Solutions of Schrödinger Equation with Generalized Inverted Hyperbolic Potential

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ABSTRACT

The bound state solutions of the Schrödinger equation with generalized inverted hyperbolic potential using the Nikiforov-Uvarov method are reported. We obtain the energy spectrum and the wave functions with this potential for arbitrary \( l \)-state. It is shown that the results of this potential reduced to the standard potentials—Rosen-Morse, Poschl-Teller and Scarf potential as special cases. We also discussed the energy equation and the wave function for these special cases.

Keywords: Schrödinger Equation; Inverted Hyperbolic Potential; Nikiforov-Uvarov Method

1. Introduction

The analytical and numerical solutions of the wave equations for both relativistic and non-relativistic cases have taken a great deal of interest recently. In many cases different attempts have been developed to solve the energy eigenvalues from the wave equations exactly or numerically for non-zero angular momentum quantum number \( l \neq 0 \) for a given potential [1-16]. It is well known that these solutions play an essential role in the relativistic and non-relativistic quantum mechanics for some physical potentials of interest, [1,2,12,17-19].

In this paper, we aim to solve the radial Schrödinger equation for quantum mechanical system with inverted generalized hyperbolic potential and show the results for this potential using Nikiforov-Uvarov method (NU), [20].

The present paper is an attempt to carry out the analytical solutions of the Schrodinger equation with the generalized inverted hyperbolic potential using the NU method.

The hyperbolic potentials under investigations are commonly used to model inter-atomic and intermolecular forces [10,21]. Among such potentials are Poschl-Teller, Rosen-Morse and Scarf potential, which have been studied extensively in the literatures, [5-8,22-25]. However, some of these hyperbolic potentials are exactly solvable or quasi-exactly solvable and their bound state solutions have been reported, [3,4,11,13,26-28]. We seek to present and study a generalized hyperbolic potential which other potentials can be deduced as special cases within the framework of Schrödinger equation with mass \( m \) and potential \( V \).

The paper is organized as follows: Section 2 is devoted to the review of the Nikiforov-Uvarov method. In Section 3 we present the exact solution of the Schrodinger equation. Discussion and results are presented in Section 4. Finally we give a brief conclusion in Section 5.

2. Review of Nikiforov-Uvarov Method

The Nikiforov-Uvarov (NU) method, [20] was proposed and applied to reduce the second order differential equation to the hypergeometric-type equation by an appropriate co-ordinate transformation \( S = S(r) \) as, [15,20].

\[
\psi''(s) + \frac{\tau(s)}{\sigma(s)}\psi'(s) + \frac{\sigma(s)}{\sigma^*(s)}\psi(s) = 0
\]  

where \( \sigma(s) \) and \( \sigma^*(s) \) are polynomials at most in the second order, and \( \tau \) is a first order polynomial. In order to find a particular solution of Equation (1) we use the separation of variables with the transformation

\[
\psi(s) = \phi(s)\chi(s)
\]  

It reduces Equation (1) to an equation as hypergeometric type

\[
\sigma(s)\chi''(s) + \tau(s)\chi'(s) + \lambda\chi(s) = 0
\]  

and \( \phi(s) \) is defined as a logarithmic derivative in the following form and its solution can be obtained from

\[
\frac{\phi'(s)}{\phi(s)} = \frac{\pi(s)}{\sigma(s)}
\]
The other part of the wave formation $\chi(s)$ is the hypergeometric type function whose polynomial solutions are given by Rodrigues relations.

$$\chi_n(s) = \frac{B_n}{\rho(s)} \frac{d^n}{ds^n} \left[ (s-\rho(s))^n \rho(s) \right]$$  \hspace{1cm} (5)

where $B_n$ is a normalization constant, and the weight function $\rho(s)$ must satisfy the condition.

$$\frac{d}{ds}(\sigma \rho) = \tau(s) \rho(s)$$  \hspace{1cm} (6)

with

$$\tau(s) = \tau(s) + 2\pi(s)$$  \hspace{1cm} (7)

The function $\pi(s)$ and the parameter $\lambda$ requires for the NU method are defined as follows:

$$\pi(s) = \frac{\sigma - \tau}{2} \pm \sqrt{\left(\frac{\sigma - \tau}{2}\right)^2 - \sigma + k \sigma}$$  \hspace{1cm} (8)

$$\lambda = k + \pi'(s)$$  \hspace{1cm} (9)

