Mathematical Derivation of Angular Momenta in Quantum Physics

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Abstract
For a two-dimensional complex vector space, the spin matrices can be calculated directly from the angular momentum commutator definition. The 3 Pauli matrices are retrieved and 23 other triplet solutions are found. In the three-dimensional space, we show that no matrix fulfills the spin equations and preserves the norm of the vectors. By using a Clifford geometric algebra it is possible in the four-dimensional spacetime (STA) to retrieve the 24 different spins 1/2. In this framework, spins 1/2 are rotations characterized by multivectors composed of 3 vectors and 3 bivectors. Spins 1 can be defined as rotations characterized by 4 vectors, 6 bivectors and 4 trivectors which result in unit multivectors which preserve the norm. Let us note that this simple derivation retrieves the main spin properties of particle physics.

Keywords: Quantum Systems; Spin 1/2; Spin 1; Particle Physics, Spacetime Algebra

1. Introduction
The spin of particles was discovered by Wolfgang Pauli toward the end of 1924 (see Fröhlich’s paper for an up-to-date discussion of spins [1]). In quantum mechanics textbooks, the matrix representation of spin is obtained from the commutator general definition of the angular momentum:

\[ [J_x, J_y] = i\hbar J_z \]
\[ [J_y, J_z] = i\hbar J_x \]
\[ [J_z, J_x] = i\hbar J_y \]

These 3 equations are called the spin equations.

The general theory of quantum angular momenta [2] uses the operator \( J^2 = J_x^2 + J_y^2 + J_z^2 \), the two shift operators \( J_x = J_x + iJ_y \), \( J_y = J_x - iJ_y \) and the positive eigenvalue \( j(j+1)\hbar \) of \( J^2 \). The values for \( j \) are 0, 1, 2, 3, 4, ... For \( j = 0 \), the spin space is of dimension 1 and reduced to a scalar equal to zero. For \( j = 1/2 \), the spin space is of dimension 2 and the matrix representation is given by

\[ J_x = \frac{1}{2} \hbar \sigma_x, J_y = \frac{1}{2} \hbar \sigma_y, J_z = \frac{1}{2} \hbar \sigma_z \] (4)

where \( \sigma_i \) are the Pauli matrices:

\[ \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \] (5)

These three \( J \) matrices describe the so called spin 1/2 quantum system.

The aim of this work is to address two questions: can the Pauli matrices or more generally any spin system be retrieved directly from the angular momentum definition and are the 3 Pauli matrices the only 2-dimensional quantum angular momenta?

2. Properties of the Minimum-Dimension Quantum Spin System
The smallest quantum angular momentum is of dimension 3 because there are 3 elements \( J_x, J_y, J_z \). As quantum mechanics acts on complex vector spaces, the minimum dimension is a two-dimensional complex space or a four-dimensional real space with two elements multiplied by \( i = \sqrt{-1} \) which is different from a true four-dimensional real space.

2.1. Matrix Description
The two components of the vectors of such a space can be written as \( (x_1 + ix_2, x_3 + ix_4) \) or in matrix notation:

\[ \begin{pmatrix} x_1 + ix_2 \\ x_3 + ix_4 \end{pmatrix} = \begin{pmatrix} \rho e^{i\theta} \\ r_2 e^{i\phi} \end{pmatrix} \] (6)
which corresponds to an ordered set of four real elements.

An operator in such a vector space is given by

$$M = \begin{bmatrix} a_{ij} \end{bmatrix}$$

a real $2 \times 2$ matrix multiplied by a complex number $a$. It can be noticed that the two parts of $a$ provide the information about the position of $i$ in the four real number set $a_{ij}$.

Quantum mechanics only studies operators which preserve the norm, and in two dimensions this is written as:

$$\alpha^* \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \alpha = \begin{pmatrix} v_1 + iv_2 \\ v_3 + iv_4 \end{pmatrix} = \begin{pmatrix} v'_1 + iv'_2 \\ v'_3 + iv'_4 \end{pmatrix}$$

with

$$v_1^2 + v_2^2 + v_3^2 + v_4^2 = v'_1^2 + v'_2^2 + v'_3^2 + v'_4^2$$

The solution is:

$$\alpha^* \begin{pmatrix} a_{11} + a_{21}^2 \\ a_{21} \end{pmatrix} = 1$$

$$\alpha^* \begin{pmatrix} a_{12} + a_{22}^2 \\ a_{22} \end{pmatrix} = 1$$

$$a_{11}a_{12} + a_{21}a_{22} = 0$$

which corresponds to

$$(\alpha M)^\dagger (\alpha M) = [1]$$

This result can be easily generalized to quantum vector spaces of any dimension. We see that $\alpha M$ is not a unitary matrix because it fulfills only one of the two conditions of a unitary matrix: $U^\dagger U = U^T U = [1]$. This implies that quantum operators need not be unitary in order to preserve the norm in the vector space contrary to frequently held views.

The problem studied was to find the 12 real elements of the three following matrices:

$$J_k = \alpha_k \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix}$$

with the three $\alpha_k$ complex numbers for $k = x, y, z$. The subscripts have presenty nothing to do with the space coordinates.

From Equation (1), it follows that the matrix $J_z$ is:

$$J_z = \frac{\alpha x \alpha y}{i\hbar} \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix}$$

with

$$z_{11} = x_{12}y_{21} - x_{21}y_{12}$$

$$z_{21} = y_{12}(x_{11} - x_{21}) + x_{12}y_{11}$$

$$z_{22} = x_{12}y_{21} - x_{21}y_{12}$$

which shows that $z_{22} = -z_{11}$. More generally it is well known that the commutator of two matrices results in a matrix with a null trace. As $J_x$ and $J_y$ are defined by a commutator, we also have $x_{22} = -x_{11}$ and $y_{22} = -y_{11}$.

