Pedestrian Analysis of Harmonic Plane Wave Propagation in 1D-Periodic Media

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Abstract

The propagation of TE, TM harmonic plane waves impinging on a periodic multilayer film made of a stack of slabs with the same thickness but with alternate constant permittivity is analyzed. To tackle this problem, the same analysis is first performed on only one slab for harmonic plane waves, solutions of the wave equation. The results obtained in this case are generalized to the stack, taking into account the boundary conditions generated at both ends of each slab by the jumps of permittivity. Differential electromagnetic forms are used to get the solutions of Maxwell’s equations.

Keywords: Periodic Slabs, Multilayer Film, TE, TM Waves, Propagation

1. Introduction

The modern approach to harmonic plane wave propagation in periodic materials such as photonic crystals [1,2] relies on the Floquet-Bloch modes [1,2,3] and on a quantum mechanics-like technique. We present here for 1D-periodic media, made of a stack of slabs with alternate but constant permittivity, a less powerful pedestrian technique but providing the explicit expressions of the electromagnetic TM and TE fields. We start with the analysis of a TM plane wave propagation inside an horizontal x,y-slab of thickness a, permittivity $\varepsilon(z)$ and afterwards, the results obtained in this case are transposed to the stack of slabs.

Harmonic plane wave propagation in a multilayer film has been known for a long time [4], the traditional approach being to consider the multiple reflections that take place at the interfaces [4], using for instance the S-matrix propagation technique [5], but because of the permittivity periodicity, we proceed differently dwelling on boundary conditions at both ends of each slab where exists a jump of permittivity. A particular attention is given to evanescent waves because of their interest in meta-materials with negative permittivity and permeability.

In addition, we start this paper with a succinct introduction of electromagnetic differential forms [6,7] more efficient than the conventional formalism to tackle the kind of problems to be discussed here. We only use the strong solutions of Maxwell’s equations supplied by the differential-form formulation so that we have no need of a computational tool as required by the weak solutions [8].

2. Differential-form Formulation of Maxwell’s Equations

We work with the subscript $j,k,l$ taking the values 1,2,3 associated respectively to the coordinates x,y,z. The summation convention is used and $\varepsilon_{jkl}$ is the antisymmetric Levi-Civita tensor.

The 3D differential-form formulation of Maxwell’s equations is [6,7] in absence of charge and current with the exterior derivative operator $d$ and $\tau = ct$

$$d \wedge E + \partial_i B = 0 \quad a) \quad d \wedge B = 0 \quad b)$$

$$d \wedge H + \partial_i D = 0 \quad a) \quad d \wedge D = 0 \quad b)$$

In these relations $d = dx_i \partial_i$, $E$, $H$ are the 1-forms $(E_i, H_i) dx_i$ (2a)

and $(B, D)$ the 2-forms

$$\left( B, D \right) = \frac{1}{2} \varepsilon_{jkl} \left( B_j, D_k \right) (dx_i \wedge dx_i)$$

(2b)

We consider these equations in a medium with permittivity $\varepsilon(x)$ ($x$ is written for $x$, $y$, $z$) and constant permeability $\mu$.

Then, let $*h$ be the Hodge star operator [6,7]
supplying the permittivity and permeability operators \( *\varepsilon, *\mu \)
\[
*\varepsilon = \varepsilon (r)^* h, *\mu = \mu ^* h
\] (3a)
from which the 2-forms \( D, B \) become
\[
D = *\varepsilon E, B = *\mu H
\] (4)
so that the coefficients of the differential terms in (2b) are
\[
(D_i, B_i) = [\varepsilon (r) E_i, \mu H_i]
\] (4a)

