{\hbar\omega}{2} \right\} dt, \\ H(p, q) &= \frac{p^2}{2m} + \frac{1}{2}m\omega^2q^2, \end{aligned} \tag{36}$$

which gives

$$A_{Q(R)} = A_c + O(\hbar), \quad O(\hbar) \sim \frac{\hbar\omega}{2} \Delta T. \tag{37}$$

4.2.2. The Induced Coherent Phase Space Geometry

The geometrical characterization of the constructed coherent states phase space is encoded by the following metric [13]

$$\tilde{c}_\gamma ds_\gamma^2 = \|d|p, x\rangle\|^2 - |\langle p, x | d|p, x\rangle|^2, \tag{38}$$

where $\tilde{c}_\gamma = c_\gamma / \hbar^2$, and

$$d|p, x\rangle = -\left(\frac{i}{\hbar}\right) [dx(P_\gamma + p) - Xdp] e^{-\frac{i}{\hbar}xP_\gamma} e^{\frac{i}{\hbar}pX} |\eta_\gamma\rangle, \tag{39}$$

$$\|d|p, x\rangle\|^2 = \frac{1}{\hbar^2} [dx^2 (D_\gamma' + 2p\gamma' + p^2) + dp^2 D_\gamma], \tag{40}$$

$$\begin{aligned} |\langle p, x | d|p, x\rangle|^2 &= \frac{1}{\hbar^2} |\langle \eta_\gamma | [dx(P_\gamma + p) - Xdp] | \eta_\gamma \rangle|^2 \\ &= \frac{1}{\hbar^2} (p + \gamma')^2 dx^2, \end{aligned} \tag{41}$$

where $D_\gamma := \langle \eta_\gamma | X^2 | \eta_\gamma \rangle$ and $D'_\gamma := \langle \eta_\gamma | P_\gamma^2 | \eta_\gamma \rangle$ are constants. Finally, we get

$$c_\gamma ds_\gamma^2 = (D'_\gamma - \gamma'^2) dx^2 + D_\gamma dp^2, \tag{42}$$

and by setting

$$x_\gamma = \sqrt{|A_\gamma|} x, \quad c_\gamma = D_\gamma, \quad A_\gamma = \frac{D'_\gamma - \gamma'^2}{D_\gamma}, \tag{43}$$

the metric reads

$$ds_\gamma^2 = A_\gamma dx^2 + dp^2 = dx_\gamma^2 + dp^2. \tag{44}$$

The metric ds_γ^2 describes the geometry of an infinite range strip of width $2a$ and can serve in a classical description as well as in a quantum description.

4.3. Enhanced Canonical Quantizations on the Line with a Point Interaction

Let us now apply the above formalism to a particle moving on a segment with a δ -point interaction at the right middle $x_o = 0$ of this finite line. The corresponding manifold which is non simply-connected is the one associated to the punctured line.

The canonical coherent states are given by

$$|p, x\rangle = e^{\frac{-i}{\hbar}xp} e^{\frac{i}{\hbar}pX} |\eta_{\lambda,\nu}\rangle, \quad p \equiv p_{\lambda,\nu} = \hbar(\lambda - \nu). \tag{45}$$

By choosing the fiducial orthonormalized states $|\eta_{\lambda,\nu}\rangle$ such as $\langle \eta_{\lambda,\nu} | X | \eta_{\lambda,\nu} \rangle = 0$ and $\langle \eta_{\lambda,\nu} | P | \eta_{\lambda,\nu} \rangle = \hbar(\lambda - \nu)$, the coherent states quantum action restricted to the phase space is given by the following expression,

$$\begin{aligned} A_Q(R) &= \int_0^T dt \langle p(t), x(t) | [i\hbar\partial_t - \mathcal{H}] | p(t), x(t) \rangle \\ &= \int_0^T dt \{ [p\dot{x} - H_{\lambda,\nu}(p(t), x(t))] + \hbar(\lambda - \nu)\dot{x} \}, \end{aligned} \tag{46}$$

with the notation

$$\langle p(t), x(t) | \mathcal{H} | p(t), x(t) \rangle = H_{\lambda,\nu}(p(t), x(t)). \tag{47}$$