From Equation (2) it follows that the matrix $J_z$ is:

$$J_z = \frac{\alpha x \alpha y}{i\hbar} \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$$

With

$$x_{11} = \frac{-2y_{12}y_{21}x_{11} + 2x_{12}y_{12} + x_{12}y_{21}}{2y_{12}y_{21} - \hbar^2}$$

$$x_{12} = \frac{2y_{12}^2 y_{21} + 4y_{12} y_{11} y_{12} - \hbar^2}{4y_{12}y_{11} + 2y_{12}y_{21} - \hbar^2}$$

$$x_{21} = \frac{2y_{12}^2 y_{21} - y_{12} y_{21}}{4y_{12}y_{11} + 2y_{12}y_{21} - \hbar^2}$$

From Equation (3) it follows that the matrix $J_y$ has the same expression as $J_x$, just by exchanging $y$ and $x$. Calculating $x_{12}y_{12} + x_{21}y_{21}$ by eliminating $x_{11}$ gives a relation between the $J_y$ elements:

$$4y_{12}^2 \left(2y_{12}y_{21} - \hbar^2\right) - 4y_{12}y_{11} \hbar^2 + \hbar^4 = 0$$

As $J_y$ is a quantum operator we have the general constraints given by Equations (9)-(11) on the $J_y$ elements:

$$y_{11}^2 + y_{22}^2 = \frac{1}{|\alpha|} = K$$

$$y_{12}^2 + y_{11}^2 = \frac{1}{|\alpha|} = K$$

$$y_{11}(y_{12} - y_{21}) = 0$$

where $K$ is a positive real constant.

Equation (19) implies that $y_{11} = 0$ or $y_{12} - y_{21} = 0$.

For $y_{11} = 0$, Equations (17) and (18) imply that $y_{21} = y_{12} = K$, therefore $y_{21} = \pm y_{12}$.

For $y_{11} = 0$ and $y_{21} = y_{12}$, Equation (16) gives $y_{12} = \pm \hbar/2$ and we obtain the second Pauli matrix

$$J_{y_3} = J_{y_2} = \frac{\hbar}{2} \left( \begin{array}{cc} 0 & -i \\ i & 0 \end{array} \right)$$

or its opposite $J_{y_2} = -J_{y_2}$.

For $y_{11} = 0$ and $y_{21} = -y_{12}$, Equation (16) gives $y_{12} = \pm \hbar/2$ and we obtain the first Pauli matrix:

$$J_{y_3} = J_{y_2} = \frac{\hbar}{2} \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right)$$

or its opposite $J_{y_4} = -J_{y_3}$.

The case $y_{11} = 0$ permits to calculate the constant $= \hbar^2/4$. 

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For $y_{11} \neq 0$ and $y_{12} = y_{12}'$, Equation (16) gives
\[ 4y_{11}^2(2y_{12}^2 - h^2) - 4y_{12}'^2h^2 + h^4 = 0 \]
and using Equation (18)
\[ y_{12}^2 + y_{12}'^2 = h^2/4 \]
we obtain $y_{12}' = 0$ and $y_{11} = \pm h/2$ which gives the third Pauli matrix:
\[ J_{yz} = J_z = \frac{h}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \] (22)
or its opposite $J_{yz} = -J_{yz}$. The 6 $J_{ik}$ matrices are $J_x, -J_y, J_y, -J_x, J_z, -J_z$.

The norm preservation and the null trace constraints applied to $J_i$ give the 6 different $J_{ik}$ matrices $J_x, -J_i, iJ_y, -iJ_y, J_z, -J_z$ which fulfill the angular momentum criteria. Let us note that all are real matrices.

All the matrix triplets which fulfill the angular momentum definition in the two-dimensional complex vector space are obtained with Equation (14). For each $J_{ik}$ there are 6 $J_{ik}$ which gives 36 possibilities for $J_z$ which fulfill the 3 Equations (1)-(3). Out of these 36 possibilities only 24 give a nonzero matrix. Obtaining the 24 solutions is straightforward and only two examples will be given explicitly. The first one is the well-known Pauli spin 1/2 matrices:
\[ J_{x1} = \frac{h}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, J_{y1} = \frac{h}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, J_{z1} = \frac{h}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \] (23)
and the second example is less conventional:
\[ J_{x13} = \frac{h}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, J_{y13} = \frac{h}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, J_{z13} = \frac{h}{2} \begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix} \] (24)

It can easily be verified that these 3 matrices fulfill the 3 angular momentum definitions given by Equations (1)-(3). The 24 solutions show that the elements of the spin 1/2 matrices can take only four values ($-1, \pm i, \pm 1$) times $h/2$ and the corresponding real matrices:
\[ J_X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, J_Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, J_Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \] (25)
Let us note that:
\[ J_X^2 = [1], J_Y^2 = [1], J_Z^2 = [1] \] (26)
and
\[ J_XJ_Y = [1], J_YJ_X = [1], J_ZJ_X = [1] \] (27)
which proves that all the solutions correspond to an operator which preserves the norm of the vectors in the two-dimensional space.

The 24 solutions of the general definition of the angular momenta for spin 1/2 are given in Table 1.

The general treatment of angular momenta uses the operator $J^2 = J_X^2 + J_Y^2 + J_Z^2$. If we calculate this operator for the 24 solutions we find $J^2 = 3/4 h^2 I_1$ in 16 cases corresponding to the first two and last two rows of Table 1.

Table 1. The 24 solutions for spin 1/2 using matrices $J_X, J_Y, J_Z$ in $h/2$ units.

<table>
<thead>
<tr>
<th>$J_X$</th>
<th>$J_Y$</th>
<th>$J_Z$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$J_{x1}$</td>
<td>$J_{y1}$</td>
<td>$J_{z1}$</td>
</tr>
<tr>
<td>$J_{x13}$</td>
<td>$J_{y13}$</td>
<td>$J_{z13}$</td>
</tr>
</tbody>
</table>

1. This result is the well-known value obtained with Pauli matrices. In the other 8 cases (rows 3 and 4 of Table 1) we obtain $J^2 = -\frac{1}{4} h^2 I_1$.