Taking into account (4), the Maxwell Equations (1a) become for harmonic fields \( \exp(\text{i} \omega t) \)
\[
d \wedge *\varepsilon E + \text{i} \omega / \varepsilon c * \mu H = 0, d \wedge H + \text{i} \omega / \varepsilon c (r) E = 0
\] (5)
that is according to (2a,b) and (4a)
\[
(\partial_j E_x - \partial_x E_j + \text{i}(\omega \mu / 2c) \varepsilon_{j,\mu} H_j)(dx_x \wedge dx_j) = 0
\] (6a)
\[
(\partial_j H_x - \partial_x H_j + \text{i}(\omega \varepsilon / 2c) \varepsilon_{j,\varepsilon} E_j)(dx_x \wedge dx_j) = 0
\] (6b)

Finally, a simple calculation gives for the second set (1b) of Maxwell’s equations
\[
\partial_j H_j (dx_x \wedge dx_x \wedge dx_x) = 0,
\]
\[
\partial_j [\varepsilon (r) E_j](dx_x \wedge dx_x \wedge dx_x) = 0
\] (7)

Electromagnetic 2-forms supply weak solutions of Maxwell’s equations by integration on 2D-manifolds, understood as limit over 2D-small simplexes made of triangular elements as used in numerical electromagnetics when physical regions are approximated by finite elements [8]. Strong solutions, the only ones considered here, are obtained by making null the coefficients of the differential terms, they are solutions of conventional Maxwell’s equations but easier to get as shown here below.

### 2.1. TM Field

The wave equation for the magnetic field is obtained by eliminating \( E \) from (5) which gives
\[
d \wedge *\varepsilon^{-1} d \wedge H - \omega^2 c^2 * \mu H = 0
\] (8)
Now, according to (2a) and (3a), the second term on the right hand side of (8) is
\[
\omega^2 c^2 * \mu H = \frac{1}{2} \omega^2 c^2 \mu \varepsilon_{j,\varepsilon} H_j (dx_x \wedge dx_x)
\] (9)
while in the first term
\[
d \wedge H = (\partial_j H_j - \partial_x H_j)(dx_x \wedge dx_x)
\] (10)
and, using the inverse Hodge star operator \( *\varepsilon^{-1} \)
\[
*\varepsilon^{-1} d \wedge H = \varepsilon^{-1} (r) \varepsilon_{j,\varepsilon} \partial_j H_j dx_x
\] (10a)
Then, we get in Appendix A
\[
d \wedge *\varepsilon^{-1} d \wedge H = \frac{1}{2} \varepsilon_{j,\varepsilon} \Psi_j (dx_x \wedge dx_x)
\] (11)
in which with the Laplacian operator \( \Delta = \partial_i \partial^i \)
\[
\Psi_j = \varepsilon^{-1} (r) \partial_j \partial^j H_j + \varepsilon^{-2} (r) \partial_j \partial^i \partial_i \partial^j H_j - \varepsilon^{-2} (r) \partial_j \partial^j \partial^i \partial_i H_j
\] (11a)
Substituting (9) and (11) into (8) gives the differential form of the wave equation
\[
\varepsilon_{j,\varepsilon} \Psi_j (\omega^2 c^2 \mu H_j)(dx_x \wedge dx_x) = 0
\] (12)
Let us now consider the TM field in which \( \phi(z) \) is an arbitrary function
\[
H_y = H_z = 0, H_x = \exp(\text{i} k z \phi(z)
\] (13)
in a medium with permittivity \( \varepsilon(z) \) depending only on \( z \).
A simple look to (12) shows that this equation reduces to
\[
(\Psi_x - \omega^2 c^2 \mu H_y)(dx_x \wedge dx_x) = 0
\] (14)
with the strong solution \( \Psi_x - \omega^2 c^2 \mu H_y = 0 \), that is according to (11a) and (13)
\[
\frac{d}{dz}[\varepsilon^{-1} (z) d\phi(z)] + [\omega^2 c^2 \mu - k^2 \varepsilon^{-1} (z)] \phi(z) = 0
\] (15)
Once obtained the solutions of (15) and consequently \( H_x \) according to (13), the electric field is provided by the 2-form (6b)
\[
E_x (dz \wedge dx) = 0, [\partial_j H_j - \text{i} \omega c^{-1} \varepsilon (z) E_j](dz \wedge dx)
\]
\[
+ [\partial_j H_j + \text{i} \omega c^{-1} \mu (z) E_j](dx \wedge dy) = 0
\] (16)
with the strong solution
\[
E_x = 0,
\]
\[
E_y = [\text{i} \omega c^{-1} \mu (z)] \partial_y H_z,
\]
\[
E_z = [\text{i} \omega c^{-1} \mu (z)] \partial_z H_y
\] (16a)
which achieves to determine the TM harmonic plane wave \( E_x, E_z, H_x \). From now on, \( \mu = 1 \).