We recognize in the relation (0.46) the classical action $A_C = \int_0^T dt [p\dot{x} - H_{\lambda,\nu}(p(t), x(t))]$, while the surface term $\hbar(\lambda - \nu)\dot{x}$ is characterized by the nontrivial holonomy parameter λ as well as the self-adjoint parameter ν . We consequently have the expected relation which expresses the coexistence of classical as well as the quantum description of the considered system,

$$A_Q(R) = A_C + O(\hbar). \tag{48}$$

The induced coherent phase space geometry is provided by the metric element

$$\tilde{C}_{\lambda,\nu} dS_{\lambda,\nu}^2 = \| |d|p, x\rangle \|^2 - \langle |p, x|d|p, x\rangle \rangle^2, \quad \tilde{C}_{\lambda,\nu} = \frac{C_{\lambda,\nu}}{\hbar^2}. \tag{49}$$

After some lines of computations, one finds

$$\begin{aligned} dS_{\lambda,\nu}^2 &= K_{\lambda,\nu}^1 dx^2 + K_{\lambda,\nu}^2 dp^2, \\ K_{\lambda,\nu}^1 &= (D'_{\lambda,\nu} - \hbar^2(\lambda - \nu)) / C_{\lambda,\nu}, \\ K_{\lambda,\nu}^2 &= (D_{\lambda,\nu} / C_{\lambda,\nu}), \end{aligned} \tag{50}$$

with the notations $D_{\lambda,\nu} := \langle \eta_{\lambda,\nu} | X^2 | \eta_{\lambda,\nu} \rangle$ and $D'_{\lambda,\nu} := \langle \eta_{\lambda,\nu} | P_\gamma^2 | \eta_{\lambda,\nu} \rangle$. A redefinition of coordinates gives the metric on the subspace of coherent states,

$$dS_{\lambda,\nu}^2 = dx_{\lambda,\nu}^2 + dp_{\lambda,\nu}^2, \quad x_{\lambda,\nu} = K_{\lambda,\nu}^1 x, \quad p_{\lambda,\nu} = K_{\lambda,\nu}^2 p. \tag{51}$$

5. Eigenstates of a Free Particle on the Two-Sphere

As the simplest next example, in the same spirit, let us now consider the case of a particle moving freely on the two-sphere of radius R . Working in spherical coordinates $(r = R, \theta, \varphi)$ with $0 \leq \varphi < 2\pi$ and $0 \leq \theta < \pi$, the invariant line element of this configuration space then reads

$$ds^2 = g_{\alpha\beta}(q) \dot{q}^\alpha \dot{q}^\beta = R^2 d\theta^2 + R^2 \sin^2\theta d\varphi^2, \tag{52}$$

thereby defining the metric structure $g_{\alpha\beta}(\theta, \varphi)$, such that

$$\sqrt{g(q)} = \sqrt{\det g_{\alpha\beta}(q)} = R^2 \sin\theta. \tag{53}$$

This space being simply-connected, there only exists a single representation of the associated Heisenberg algebra, namely that in the trivial $U(1)$ holonomy class $A_\alpha(\theta, \varphi) = 0$. Consequently, the configuration space representation is given by the basis of positions eigenstates $|\theta, \varphi\rangle$ such that

$$\langle \theta, \varphi | \theta', \varphi' \rangle = \frac{1}{R^2 \sin\theta} \delta(\theta - \theta') \delta(\varphi - \varphi'), \tag{54}$$

$$\mathbf{1} = R^2 \int_0^\pi d\theta \sin\theta \int_0^{2\pi} d\varphi |\theta, \varphi\rangle \langle \theta, \varphi|,$$

Thus implying the following inner product for configuration space wave functions of states,

$$\langle \psi | \chi \rangle = R^2 \int_0^\pi d\theta \sin\theta \int_0^{2\pi} d\varphi \chi(\theta, \varphi) \psi^*(\theta, \varphi), \tag{55}$$

while the momentum operators are issued by the following configuration space representations,

$$\langle \theta, \varphi | P_\theta | \psi \rangle = \frac{-i\hbar}{(\sin^2\theta)^{\frac{1}{4}}} \frac{\partial}{\partial\theta} \left[\sin^{\frac{1}{2}}\theta \psi(\theta, \varphi) \right], \tag{56}$$

$$\langle \theta, \varphi | P_\varphi | \psi \rangle = -i\hbar \frac{\partial}{\partial\varphi} \psi(\theta, \varphi). \tag{57}$$

Within this configuration space representation, the general diffeomorphic invariant quantum Hamiltonian in (9) corresponds in the present case to the following dif-

ferential operator, acting on configuration space wave functions $\psi(\theta, \varphi)$,

$$H : -\frac{\hbar^2}{2m_0 R^2} \left[\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \varphi^2} \right]. \quad (58)$$