We can associate the spin 1/2 with components $J_{ik}$, $J_{ik}'$, and $J_{ik}$ to the space coordinates in the laboratory frame $(O,u_x,u_y,u_z)$ to define the spin vector of a particle:
\[ J = J_{ik} u_i + J_{ik}' u_i + J_{ik} u_i \] (28)

2.2. Dirac Equation

It is well known [2] that the spin of a particle can be retrieved using an equation which satisfies the special theory of relativity and quantum mechanics postulates, such as the Dirac equation. In Dirac’s book [3], the Pauli matrices are obtained from the commutation definition with the additional constraint:
\[ J^2 = J_X^2 + J_Y^2 = \frac{\hbar^2}{4} \] (29)

Owing to the fact that each observable has only two eigenvalues $h/2$ and $-h/2$. He obtained
\[ \sigma = \begin{pmatrix} 0 & e^{i\alpha} \\ e^{-i\alpha} & 0 \end{pmatrix} \] (30)
and stated that the phase $\alpha$ could be adequately chosen so as to obtain the three Pauli matrices. It is clear that Dirac did not consider non observable spins and he only took the 3 Pauli matrices as components of the spin vector.

In his relativistic theory of the electron, he found that in order to obtain a linear wave equation it was necessary to introduce four $4 \times 4$ matrices solving the following equations:
\[ \alpha^2 = 1, \quad \alpha_x \alpha_y + \alpha_y \alpha_x = 0 \] (31)
\[ \beta^2 = m^2 c^2, \quad \alpha_x \beta + \beta \alpha_x = 0 \]
and the same relations obtained by permuting $x,y$ and $z$.

A solution given by Dirac to these four matrices was:
\[ \alpha = \begin{pmatrix} 0 & \sigma_z \\ \sigma_z & 0 \end{pmatrix}, \beta = \begin{pmatrix} I_1 & \sigma_z \\ \sigma_z & -I_1 \end{pmatrix} \] (32)
with $k = x, y, z$, which nowadays is referenced as the standard form. But it is well known that any set of ma-
traces obtained with:
\[ \alpha' = U \alpha U^\dagger \quad \text{and} \quad \beta' = U \beta U^\dagger \]  \tag{33}

where \( U \) is any unitary matrix \( (U^\dagger U = UU^\dagger = 1) \), is also a solution to Equations (31). It can be shown that for a homogeneous magnetic field, the energy of the interaction of the intrinsic momentum with the magnetic field is
\[ E_i = \frac{\hbar q}{2m_B} \sigma B \]  \tag{34}

and the spin definition can also be retrieved as
\[ S = \frac{1}{2} \hbar \sigma \]  \tag{35}

where \( \sigma \) is the spin vector defined by the 3 Pauli matrices as in the non relativistic case. Dirac’s equation has to be discussed more precisely only when time evolution is introduced.

An important question is why does the general treatment of quantum angular momentum give only one solution? This comes from the definition of \( J^2 = J_x^2 + J_y^2 + J_z^2 \), which is a dot product and not the matrix product of two angular momentum components. This definition introduces some constraints which permit to eliminate the solutions where \( J^2 < 0 \), but by taking only the positive values of \( j \), also eliminates the negative components in the combination of the Pauli matrices. Therefore only the solution \( J_i(\sigma_x, \sigma_y, \sigma_z) \) is obtained.

### 3. Angular Momenta for 3-Dimensional Quantum System

If we consider the extension to three dimensions, a vector has now 3 components and any quantum operator is described by a real \( 3 \times 3 \) matrix multiplied by a complex number \( \alpha' \). We can choose a basis for the vector space where one angular momentum is diagonal. If we take the \( z \) direction for this, the angular momentum is:
\[ J_z = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \]  \tag{36}

where the \( \lambda_k \) are three complex numbers.

As the angular momenta are defined by Equations (1)-(3) which correspond to non commuting matrices the trace of the 3 angular momentum matrices \( J_x, J_y, J_z \) is null. If we suppose that in the same basis and for the most general case \( J_z = [\beta_{ij}] \), where \( \beta_{ij} \) are complex numbers, we obtain according to Equation (2):
\[ J_z = \frac{1}{\hbar} \begin{pmatrix} 0 & \beta_{12}(\lambda_2 - \lambda_1) & \beta_{13}(\lambda_3 - \lambda_1) \\ -\beta_{12}(\lambda_2 - \lambda_1) & 0 & \beta_{23}(\lambda_3 - \lambda_2) \\ -\beta_{13}(\lambda_3 - \lambda_1) & -\beta_{23}(\lambda_3 - \lambda_2) & 0 \end{pmatrix} \]  \tag{37}

and Equation (3) gives:
\[ J_y = -\frac{1}{\hbar} \begin{pmatrix} 0 & -\beta_{12}(\lambda_2 - \lambda_1)^2 & -\beta_{13}(\lambda_3 - \lambda_1)^2 \\ -\beta_{12}(\lambda_2 - \lambda_1)^2 & 0 & -\beta_{23}(\lambda_3 - \lambda_2)^2 \\ -\beta_{13}(\lambda_3 - \lambda_1)^2 & -\beta_{23}(\lambda_3 - \lambda_2)^2 & 0 \end{pmatrix} \]  \tag{38}

and Equation (1):
\[ J_x = \frac{1}{\hbar^2} \begin{pmatrix} \alpha_{ij} \end{pmatrix} \]  \tag{39}