### 2.2. TE Field

The wave equation for the electric field is obtained by eliminating \( H \) from (5)
\[
d \wedge *\mu^{-1} d \wedge E - \omega^2 c^2 * \varepsilon (r) E = 0
\] (17)
According to (2a) and (3a), the second term in (17) is
\[ \alpha^2 c^{-2} \varepsilon(r) E = \frac{1}{2} \alpha^2 c^{-2} \varepsilon(r) \varepsilon_{\mu \nu} E_{\mu} \left( dx_\mu \wedge dy_\nu \right) \] (18)

while in the first term

\[ d \wedge E = (\partial_j E - \partial_j E_j) \left( dx_j \wedge dx_k \right) \] (19)

and using the inverse Hodge star operator \( \ast \mu^{-1} \)

\[ \ast \mu^{-1} d \wedge E = \mu^{-1} \varepsilon_{\mu \nu} \partial_\mu E dx_j \] (19a)

Then

\[ d \wedge \ast \mu^{-1} d \wedge E = \frac{1}{2} \partial_\mu dx_j \left( \mu^{-1} \varepsilon_{\mu \nu} \partial_\mu E dx_j \right) \]

\[ = \frac{1}{2} \epsilon_{\mu \nu} \partial_\mu \partial_\mu E \left( dx_\mu \wedge dx_\nu \right) \] (20)

so that with \( A_2, A_3 \) deduced from \( A_1 \) by a circular permutation of x,y,z; a simple calculation gives

\[ d \wedge \ast \mu^{-1} d \wedge E = A_1 (x, y, z) + A_2 (y, z, x) + A_3 (z, x, y) \] (21)

with

\[ A_i (x, y, z) = \Phi_i (x, y, z) (dx \wedge dy) \] (22)

in which

\[ \Phi_i (x, y, z) = \mu^{-1} \left[ \partial_i \left( \partial_j E_j + \partial_j E_j \right) - \partial_i E_i - \partial_i \left( \partial_j E_j \varepsilon_j \right) \right] \] (22a)

and, taking into account the divergence Equation (7)

\[ \partial_j E_j + \partial_j E_j = 0 \], this expression be comes

\[ \Phi_i (x, y, z) = -\mu^{-1} \left[ \Delta E_i + \partial_i \left( \partial_j E_j \varepsilon_j \right) \right] \] (23)

supplying \( \Phi_0, \Phi_1 \) by a circular permutation of x, y, z so that according to (21), (22):

\[ d \wedge \ast \mu^{-1} d \wedge E = \frac{1}{2} \epsilon_{\mu \nu} \Phi_i (dx_k \wedge dx_j) \] (24)

Substituting (18) and (24) into (17) gives the differential form of the wave equation for the electric field

\[ \epsilon_{\mu \nu} \left[ \Phi_j - \alpha^2 c^{-2} \varepsilon(r) \varepsilon(r) E_{\mu} \right] (dx_\mu \wedge dx_\nu) = 0 \] (25)

For the TE wave

\[ E_y = E_z = 0, \ E_x = \exp(ik_x y) \phi(z) \] (26)

in a medium with permittivity \( \varepsilon(z) \) depending only on \( z \), \( E_j \partial_j \varepsilon = 0 \) in (23); Equation (25) reduces to

\[ \left[ \Delta E_{\mu} + \alpha^2 c^{-2} \varepsilon(z) E_{\mu} \right] (dy \wedge dz) = 0 \] (27)

and, taking into account (26), this differential form has the strong solution

\[ \partial_z^2 \phi(z) + \left[ \alpha^2 c^{-2} \varepsilon(z) \right] - k^2 \phi(z) = 0 \] (28)