The configuration space wave functions of the energy eigenstates of this system are of course nothing else than the spherical harmonics, labeled by the angular momentum values $\ell = 0, 1, 2, \dots$ and the integer magnetic moment values $-\ell \leq m \leq \ell$ of its projection on the angular momentum quantization axis. The corresponding states $|\ell, m\rangle$ have their wave functions given by $\langle \theta, \varphi | \ell, m \rangle = \psi_\ell^m(\theta, \varphi)$ such as

$$\psi_\ell^m(\theta, \varphi) = \sqrt{\frac{2\ell+1}{4\pi R^2} \frac{(\ell-|m|)!}{(\ell+|m|)!}} e^{im\varphi} P_\ell^{|m|}(\cos \theta), \quad (59)$$

$P_\ell^{|m|}(\cos \theta)$ being the associated Legendre functions [14]. The normalization of these states is chosen such that $\langle \ell, m | \ell', m' \rangle = \delta_{\ell, \ell'} \delta_{m, m'}$, while their energy eigenvalues read

$$\hat{H} |\ell, m\rangle = E_\ell |\ell, m\rangle, \quad E_\ell = \frac{\hbar^2}{2m_0 R^2} \ell(\ell+1), \quad (60)$$

with the usual degeneracy in $-\ell \leq m \leq \ell$; $\ell = 0, 1, 2, \dots$, due to the $SO(3)$ symmetry of the sphere.

It is to be noted that if this spectrum is well known, the method used here which is based on the most general representation of the Heisenberg algebra is not common although it is associated to a trivial holonomy in this case.

Let us finally sketch the spectrum of the momentum operator P . The corresponding wave function $\Phi(\theta, \varphi) = \langle \theta, \varphi | P_\theta, P_\varphi \rangle = F(\varphi) \times T(\theta)$ is obtained by solving the following differential equations,

$$\frac{\partial}{\partial \theta} \left[(\sin^2 \theta)^{\frac{1}{4}} \Phi(\theta, \varphi) \right] = \frac{i}{\hbar} P_\theta (\sin^2 \theta)^{\frac{1}{4}} \Phi(\theta, \varphi), \quad (61)$$

$$\frac{\partial}{\partial \varphi} \left[\Phi(\theta, \varphi) \right] = \frac{i}{\hbar} P_\varphi \Phi(\theta, \varphi), \quad (62)$$

which gives the following P -wave function, properly normalized by the means of conditions in (0.54),

$$\Phi(\theta, \varphi) = \langle \theta, \varphi | p_\theta, p_\varphi \rangle = \frac{1}{\pi R \sqrt{2}} \frac{e^{if_o}}{(\sin^2 \theta)^{\frac{1}{4}}} e^{\frac{i}{\hbar}(\theta p_\theta + \varphi p_\varphi)}, \quad (63)$$

where f_o is an arbitrary function. The corresponding eigenvalues are given by

$$P |p_\theta, p_\varphi\rangle = p |p_\theta, p_\varphi\rangle, \quad p_\theta = \hbar(2n+1), \quad p_\varphi = \hbar m; \quad n, m \in \mathbb{Z} \quad (64)$$

6. Discussions and Outlooks

In this paper, we have constructed the spectra of some simple models, namely a particle on the line and on the two-sphere, within the framework of the well-known conventional quantization as well as that of the recently introduced promising enhanced quantization procedure. The model described by the particle on the finite line is extended by the addition of a delta-point interaction which, mathematically, boils down to consider a punctured line which is related to a non simply-connected manifold. It appeared that the representations of the Heisenberg algebra—with their topological classifications—are indissociable of such constructions if one needs to be exhaustive. We found that the quantum action functional restricted to phase space coherent states, is expressed as the classical action plus \hbar -corrections. In this view, classical theory appears clearly as a subset of quantum theory, and they both co-exist as in the real world where $\hbar > 0$. Thus, the main result of the enhanced canonical quantization is that, we have constructed a set of coherent states which serve to remove the barrier between classical and quantum theories. This result adds an example to the existing ones [6,15], extending then the explorations of the implications of this new theory. Many confined systems could be concerned by our study like harmonic oscillators in the physical one-dimensional space.