On the other hand in order to find the value of $k$, the expression under the square of polynomial must satisfy the condition.

$$\frac{d}{ds}s \sigma \rho \sigma = (s-\rho(s))^{n-1}$$  \hspace{1cm} (10)

where the derivative $\frac{d\pi(s)}{ds}$ is negative. By comparing Equations (9) and (10), we obtained the energy eigenvalues.

$$R^*(r) + \frac{2m}{\hbar^2} [E + aV_0 \coth(\alpha r) - bV_1 \coth^2(\alpha r) + cV_2 \coth^2(\alpha r) - d] R(r) = 0$$  \hspace{1cm} (11)

where the prime indicates differentiation both respect to $r$.

Now using a new ansatz for the wave function in the form $[3,4,11]$

$$\frac{d^2R(r)}{dr^2} - \frac{\beta dE}{dr} + \frac{2m}{\hbar^2} \left[ E + aV_0 \coth(\alpha r) - bV_1 \coth^2(\alpha r) + cV_2 \coth^2(\alpha r) - d \right] R(r) = 0$$  \hspace{1cm} (12)

and including the centrifugal term, reduces Equation (12) into the following differential equation,

$$\frac{d^2F(r)}{dr^2} - \frac{\beta dF(r)}{dr} + \frac{2m}{\hbar^2} \left[ E + aV_0 \coth(\alpha r) - bV_1 \coth^2(\alpha r) + cV_2 \coth^2(\alpha r) - d \right] F(r) = 0$$  \hspace{1cm} (13)

Because of the centrifugal term in Equation (13), this equation cannot be solved analytically when the angular momentum quantum number $\ell \neq 0$. Therefore, in order to find the approximate analytical solution of Equation (13) with $\ell \neq 0$, we must make an approximation for the centrifugal term. Thus, when $\alpha r \ll 1$ we use the approximation scheme $[9,29]$ for the centrifugal term,

$$\frac{1}{r^2} \approx \alpha^2 \coth^2(\alpha r)$$  \hspace{1cm} (14)

Substituting Equation (14) into Equation (13), we get

$$\frac{d^2F(r)}{dr^2} - \frac{\beta dF(r)}{dr} + \frac{2m}{\hbar^2} \left[ E + \frac{\beta^2}{\alpha^2} + aV_0 \coth(\alpha r) - bV_1 \coth^2(\alpha r) + cV_2 \coth^2(\alpha r) - d \right] F(r) = 0$$  \hspace{1cm} (15)
Now making the change of variable

\[ s = \coth(\alpha r) \]  

we obtain

\[
\left(1 + s^2\right)^2 \frac{d^2 F}{ds^2} + \left(1 + s^2\right)(\beta + 2s) \frac{dF}{ds} + \frac{2m}{\alpha^2 h^2} \left[E + \left(\frac{\beta}{2}\right)^2 + aV_s - bV_s^2 + cV_s^4 \left(1 + s^2\right) - \alpha^2 l(l+1)\left(1 + s^2\right)^2 - d\right]F(s) = 0
\]

Simplifying Equation (23), we have

\[
\frac{d^2 F}{ds^2} + \frac{\beta + 2s}{\left(1 + s^2\right)} \frac{dF}{ds} + \frac{1}{\left(1 + s^2\right)^2} \left[-\epsilon^2 + \beta^2 s + \gamma^2 s^2\right]F(s) = 0
\]

where the following dimensionless parameters have been employed:

\[ \epsilon^2 = -\frac{2m}{\alpha^2 h^2} \left[E + \left(\frac{\beta}{2}\right)^2 + cV_s - \alpha^2 \ell (\ell + 1 + d)\right], \]

\[ \beta^2 = \frac{2maV_s}{h^2 \alpha^2}, \]

\[ \gamma^2 = \frac{2m}{h^2 \alpha^2} \left(cV_s - bV_s - \alpha^2 \ell (\ell + 1)\right) \]  

Comparing Equations (1) and (24), we obtain the following polynomials,

\[ \sqrt{(u+v)s} + \sqrt{(u-v)} \]

\[ \pi(s) = -\frac{\beta}{2} \pm \frac{1}{2} \left[\sqrt{(u+v)s} + \sqrt{(u-v)}\right], \text{ for } k = \gamma^2 - \epsilon^2 - \left(\frac{\beta}{2}\right)^2 + \sqrt{u^2 - v^2} \]

where \( u = \frac{\epsilon^2 + \beta^2}{2\epsilon^2 + \gamma^2} \) and \( V = i\beta \sqrt{\gamma^2 + \frac{5}{2}\beta^2} \).