The components \( H_j, H_z \) of this TE field are in terms of \( E_x \):

\[ \partial_z E_x + i \alpha \mu \mu / c H_x = 0, \ \partial_z E_x + i \alpha \mu \mu / c H_x = 0 \] (29)

3. Harmonic Plane Waves in a 1D-Periodic Medium

3.1. TM Plane Wave in a Slab \((0 < z < a)\)

Suppose that the plane wave (13), (16a) with \( \phi(z) = \exp(ikz) \) impinges of the \( z = 0 \) face of an horizontal slab of thickness \( a \) and constant permittivity \( \varepsilon_i \) endowed in a medium with permit tivity \( \varepsilon_0 \) for \( z < 0 \) and \( \varepsilon_2 \) for \( z > a \), that is

\[ \varepsilon(z) = \begin{cases} \varepsilon_0 : z < 0; \\ \varepsilon_1 : 0 < z < a \\ \varepsilon_2 : z > a \end{cases} \] (30)

So, according to (13) and (16a), the components of the incident and reflected fields in the half space \( z < 0 \) are \( (i = \sqrt{-1}) \)

\[ H_y = A \exp(\mu r), \]

\[ E_x = -(ck_x A / \omega e_0) \exp(\mu r), \]

\[ E_y = -(ck_x A / \omega e_0) \exp(\mu r), \]

\[ r_x = k_{z} y + k_{x} z, \]

\[ k_{y}^2 + k_{z}^2 = \alpha^2 c^{-2} e_0, \] (31a)

\[ H_y = A \exp(\mu r), \]

\[ H_x = -(ck_x A / \omega e_0) \exp(\mu r), \]

\[ H_y = -(ck_x A / \omega e_0) \exp(\mu r), \]

\[ r_x = k_{z} y + k_{x} z, \]

\[ k_{y}^2 + k_{z}^2 = \alpha^2 c^{-2} e_0, \] (32a)

Now, in the other two intervals (30) where \( e_j \), \( j = 1, 2 \) is constant, Equation (15) reduces to

\[ d^2 \phi/dz^2 + \Omega^2 \phi = 0, \]

\[ \Omega_j = \alpha^2 c^{-2} \varepsilon_j - k^2 \] (33)

and, we shall consider the three situations

\[ \Omega^2_1 > 0, \ \Omega^2_2 > 0; \ \Omega^2_1 > 0, \ \Omega^2_2 < 0; \ \Omega^2_1 < 0, \ \Omega^2_2 > 0 \] (33)

1) taking into account (30), the solutions of Equation (33) in the first situation \( \Omega^2_j > 0, \ j = 1, 2 \), are with amplitudes \( A, B, A_t \)

\[ \phi(z) = A \exp(ikz) + B \exp(-ikz), \]

\[ k_{y}^2 + k_{z}^2 = \alpha^2 c^{-2} e_0, 0 < z < a \] (34a)
\( \phi(z) = A \exp(ikz) \),  
\( k_y^2 + k_z^2 = \sigma^2 \epsilon^{-2} \epsilon_z, z > a \) \hspace{1cm} (34b)

Remark 1. If the region with permittivity \( \epsilon_o \) above the slab is limited at \( z = 2a \), \( \phi(z) \) is changed into  
\( \phi(z) = A \exp(ikz) + B \exp(-ikz), a < z < 2a \) \hspace{1cm} (34c)

a remark of interest in the next section. Now, taking into account (34a), the components of the TM wave for \( 0 < z < a \) are, according to (13) and (16a)

\[ H_z = A \exp(i\tau) + B \exp(i\tau'), \]

\[ E_z' = -(ck/\omega c_\gamma) \left[ A \exp(\tau) + B \exp(\tau') \right] \]