Let's discuss some further potential applications of the new tool that constitutes the enhanced quantization, in providing some intuitive models that could be deciphered by the means of this procedure. Our first example is the famous Aharonov-Bohm (AB) effect. It's well-known that this effect occurs only at quantum level and is essentially topological. Hence the enhanced canonical quantization that gives to a quantum action combining the classical as well as the quantum terms, could be useful to verify if this AB effect is only quantum and nonobservable at classical level and is topological. As a matter of fact, it is the case. Indeed, it has been proved in Refs. [1,5] that the quantum Hamiltonian recovered in paying the required attention to the nontrivial holonomy associated to the punctured Euclidean plane is mathematically equivalent to that associated to the AB effect and the suitable representation of the momentum operators is provided in polar coordinates (r, θ) by

$p_r : -i \frac{\hbar}{\sqrt{r}} \partial_r \sqrt{r}$, $p_\theta : -i\hbar(\partial_\theta + i\chi)$. The quantum parameter χ being the representative of the magnetic field or precisely the vector gauge potential. It appears that this physical phenomenon observes really only at quantum level. It suffices to observe the expression of the restricted to phase space quantum action obtained on the circle which has a similar topology to that of the punctured Euclidean plane [15]. Explicitly, the holonomy

parameter λ which induces the surface term in the restricted enhanced quantum action makes no influence on the enhanced classical equation of motion whatsoever (see P. 4 of Ref. [15]). Furthermore, this so-called enhanced quantization procedure has been conceived some years ago even though it has not been called enhanced quantization and its methods were applied to several physical systems (see Ref. [16] for an account).

We note that the presence of certain interactions can change the considered topology of the manifold associated to a physical configuration space; this is actually the case of singular delta interactions as our study has shown. Among these, we also have the delta-sphere interactions. These interactions which were introduced first by Green and Moszowski [17] have intensively been used in nuclear, molecular as well as in solid states physics. Let us mention that one should study any Hamiltonian containing delta interactions by considering the most general diffeomorphic covariant representations of the momentum operator given in the relation (6). Ignoring this and using the commonly encountered textbook representation $\hat{p}_\alpha : -i\hbar\nabla$, reduces the study to the trivial case ($A_\alpha(q) = 0$). Likely, it has been proved [1,5] that a punctured hole in the bidimensional Euclidean plane induces mathematically, at quantum level, the Aharonov-Bohm flux line threading this plane. This hole makes non simply-connected the manifold related to the plane and creates, at quantum level, a magnetic interaction via the nontrivial holonomy quantum variable which possesses the properties of the magnetic vector gauge potential. In this connection, let's recall the spectrum on the circle [4, 15]. In conventional quantization, the P -spectrum is given by

$$\Phi(q) = \frac{1}{\sqrt{2L}} e^{i q \left(p - \frac{\pi\hbar}{L} \lambda \right)}, \quad p_m = \frac{\pi\hbar}{L} (m + \lambda) \quad (65)$$

while the H -spectrum is issued by

$$\begin{aligned} \psi(q) = & A \exp \left\{ i \left(\sqrt{\frac{2ME}{\hbar^2}} - \frac{\pi\lambda}{L} \right) q \right\} \\ & + B \exp \left\{ -i \left(\sqrt{\frac{2ME}{\hbar^2}} + \frac{\pi\lambda}{L} \right) q \right\}, \quad (66) \\ E_m = & \frac{\pi^2 \hbar^2}{2ML^2} (m + \lambda)^2, \quad m \in \mathbb{Z} \end{aligned}$$

$2L$ being the circumference of this circle and A, B are constants. At the beginning, the particle of mass M is supposed to be free of any interaction. But after quantization, it occurs that the quantum parameter λ in this spectrum which renders mathematically the influence of an Aharonov-Bohm flux line on this particle. The situation is analogous to that of the enhanced quantization on the circle [15]. It thus appears that the quantum mechan-

ics on non simply-connected manifolds possesses hidden quantum degrees of freedom, which necessarily must also govern in an essential way the quantum physics properties of such systems. Finally, at quantum level, the topology is able to generate an interaction, and interactions can modify the topology. It would be worth looking at how the enhanced quantization applied to the two-sphere which is a system invariant under rotations in three dimensional physical space, whose quantum states thus fall into specific $SO(3)$ spin representations, providing probably some complications in the phase space coherent states in this case. A small simulation leads us to the following steps. One has first to extend the operators p_θ and p_φ which are the conjugate momenta associated to the position variables θ and φ respectively so that they are self-adjoint on the sphere. This first step is rather straightforward and $\text{def}(\hat{p}) = (1,1)$ leading to a self-adjoint boundary condition characterized by a phase factor for this operator. The second step is the construction of the phase space coherent states, that could be noted $|\mathbf{r}, \mathbf{p}\rangle$, $\mathbf{r} = (\theta, \varphi)$, $\mathbf{p} = (p_\theta, p_\varphi)$. This crucial step could be made possible by inspiring oneself of the work by Kowalski *et al.* [18]. In this paper, the authors have discussed such a construction similarly to the case of the circle [15,19]. Then, the restricted quantum action could be computed for the model. The two-torus could also provide a good laboratory for testing these methods.

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