For the polynomial of \( \tau = \pi + 2\pi \) which has a negative derivative, we get

\[ k = \gamma^2 - \epsilon^2 - \left(\frac{\beta}{2}\right)^2 + \sqrt{u^2 - v^2} \]

\[ \pi(s) = -\frac{\beta}{2} - \frac{1}{2} \left[\sqrt{(u+v)s} - \sqrt{(u-v)}\right] \]  

Now using \( \lambda = k + \pi'(s) \), we obtain \( \tau(s) \) and \( \lambda \) valued as

\[ \tau(s) = 2s - \sqrt{(u+v)s} + \sqrt{u-v} \]

\[ \lambda(s) = \gamma^2 - \epsilon^2 - \left(\frac{\beta}{2}\right)^2 - \sqrt{u^2 - v^2} - \frac{1}{2} \sqrt{u+v} \]  

Substituting these polynomials into Equation (8), we obtain the \( \pi(s) \) function as

\[ \pi(s) = -\frac{\beta}{2} \pm \sqrt{\frac{1}{2}(4k - 4\gamma^2)s^2 - 4\beta^2 s + \beta^2 + 4\epsilon^2 + 4k} \]  

The expression in the square root of Equation (27) must be square of polynomial in respect of the NU method. Therefore, we determine the \( \pi(s) \) -values as

Another definition of \( \lambda_n \) is as given in Equation (10), thus using values of \( \tau(s) \) and \( \sigma(s) \), we get,

\[ \lambda = \lambda_n = u\sqrt{u+v} - u(u+1) \]  

Comparing Equations (32) and (33), we obtain the energy eigenvalue equation as

\[ \left[\frac{(n+1)}{8\sqrt{2}\beta}\right] + i\left(\frac{\gamma}{8\sqrt{2}\beta} - \frac{1}{2v}\right) \epsilon^2 - \left[1 + \frac{i\beta^2}{4}(1 + \frac{1}{v})\right] \epsilon^2 \]

\[ - \left[\Sigma - \frac{(n+1)}{2}\sqrt{v} + \frac{By}{2\sqrt{v}}((n+1) + iy^2) - \frac{i\gamma^2}{2}\right] = 0 \]

where \( \Sigma = \left(\frac{\beta}{2}\right)^2 - \gamma^2 - n(n+1) \).

Solving the energy eigenvalue equation explicitly, we obtain the energy eigenvalues as
\[\varepsilon^2 = \frac{1 + \frac{i\beta^2}{4} \left(1 + \frac{1}{v}\right)}{2 + \frac{(n+1)}{8\sqrt{2}\beta r} + i\left(\gamma - \frac{1}{2}\right)} + \frac{1}{2}
abla^2 \left(\frac{n+1}{8\sqrt{2}\beta r} + i\left(\gamma - \frac{1}{2}\right)\right)\]

Now using these quantities of Equation (20) and the definition for \(\sum\) and \(V\) given as

\[\sum = \frac{2m}{\hbar^2 \alpha^2} \left[\frac{a V_2}{2} - c V_1 + \alpha^2 l(\ell+1) + \frac{\hbar^2 \alpha^2 n(n+1)}{2m}\right]\]

\[V = i \left(\frac{2m}{\hbar^2 \alpha^2}\right) \sqrt{a V_2 + c V_1 - \alpha^2 l(\ell+1)}\]

We obtain the energy spectrum of the Equation (35) for the Schrödinger equation with the generalized inverted hyperbolic potential as

\[E_{nl} = -\frac{\hbar^2 \alpha^2}{2m} \left[1 + \frac{i\beta^2}{4} \left(1 + \frac{1}{v}\right)\right] + \frac{\hbar^2 \alpha^2}{2m} \left[1 + \frac{i\beta^2}{4} \left(1 + \frac{1}{v}\right)\right] + 4 \left(\frac{n+1}{8\sqrt{2}\beta r} + i\left(\frac{r}{8\sqrt{2}\beta} - \frac{1}{2}\right)\right)\]

\[\sum = \frac{n+1}{8\sqrt{2}\beta r} + i\left(\frac{r}{8\sqrt{2}\beta} - \frac{1}{2}\right)\]

where \(B = \frac{\mu + \beta}{2i}\).