\[ \begin{align*}
\alpha_{11} &= 2\left(\beta_{12}\beta_{13}(\lambda_1 - \lambda_2)^2 + \beta_{13}\beta_{23}(\lambda_1 - \lambda_3)^2\right) \\
\alpha_{22} &= 2\left(\beta_{12}\beta_{23}(\lambda_2 - \lambda_3)^2 + \beta_{13}\beta_{23}(\lambda_2 - \lambda_3)^2\right) \\
\alpha_{33} &= 2\left(\beta_{31}\beta_{23}(\lambda_3 - \lambda_1)^2 + \beta_{31}\beta_{23}(\lambda_3 - \lambda_2)^2\right) \\
\alpha_{12} &= \beta_{12}\beta_{23}(\lambda_1 - \lambda_2)(\lambda_2 - \lambda_3)(\lambda_1 + \lambda_2 - 2\lambda_3) \\
\alpha_{13} &= \beta_{13}\beta_{23}(\lambda_1 - \lambda_2)(\lambda_2 - \lambda_3)(\lambda_2 + \lambda_3 - 2\lambda_2) \\
\alpha_{21} &= \beta_{31}\beta_{23}(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_1 + \lambda_2 - 2\lambda_3) \\
\alpha_{23} &= \beta_{31}\beta_{23}(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_2 + \lambda_3 - 2\lambda_2) \\
\alpha_{31} &= \beta_{13}\beta_{23}(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)(\lambda_1 + \lambda_2 - 2\lambda_3) \\
\alpha_{32} &= \beta_{13}\beta_{23}(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)(\lambda_2 + \lambda_3 - 2\lambda_2)
\end{align*} \]  \tag{40}

To find the angular momenta, Equation (39) must be equal to Equation (36). There are several solutions obtained from the non diagonal \( J_z \) matrix elements equal to zero.

If \( \beta_{31} = \beta_{32} = \lambda_2 = 0 \), then \( \lambda_3 = \pm \hbar \). For \( \lambda_1 = \hbar \) we obtain a set of three matrix solutions to the spin equation:
\[ J_z = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} J_z = \begin{pmatrix} 0 & i\beta_{12} & 0 \\ -i\hbar^2 & 2\beta_{12} & 0 \\ 0 & -i\hbar^2 & 2\beta_{23} \end{pmatrix} \]  \tag{41}

\[ J_y = \begin{pmatrix} 0 & \beta_{12} & 0 \\ \frac{\hbar^2}{2\beta_{12}} & 0 & \beta_{23} \\ 0 & \frac{\hbar^2}{2\beta_{23}} & 0 \end{pmatrix} \]  \tag{42}

For \( \lambda_1 = -\hbar \) the sign of \( J_z \) and \( J_x \) is changed but \( J_y \) is unchanged. If \( \beta_{12} = \beta_{23} = -i\hbar/\sqrt{2} \) we retrieve the conventional matrices for a spin 1:
Another solution is obtained with \( \beta_3 = \beta_2 = \lambda_1 = 0 \), then \( \lambda_2 = \pm h \) and for \( \lambda_2 = h \) the angular momenta are:

\[
J_z = h \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad J_x = \frac{h}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad J_y = \frac{h}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix},
\]

(42)

The third set of solutions is obtained with \( \beta_3 = \beta_2 = \lambda_0 = 0 \), then \( \lambda_1 = \pm h \) and for \( \lambda_1 = h \) the angular momenta are:

\[
J_z = h \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad J_x = \begin{pmatrix} 0 & -i\beta_2 & -i\beta_3 \\ i\beta_2 & 2\beta_1 & 0 \\ i\beta_3 & 0 & 2\beta_2 \end{pmatrix}, \quad J_y = \begin{pmatrix} 1 & \beta_2 & \beta_3 \\ \beta_2 & 0 & 0 \\ \beta_3 & 0 & 0 \end{pmatrix},
\]

(43)

The choice of the basis of the 3-dimensional vector space can also be made so as to have a diagonal angular operator in the \( x \) or \( y \) directions, then the equivalent sets of solutions will be obtained by permuting the \( x \), \( y \) and \( z \) subscripts. The important result of this derivation is that, contrary to the 2-dimensional case, there is no solution where the derived angular momentum operator preserves the norm of the vectors in the 3-dimensional vector space. Indeed, \( J_i^T J_i \) is never equal to 1.

Therefore, the spin of any boson in particle physics requires a description other than the matrix representation of the angular momentum.

An interesting way to define the spin is to use the framework of a Clifford algebra defined in the quantum vectorial space.

4. Clifford Algebra and Spins

In his paper, in order to define the spin, Fröhlich [1] used a Clifford algebra involving \( 2^k \times 2^k \) matrices over complex numbers, which has to be compared to the geometric algebra of spacetime initiated by Hestenes [4-6] and also developed by the Cambridge astrophysical group [7-9].

Grassmann and Clifford’s geometric algebra is based on the definition of the geometric product \( uv \) for vectors \( u, v, w \) obeying the following rules:

\[
(uv)w = u(vw)
\]

(45)

\[
u(v + w) = uv + wu
\]

(46)

\[
v^2 = e_s |v|^2
\]

(48)

where \( e_s \) is the signature of \( v \) and the magnitude \( |v| \) is a real positive scalar. The geometric product can be decomposed into a symmetric inner product:

\[
u \cdot v = v \cdot u \]

(49)

and an antisymmetric outer product:

\[
u \wedge v = -v \wedge u
\]

(50)

such that

\[
uv = u \cdot v + u \wedge v.
\]

(51)

Orthonormal vectors are defined by:

\[
u_1 \cdot u_1 = 1 \quad u_i \wedge u_i = 0
\]

(52)

and

\[
u_1 \cdot u_2 = 0
\]

(53)

The signature of orthonormal vectors \( u_k \) are \( u_k^2 = \pm 1 \). The fact that real numbers can give negative squares is a little surprising but it is well illustrated by matrix \( J_y \) of Equation (25).