\[ E_z = \left( ck/\omega c_\gamma \right) \left[ A \exp(\tau) + B \exp(\tau') \right] \]

\[ \tau = k_y y + kz, \]

\[ \tau' = k_y y - kz, \] \hspace{1cm} (35a)

while for \( z > a \), taking into account (34b)

\[ H_z' = A \exp(i\tau), \]

\[ E_z' = -(ck/\omega c_\gamma) A \exp(i\tau), \]

\[ E_z = -(ck/\omega c_\gamma) A \exp(i\tau), \]

\[ \tau = k_y y + k_z, \]

\[ k_y^2 + k_z^2 = \sigma^2 \epsilon^{-2} \epsilon_z, z > a \] \hspace{1cm} (35b)

We now have to take into account the boundary conditions at \( z = 0 \) and \( z = a \), imposed by the continuity of the \( H_x, E_y \) components of the electromagnetic field at permittivity jumps.

Then, according to (31), (32), (35), we get at \( z = 0 \) the two relations

\[ A_1 + A_2 = A + B, k_1 \epsilon_o^{-1} (A_1 - A_2) = \epsilon_1^{-1} (A - B) \] \hspace{1cm} (37a)

while at \( z = a \), taking into account (35), (36) it comes

\[ A \exp(ika) + B \exp(-ika) = A \exp(ika) \]

\[ Ak \epsilon_1^{-1} \exp(ika) - Bk \epsilon_1^{-1} \exp(-ika) = A k \epsilon_1^{-1} \exp(ika) \] \hspace{1cm} (37b)

The four relations (24(a,b)) supply in Appendix B the four amplitudes \( A_1, A_2, B_1, B_2 \) in terms of the incident amplitude \( A_1 \), which achieves to determine the fields (32), (35), (36). These boundary conditions impose no constraint on frequency when all the possible values of \( k_y \) are considered.

2) In the second situation, \( \Omega_y^2 > 0, \Omega_z^2 < 0 \), the component \( k_y \) of the wave vector is pure imaginary and, according to (36a)

\[ \phi(z) = A \exp[-k_y(z - a)], z > a \] \hspace{1cm} (38a)

Remark 2: similarly to the previous remark, for a upper region bounded at \( z = 2a \)

\[ \phi(z) = A \exp[-k_y(z - a)] + B \exp[-k_y(2a - z)], a < z < 2a \] \hspace{1cm} (38a)

So, according to (38), the field above the slab is evanescent, does not propagate and the components (36) \( H_z', E_z' \) become

\[ H_z' = A \exp(ik_y y) \exp[-k_y(z - a)] \]

\[ E_z' = (-ic k_1 c_\gamma) A \exp(ik_y y) \exp[-k_y(z - a)], z > a \] \hspace{1cm} (39)

and, using (35) in \( 0 < z < a \), the boundary conditions at \( z = a \) imply

\[ A_1 = A \exp(ika) + B \exp(-ika), \]

\[ -ik_1 \epsilon_1^{-1} A_1 = -k_1 \epsilon_1^{-1} \left[ A \exp(ika) - B \exp(-ika) \right] \] \hspace{1cm} (40)

from which we get

\[ 2A_1 = \left( 1 - ik c_1 / k_1 c_\gamma \right) \exp(ika) \exp(-ika) \] \hspace{1cm} (41)

\[ + (1 + ik c_1 / k_1 c_\gamma) B \exp(-ika) \]

\[ + (1 + ik c_1 / k_1 c_\gamma) A \exp(ika) \exp(-ika) \] \hspace{1cm} (42)

These relations have to be made complete with the boundary conditions (37a) at \( z = 0 \) from which \( A, B \) are provided in terms of \( A_1, A_2 \), so that according to (41), \( A, B, A_1, A_2 \) are obtained in terms of \( A_1 \). Explicitly, substituting into (41) the relations (B.1) of Appendix B, we get with the \( \gamma \) functions \( \gamma_\pm = 1/2 (1 \pm \epsilon c_1 / k c_\gamma) \)

\[ (1 + ik c_1 / k_1 c_\gamma) \left[ \gamma_+ A_1 + \gamma_- A_2 \right] \exp(ika) \]

\[ + (1 + ik c_1 / k_1 c_\gamma) \left[ \gamma_+ A_1 + \gamma_- A_2 \right] \exp(-ika) = 0 \] \hspace{1cm} (43)

supplying \( A_1 \) from which the amplitudes \( A, B, A, A_2 \) are obtained.