Combining the Jacobi polynomials of Equations (40) and (41), we obtain the radial wave function of the Schrödinger equation with inverted generalized hyperbolic potential as

\[F_{nl} (x) = N_n \left[1 + x \left(\frac{\mu B}{2}\right) \left(1 - x\right)^{\frac{\mu B}{2} - 2} P_n^{\mu,2,\frac{1}{2}} (x)\right] \]

where \(N_n\) is a new normalization constant and obeys the condition

\[\int_{-\infty}^{\infty} R_{nl}^2 (x) dx = 1.\]

The total radial wave function is obtain using Equations (18) and (22) as

\[R_{nl} (r) = N_n \left[1 + \text{coth}(\alpha r)\right]^{\frac{\mu B}{2}} \left(1 + \text{coth}(\alpha r)\right)^{\frac{\mu B}{2} - 2} \]

\[\cdot P_n^{\mu,2,\frac{1}{2}} \left(\text{coth}(\alpha r)\right)\]

4. Results and Discussion

The well-known potentials are obtained from the generalized inverted hyperbolic potential if we make appropri-
ate choose for the values of the parameters in the generalized inverted potentials as stated in Section 3. We plotted the variation of the generalized inverted hyperbolic potential as a function of \( r \) for \( a = 1, b = 0.01, V_0 = 1 \text{MeV}, V_1 = 0.5 \text{MeV}, C_2 = 2, V_2 = 0.02 \text{MeV}, d = 2 \text{MeV} \) at different parameters of \( \alpha = 1, 2, 3, \) and 4 as display in Figure 1.

**Rosen-Morse Potential:** For \( b = d = 0 \), the Rosen-Morse Potential is obtained as given in Equation (13). We plotted the variation of Rosen-Morse \( V(r) \) with \( r \) for \( a = -1, V_0 = 1 \text{MeV}, c = 2 \) and \( V_2 = 0.02 \text{MeV} \) with different \( \alpha \) parameters of \( \alpha = 1, 2, 3, \) and 4 in Figure 2. Substituting \( b = d = 0 \) in Equations (38) and (43), we obtain the energy spectrum and the wave function of the Rosen-Morse potential respectively.

**Poschl-Teller Potential:** Poschl-Teller Potential is obtained from the generalized inverted hyperbolic potential by setting \( a = b = c = d = 0 \) and \( c = -c \) as given in Equation (14). The Poschl-Teller potential is plotted as a function of \( r \) for \( c = -2 \) and \( V_2 = 0.02 \text{MeV} \) in Figure 3. Substituting these parameters in the energy equation of Equation (38) and wave function (43), we obtain the desired energy spectrum and the wave function of the Poschl-Teller potential.

**Scarf Potential:** We can deduce the Scarf potential from the generalized inverted hyperbolic potential by setting \( a = c = d = 0 \). We display in Figure 4 the plot of Scarf potential as a function of \( r \) for \( b = 0.05, V_1 = 0.5 \text{MeV} \) with various parameter of \( \alpha = 1, 2, 3, \) and 4. Setting the above limiting values in Equations (38) and (43) we obtain the energy eigen-values and wave function for the Scarf potential respectively.

### 5. Conclusion

The bound state solutions of the Schrödinger equation with a generalized inverted hyperbolic potential have been investigated within the framework of the NU method. Three well-known potential have been deduced from this potential. We discussed the energy spectrum and the wave function of the SE with this potential for an arbitrary \( l \)-state. We also discussed the special cases of the generalized inverted hyperbolic potential: Rosen Morse, Poschl-Teller and Scarf potentials. Finally, we plotted the effective potential as a function of \( r \) for different \( l = 1, 2, 3, \) and 4 as shown in Figure 5.
Figure 5. A variation of the effective potential as a function of \( r \) for \( l = 1, 2, 3 \) and \( 4 \) with \( \alpha = 1 \).

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REFERENCES