4.1. Basis of Geometric Algebra in Two and Three Dimensions

In two dimensions we have the geometric algebra of the plane with four elements:

\[
l, \quad u_1, \quad u_2, \quad u_1u_2 = i.
\]

(54)

scalar, vector, vector, bivector.

The bivector \( u_1u_2 \) has a property such that its square \( (u_1u_2)^2 = u_1u_2u_1u_2 = -u_1^2u_2^2 \). If the signatures \( u_1^2 = u_2^2 = 1 \)
or \( u^2 = u_2^2 = -1 \), \( u_1 u_2 = 1 \) form a natural subalgebra equivalent to complex numbers \( z = x + yu_2 \). The bivector \( i \) is called a pseudoscalar which anticommutes with vectors \( u_1 \) and \( u_2 \).

An arbitrary linear sum over the four basis elements is called a multivector:

\[
A = a_0 + a_1 u_1 + a_2 u_2 + a_3 i
\]

with components \( a_i \in \mathbb{R} \).

The sum of two multivectors is obtained by adding each component and the geometric product is obtained from the multiplication table (Table 2).

We can notice that the same algebraic properties are obtained by using the \( \{1, u_1, u_2, i = u_1 u_2 \} \) basis vectors (where \( \sim \) indicates a reversion of vector products). Again \( i^2 = -u_1^2 u_2^2 = -1 \) if the signatures are \( u_1^2 = u_2^2 = 1 \) or \( u_1^2 = u_2^2 = -1 \).

The product of two multivectors \( AB \) is explicitly given by:

\[
AB = (a_0 b_0 + a_1 b_1 u_1^2 + a_2 b_2 u_2^2 - a_3 b_3)1 + (a_0 b_1 - a_1 b_0 - a_2 b_3 + a_3 b_2)u_1 + (a_0 b_2 + a_1 b_3 - a_2 b_1 + a_3 b_0)u_2 + (a_0 b_3 - a_1 b_2 + a_2 b_1 - a_3 b_0)i
\]

and the square of a multivector is

\[
A^2 = (a_0^2 + a_1^2 u_1^2 + a_2^2 u_2^2 - a_3^2)1 + 2(a_0 a_1 u_1 + a_0 a_2 u_2 + a_0 a_3 i)
\]

Unit multivectors are characterized by:

\[
A^2 = 1 \iff (a_0 = \pm 1, a_1 = a_2 = a_3 = 0) \forall u_1^2, u_2^2
\]

or \( a_0 = 0, a_1 u_1^2 + a_2 u_2^2 - a_3^2 = 1 \) (58)

Rotation in geometric algebra is based on a theorem by Hamilton: given any unit vector \( n (n^2 = 1) \), we can resolve an arbitrary vector \( x \) into parts parallel and perpendicular to \( n \): \( x = x_n + x_p \). These components are identified algebraically through their commutation properties:

\[
x_n = x \cdot n
\]

The vector \( x_n - x_p \) can therefore be written \( -n x_n \). Geometrically, the transformation \( x \rightarrow -n x_n \) represents a reflection in the plane \( x, n \). To make a rotation we need two such reflections:

\[
x \rightarrow n n x n
\]

with \( mn nm = 1 \) [7]. The multivector \( R = mn \) is called a rotor and \( R = mn \) is called the “reverse” of \( R \). \( R \) satisfying \( R R = R R = 1 \) is defined as “unimodular”.

The bilinear transformation of vectors is a very general way of handling rotations which can easily be generalized to vector spaces of any dimension [7].

In three dimensions we have the geometric algebra of space with 8 elements:

\[
1, \{u_1, u_2, u_3\}, \{u_1 u_3, u_2 u_3, u_1 u_2\}, u_1 u_2 u_3 = i
\]

1 scalar, 3 vectors, 3 bivectors, 1 trivector

Again the pseudoscalar \( i = u_1 u_2 u_3 \) is defined by its property such that \( i^2 = u_1 u_2 u_3 u_1 u_2 u_3 = -1 \), therefore the signature of the orthonormal vectors must be one of the four possibilities: \( u_1^2 = u_2^2 = u_3^2 = 1 \) or \( u_1^2 = u_2^2 = -1 \) and \( u_3^2 = 1 \) or \( u_1^2 = u_2^2 = 1 \) and \( u_3^2 = -1 \) or \( u_1^2 = u_2^2 = -1 \) and \( u_3^2 = -1 \).

Contrary to the 2-dimensional case, varying the order of multivectors does not give the same algebraic properties. By choosing the bivectors:

\[
\{\rho_1 = u_1 u_2, \quad \rho_2 = u_1 u_3, \quad \rho_3 = u_2 u_3\}
\]

for the basis of the 3-dimensional vector space, we have the relation defining the quaternion algebra:

\[
\rho_1^2 = \rho_2^2 = \rho_3^2 = \rho_1 \rho_2 \rho_3 = -1
\]

which is only valid with the signatures:

\[
u_1^2 = u_2^2 = u_3^2 = 1
\]

The important property given by Equation (64) is that the trivector \( i \) commutes not only with vectors but also with bivectors.

The multiplication table for 3-dimensional geometric algebra with the signatures, in order to have the general case, is given in Table 3.

The product of two multivectors is given explicitly by:

\[
R = AB = R_{01} + R_{1} u_1 + R_{2} u_2 + R_{3} u_3 + R_{4} \rho_1 + R_{5} \rho_2 + R_{6} \rho_3 + R_{7} i
\]

with

\[
x_n = -x_i n
\]
Table 3. Multiplication table for geometric algebra in three dimensions.