In this case also, the boundary conditions impose no constraint on \( k_y \) when \( k_y \) takes all the possible values but, the situation is different when \( k_y = 0 \) for propagation in the \( z \)-direction. Then \( k_y = k \) and \( \omega = ck / \sqrt{\epsilon}, j = 1, 2 \), with \( j = 1 \) in \( 0 < z < a \) and \( j = 2 \) for \( z > a \) so that if \( \epsilon_o > \epsilon_i \) there is a frequency band gap in the interval \( (ck / \sqrt{\epsilon_i}, ck / \sqrt{\epsilon_o}) \).

3) Finally in the third situation: \( \Omega_y^2 < 0, \Omega_z^2 > 0 \), the TM plane wave (13), (16a) generates in the slab an evanescent plane wave with the components \( H_z', E_z' \) deduced from (35), (35a) by changing \( k \) into \( ik \) and it comes (for simplification, the coefficient \( \exp(ik_y y) \) pre-
sent in each component is discarded)

\[ H_z = A \exp(-kz) + B \exp[-k(a-z)] \]

\[ E_z = (-ik/\omega \varepsilon) \left\{ A \exp(-kz) - B \exp[-k(a-z)] \right\}, \quad (44) \]

\[ 0 < z < a \]

Then, according to (31), (32), (44), the boundary conditions at \( z = 0 \) supply the two relations

\[ \begin{align*}
A + A' &= A + B \exp(-ka), \\
k_0 A' &= -ik_0 A \exp(-ka)
\end{align*} \quad (45) \]

from which we get

\[ \begin{align*}
2A &= (1 - ik_0/k_0) A + (1 + ik_0/k_0) B, \\
2A' &= (1 + ik_0/k_0) A + (1 - ik_0/k_0) B
\end{align*} \quad (46) \]

Now, at \( z = a \), the boundary conditions imply

\[ \begin{align*}
A \exp(-ka) + B &= A \exp(ikz), \\
-ik_0 A' \exp(-ka) &= -k_0 A \exp(ikz)
\end{align*} \quad (47) \]

which gives

\[ A \exp(-ka)(k_0/k_0 + ik_0/k_0) + B(k_0/k_0 - ik_0/k_0) = 0 \quad (48) \]

together with (46), this last relation supplies \( A, B \) in terms of \( A' \) which achieves to determine \( A, A' \) according to (46), (47).

Of course, if the region with permittivity \( \varepsilon_2 \) is bounded at \( z = 2a \), \( \phi = A \exp(ikz) \) is changed into \( \phi = A \exp(ikz) + B \exp(-ikz) \). In this situation also, there is a frequency band gap if \( \alpha_1 > \alpha_2 \).

### 3.2. TM Wave Propagation in a Periodic Multilayer Film

We now consider a stack of slabs with each the same thickness \( a \) but with an alternate value \( \varepsilon(z) \) constant inside the slabs.

\[ \varepsilon(z) = \begin{cases} 
\varepsilon_1: & 2ma < z < (2m+1)a \\
\varepsilon_2: & (2m+1)a < z < (2m+2)a
\end{cases} \quad m = 0, 1, 2, \cdots \quad (49) \]

so that \( \varepsilon(z + 2ma) = \varepsilon(z) \).