<table>
<thead>
<tr>
<th></th>
<th>(u_1)</th>
<th>(u_2)</th>
<th>(u_3)</th>
<th>(u_1 u_2)</th>
<th>(u_2 u_3)</th>
<th>(u_1 u_3)</th>
<th>(u_1 u_2 u_3)</th>
<th>(i)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(u_1)</td>
<td>(u_1)</td>
<td>(u_2)</td>
<td>(u_3)</td>
<td>(u_1)</td>
<td>(u_2)</td>
<td>(u_3)</td>
<td>(-u_1 u_2 u_3)</td>
<td>(i)</td>
</tr>
<tr>
<td>(u_2)</td>
<td>(-u_1 u_2)</td>
<td>(u_1)</td>
<td>(u_2)</td>
<td>(-u_1)</td>
<td>(u_2)</td>
<td>(u_3)</td>
<td>(-u_1 u_2 u_3)</td>
<td>(-u_1 u_2)</td>
</tr>
<tr>
<td>(u_3)</td>
<td>(-u_1 u_3)</td>
<td>(-u_2 u_3)</td>
<td>(u_3)</td>
<td>(-u_1 u_3)</td>
<td>(-u_2 u_3)</td>
<td>(u_3)</td>
<td>(-u_1 u_2 u_3)</td>
<td>(-u_1 u_3)</td>
</tr>
<tr>
<td>(u_1 u_2)</td>
<td>(i)</td>
<td>(-u_2 u_3)</td>
<td>(-u_1 u_3)</td>
<td>(u_1 u_2 u_3)</td>
<td>(i)</td>
<td>(-u_2 u_3)</td>
<td>(-u_1 u_2 u_3)</td>
<td>(-u_2 u_3)</td>
</tr>
</tbody>
</table>

\[
R_0 = a_0 b_0 + a_1 b_1 u_1^2 + a_2 b_2 u_2^2 + a_3 b_3 u_3^2 - a_2 b_1 u_3^2 - a_3 b_1 u_2^2 - a_3 b_2 u_1^2 \]
\[
R_1 = a_0 b_0 - a_1 b_1 u_1^2 + a_2 b_2 u_2^2 + a_3 b_3 u_3^2 + a_1 b_2 u_3^2 - a_2 b_3 u_1^2 - a_3 b_2 u_2^2 \]
\[
R_2 = a_0 b_0 + a_1 b_1 u_1^2 + a_2 b_2 u_2^2 + a_3 b_3 u_3^2 - a_1 b_2 u_3^2 - a_2 b_3 u_1^2 - a_3 b_2 u_2^2 \]
\[
R_3 = a_0 b_0 - a_1 b_1 u_1^2 + a_2 b_2 u_2^2 + a_3 b_3 u_3^2 + a_1 b_2 u_3^2 - a_2 b_3 u_1^2 - a_3 b_2 u_2^2 \]
\[
R_4 = a_0 b_0 + a_1 b_1 u_1^2 + a_2 b_2 u_2^2 + a_3 b_3 u_3^2 - a_1 b_2 u_3^2 - a_2 b_3 u_1^2 - a_3 b_2 u_2^2 \]
\[
R_5 = a_0 b_0 - a_1 b_1 u_1^2 + a_2 b_2 u_2^2 + a_3 b_3 u_3^2 + a_1 b_2 u_3^2 - a_2 b_3 u_1^2 - a_3 b_2 u_2^2 \]
\[
R_6 = a_0 b_0 + a_1 b_1 u_1^2 + a_2 b_2 u_2^2 + a_3 b_3 u_3^2 - a_1 b_2 u_3^2 - a_2 b_3 u_1^2 - a_3 b_2 u_2^2 \]

According to the previous mentioned Hamilton’s theorem, any multivector \(A\) can be resolved in a commuting (perpendicular) part: \(C_a\) and anticommuting (parallel) part: \(J_a\), therefore:

\[
A = C_a + J_a \tag{67}
\]

with

\[
C_a = a_0 1 + a_1 i
\]

\[
J_a = a_1 u_1 + a_2 u_2 + a_3 u_3 + a_1 \sigma_1 + a_2 \sigma_2 + a_3 \sigma_3
\]

because \(1\) and \(i\) commute with all the elements of a multivector.

If we look at the spin definition in natural units \((\hbar = 1)\) given by the spin equations in a geometric algebra framework, \(J_x J_y\) can be considered as anticommuting multivectors defining a rotation \(R_z\) in the \(x, y\) plane.

To prove the validity of this unconventional definition of spin 1/2, we consider the first spin equation, \(J_y J_x - J_x J_y = i \hbar J_z\), which can be easily calculated with Equation (66). It appears that the scalar and the pseudo-scalar components of \(J_z\) are equal to zero. Since this relation applies to the other two spin equations, every \(J_x\), \(J_y\), and \(J_z\) have no scalar and no pseudoscalar components.

The first spin equation gives the following 6 equations:

\[
2 \left( x_3 y_1 u_3^2 - x_1 y_3 u_1^2 + x_1 y_2 u_1^2 - x_3 y_2 u_3^2 \right) = -z_2 u_2^2 u_3^2 \]
\[
2 \left( x_1 y_3 u_2^2 - x_3 y_1 u_2^2 + x_3 y_2 u_1^2 - x_1 y_2 u_3^2 \right) = -z_2 u_1^2 u_3^2 \]
\[
2 \left( x_3 y_1 u_3^2 - x_1 y_3 u_3^2 + x_1 y_2 u_3^2 - x_3 y_2 u_1^2 \right) = -z_2 u_1^2 u_2^2 \]

\[
(68)
\]

Calculating the components of the product of two multivectors \(R = J_x J_y\) allows to retrieve the 6 previous \(R_{xz}\) factors and two new factors \(r_{z0}\) and \(r_{z7}\) for the scalar and the pseudoscalar components of the product:

\[
R_z = r_{z0} u_0 + r_{z1} u_1 + r_{z2} u_2 + r_{z3} u_3 + r_{z4} \rho_1 + r_{z5} \rho_2 + r_{z6} \rho_3 + r_{z7} i
\]

(69)

with

\[
r_{z0} = x_3 y_1 u_2^2 + x_3 y_2 u_3^2 - x_1 y_3 u_1^2 - x_2 y_3 u_2^2 \]

\[
r_{z7} = x_3 y_3 + x_3 y_2 u_3^2 - x_1 y_2 u_3^2 - x_2 y_3 u_2^2 \]

therefore

\[
R_z = J_x J_y = r_{z0} 1 + \frac{1}{2} \left( J_x J_y - J_y J_x \right) + r_{z7} i \tag{70}
\]

An analogous derivation gives:

\[
\widetilde{R}_z = J_x J_z = r_{z0} 1 + \frac{1}{2} \left( J_x J_z - J_z J_x \right) + r_{z7} i \tag{71}
\]

Therefore

\[
R_z \widetilde{R}_z = \tilde{R}_z R_z = (r_{z0} 1 + r_{z7} i)^2 - \frac{1}{4} i^2 J_z^2 \tag{72}
\]

where \(J_z^2\) (with \(k = x, y, z\)) have only a scalar and a
pseudoscalar component. The important consequence is that the $J^2$ commute.

Hamilton’s theorem proves that there exists a unit vector which resolves any vector into a perpendicular and a parallel part and $u_3$ is a good candidate to play this role for $R_z$ because $u_3$ is perpendicular to any multivector $J_x$ and any multivector $J_y$. The permutation of $x,y,z$ shows that there is no incompatibility in doing so for $J_x J_y$ and $J_x J_y$. We retrieve the relation (65):

$$u^2 = u^2 = u^2 = 1$$

and find the 3 relations for $R_x R_y = 1$:

$$R_x R_y = (r_0 + r_i r_j)^2 + \frac{1}{4} J^2 = 1$$

$$R_y R_x = (r_0 + r_i r_j)^2 + \frac{1}{4} J^2 = 1$$

$$R_x R_y = (r_0 + r_i r_j)^2 + \frac{1}{4} J^2 = 1$$

Therefore a spin 1/2 (S1/2) can be defined in geometric algebra as a set of 3 rotors:

$$R_x = J_x J_y, \quad R_y = J_y J_z, \quad R_z = J_z J_z$$

As $R_k$ are unimodular vectors, they preserve the magnitude of multivectors. Equations (68) and (69) show that the solution for the component values of the set of the 3 rotors depends on the signatures of the orthonormal vectors.

In summary, in geometric algebra a spin 1/2 is a set of 3 rotations obtained from the product of two multivectors which are the linear summation of 3 orthonormal vectors $u_1, u_2, u_3$ and 3 bivectors $\rho_1, \rho_2, \rho_3$ according to:

$$J_k = k_1 u_1 + k_2 u_2 + k_3 u_3 + k_4 \rho_1 + k_5 \rho_2 + k_6 \rho_3$$

$$k = \{x,y,z\}$$

$$\rho_1^2 = \rho_2^2 = \rho_3^2 = \rho_1 \rho_2 \rho_3 = -1$$

which fulfill the spin equations:

$$[J_x J_y] = u_i u_j u_k h J_z$$

and the permutation of $(x,y,z)$.

It can be noticed that the only solution if we try to solve the spin equations in the two-dimensional case, is $J_x = J_y = J_z = 0$. This result is not surprising because in two dimensions, rotations commute and $J_x J_y - J_y J_x = 0$.

4.2. Spacetime Algebra

According to Hestenes [5], the standard model for spacetime is a real 4D Minkowski spacetime with vector addition and scalar multiplication where we can impose the geometric product defined by equations 45 to 51 in order to generate a geometric algebra called spacetime algebra (STA).

A basis for STA can be generated by a frame $\{u_i, u_2, u_3, u_4\}$ of orthonormal vectors which determines the pseudoscalar:

$$i = u_i u_2 u_3 u_4$$

In order for the pseudoscalar $i$ to keep the same properties as in the two- and three-dimensional cases, $i^2$ must be equal to $-1$, which imposes two signature combinations which are called the metric of STA. One possibility to fulfill $i^2 = -1$ is that three signatures should be equal to 1 and the fourth equal to −1, the second possibility is that three signatures should be equal to −1 and the fourth equal to 1. The metrics are defined by $\{1,1,1,1\}$ and $\{-1,1,-1,1\}$. The 3 orthonormal vectors with identical signature will be associated with the space components $x,y,z$ and the other with the time component $t$ so that:

$$i^2 = u^2 u^2 u^2 u^2 = -1$$

By forming all distinct products of the four $u_k$ we obtain a complete basis for STA consisting of $2^4 = 16$ linearly independent elements.

There are 12 bivectors obtained from the arrangement of 2 out of the 4 orthonormal vectors, but there are only 6 with different magnitudes:

$$u_i u_j = -u_j u_i = \rho_1 \quad u_i u_j = -u_j u_i = \sigma_1$$

$$u_i u_j = -u_j u_i = \rho_2 \quad u_i u_j = -u_j u_i = \sigma_2$$

$$u_i u_j = -u_j u_i = \rho_3 \quad u_i u_j = -u_j u_i = \sigma_3$$

As regards trivectors, there are 24 products to be obtained from the arrangement of 3 out of the 4 orthonormal vectors. There are 8 distinct products and only four having different magnitudes:

$$u_i u_j u_k = -u_k u_j u_i = \frac{i u_i}{u_k} = \xi_0$$

$$u_i u_j u_k = -u_j u_i u_k = \frac{i u_i}{u_k} = \xi_1$$

$$u_i u_j u_k = -u_j u_i u_k = \frac{i u_i}{u_k} = \xi_2$$

$$u_i u_j u_k = -u_k u_j u_i = \frac{i u_k}{u_j} = \xi_3$$

We can verify that $i$ anticommutes with vectors and trivectors but commutes with bivectors.

A complete basis for STA is:

1 scalar : 1
4 vectors : $\{u_1, u_2, u_3, u_4\}$
6 trivectors : $\{3 \rho, 3 \sigma\}$
4 pseudovectors : $\{\xi\}$
1 pseudoscalar : $i$
According to the 3-dimensional geometric algebra case we have the quaternion relation (64) for the $\rho_\xi$ bivectors. By adequately choosing the bivectors of the vector space basis, the 3 new bivectors $\sigma_\xi$ fulfill the following quaternion relation:

\[(i\sigma_\xi)^2 = (i\sigma_\xi)^2 = (i\sigma_\xi)(i\sigma_\xi) = -1\]  

(83)

The four pseudovectors are: $\zeta^_\xi$ the pseudoscalar of dimension 3 and the three other $\zeta^_\xi$ which, again, fulfill a quaternion relation between the geometric product of the time orthonormal vector and $\zeta^_\xi$:

\[(u^_\xi \zeta^_\xi)^2 = (u^_\xi \zeta^_\xi)^2 = (u^_\xi \zeta^_\xi)(u^_\xi \zeta^_\xi) = -1\]  

(84)

The 3 quaternion relations are verified with the metric \{1,1,1,–1\} and the choice of the basis defined by Equations (80) and (81).

For the other metric, Equation (84) is still valid, but to keep the quaternion relations (63) and (83) for bivectors, we have to consider another basis where $\rho_\xi = -\rho_\xi = u^_\xi u^_\xi$, which corresponds to reverse rotations in the $x_2$ plane.

If we consider the geometric algebra definition of spin 1/2 given by Equations (76) to (77) we retrieve the spin 1/2 characterized by the 6-element multivector:

\[S_{x,y,z} = \{u^_1, u^_2, u^_3, \rho_1, \rho_2, \rho_3\}\]  

(85)

but in the 4-dimensional vector space there are two other spins 1/2:

\[S_{x,y,z} = \{u^_1, u^_2, u^_3, i\sigma_2, i\sigma_3\}\]  

(86)

\[S_{x,y,z} = \{u^_1, u^_2, u^_3, u^_4, \zeta^_1, \zeta^_2\}\]  

(87)

for the metric \{1,1,1,–1\}. For the other metric with the other basis there are three new spins 1/2 obtained from the 3 vectors $u^_1$, $u^_2$, and $u^_3$:

\[S_{x,y,z} = \{u^_1, u^_2, \rho_1, \tilde{\rho}_2, \rho_3\}\]  

(88)

\[S_{x,y,z} = \{u^_1, u^_2, u^_3, -u^_4, \zeta^_1, \zeta^_2\}\]  

(89)

\[S_{x,y,z} = \{u^_1, u^_2, u^_3, \rho_1, -\tilde{\rho}_2, \rho_3\}\]  

(90)

There are 18 other spins 1/2 obtained with the different vectors of the 4-dimensional case and the same bivectors:

\[S_{x,y,z} = \{u^_1, u^_2, \rho_1, \tilde{\rho}_2, \rho_3\}\]  

(91)

Therefore the 24 spins 1/2 found with the matrix derivation are retrieved in the STA description but here with a clear distinction between spins 1/2 which fall in two metric-dependent categories. If the 12 spins 1/2 of each category were sorted as a function of bivectors we would find 3 families of 4 spins 1/2 as in elementary particle physics.

As the definition of rotation given by Equation (61) applies whatever the dimension of the vector space, in four dimensions the spin 1 can be defined by set of 3 rotors:

\[R_{x_4} = J_{x_4} J_{x_4}, \quad R_{y_4} = J_{y_4} J_{y_4}, \quad R_{z_4} = J_{z_4} J_{z_4}\]  

with $J_{k_4} = k^_1 u^_1 + k^_2 u^_2 + k^_3 u^_3 + k^_4 u^_4 + k^_5 \rho_3$

\[+ k^_6 \rho_2 + k^_7 \rho_1 + k^_8 \sigma_1 + k^_9 \sigma_2 + k^_10 \sigma_3 + k^_11 \zeta^_1 + k^_12 \zeta^_2 + k^_13 \zeta^_3\]  

(92)

where the $J_k$ fulfill the spin equations. From the generalization to 4 dimensions of the vector product given by Equation (66), it is easy to see that the scalar component is null because real components commute and the pseudo scalar component is null because its parts are independent of the signatures. As the $R_k$ are unimodular vectors, they induce rotations which preserve the norm of any vector. Again the solution to the equations which determine the components of the multivectors $J_k$ is dependent on the signature of the orthonormal vectors. Therefore there are 4 different solutions with space-like signatures $u^_2 = u^_2 = \pm 1$ and time-like signature $u^_3 = \pm 1$ and 12 different solutions if we consider each signature independently.

5. Conclusion

We have shown that for the two-dimensional complex vector space, the spin matrices can be calculated directly from the angular momentum commutator definition. We have retrieved the 3 Pauli matrices and found 23 other triplet solutions. When extended to the three-dimensional space, we have shown that there is no matrix which preserves the norm of the vectors and fulfills the spin equations. By using a geometric algebra with a vector product which combines a commuting product and an anticommuting product it has been possible in four-dimensional spacetime to retrieve the 24 different spins 1/2 defined as 12 clockwise and 12 counter-clockwise rotations. These rotations are characterized by anticommuting parts composed of 3 vectors and 3 bivectors which fulfill the spin equations. Spin 1 can be defined as 3 rotations characterized by 4 vectors, 6 bivectors and 4 trivectors which fulfill the spin equations. These unimodular spin 1 rotations preserve the magnitude of multivectors. There are 12 different spins 1 depending on the signature of the 4 orthonormal vectors of the four-dimensional vector space. The correspondence between this derivation and particle physics is perhaps fortuitous but the use of STA offers the advantage of formulating...
conventional relativistic physics in invariant form without reference to a coordinate system [5] and it seems promising to analyze how time evolution and spin interaction can be used in order to predict the gyromagnetic ratio of the proton and the neutron.

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REFERENCES