The TM plane wave (13) (16a) is assumed to impinge on the \( z = 0 \) face of this stack (\( m = 0 \)) and the following notations are used for the field \( \phi(z) \).

\[ \phi(z) = \begin{cases} 
\phi_{2m}(z): & 2ma < z < (2m+1)a \\
\phi_{2m+1}(z): & (2m+1)a < z < (2m+2)a
\end{cases} \quad (50) \]

The equation (15) becomes inside the slabs since \( d\varepsilon/dz = 0 \) and \( \Omega_i^2 = \omega^2 c^2 \varepsilon_j - k_j^2, \quad j = 1, 2 \)
(C.3), (C.6) of Appendix C for $m = 0, 1, 2, \cdots, M$. Then, $A_i \exp\{ik_x y + ik_z z\}$ being the field outside the stack for $z > 2Ma$, the boundary conditions are similarly to (37b), $\varepsilon_0$ being the permittivity outside the stack

$$A_{2M} \exp\{2Mik_a\} + B_{2M} \exp\{-2Mik_a\} = A_i \exp\{2Mik_a\}$$

$$A_{2M} \exp\{2Mik_a\} - B_{2M} \exp\{-2Mik_a\} = A_i \exp\{2Mik_a\}$$

(56)

from which $(A_i, A_i)$ are obtained in terms of $A_i$ which achieves to determine the amplitudes $(A_{2m}, B_{2m}), (A_{2m+1}, B_{2m+1}) m = 1, 2, \cdots M - 1$

$$\phi_{2m-1}(z) = A_{2m-1} \exp\{-k_z\left\{z - (2m - 1)a\right\}\} + B_{2m-1} \exp\{-k_z(2ma - z)\}$$

(57)

and the boundary conditions (54) at $z = 2ma$ are changed into

$$A_{2m} \exp\{2mik_z\} + B_{2m} \exp\{-2mik_z\} = A_{2m-1} \exp\{2mik_z\} + B_{2m-1}$$

$$A_{2m} \exp\{2mik_z\} - B_{2m} \exp\{-2mik_z\} = (ik_z/\varepsilon_z) [A_{2m-1} \exp\{2mik_z\} - B_{2m-1}]$$

(58)

from which we get $(A_{2m}, B_{2m})$ in terms of $(A_{2m-1}, B_{2m-1})$

$$2A_{2m} \exp\{2mik_z\} = (1 + ik_z/\varepsilon_z/k_z) \exp\{-k_z a\} A_{2m-1} + (1 - ik_z/\varepsilon_z/k_z) B_{2m-1}$$

$$2B_{2m} \exp\{-2mik_z\} = (1 - ik_z/\varepsilon_z/k_z) \exp\{-k_z a\} A_{2m-1} + (1 + ik_z/\varepsilon_z/k_z) B_{2m-1}$$

(59)

which takes the place of (C.3) in Appendix C while in (C.6) $k_z$ has to be changed into $ik_z$ to get $(A_{2m+1}, B_{2m+1})$ in terms of $(A_{2m}, B_{2m})$.

The relations (55), (56) achieve to determine the amplitudes $(A_{2m-1}, B_{2m-1}), (A_{2m}, B_{2m}), (A_{2m+1}, B_{2m+1}), m = 1, 2, \cdots M$ in terms of the amplitude $A_i$ of the incident field. If $\varepsilon_z > \varepsilon_i$, there exists a frequency band gap for propagation in the z-direction as discussed in Sec.3.1.

3) In the third situation $\Omega_z^2 > 0, \Omega_z^3 < 0$, $k_z$ is pure imaginary and in the intervals $2ma < z < (2m + 1)a$ and $\phi_{2m}(z)$ becomes

$$\phi_{2m}(z) = A_m \exp\{-k_z\left\{z - 2ma\right\}\} + B_{2m} \exp\{-k_z\left\{(2m + 1)a - z\right\}\}$$

(60)

and the boundary conditions at $z = (2m + 1)a$ are

$$A_{2m+1} \exp\{(2m + 1)ik_z\} + B_{2m+1} \exp\{-(2m + 1)ik_z\} = A_{2m} \exp\{-(k_z a) + B_{2m}$

$$A_{2m+1} \exp\{(2m + 1)ik_z\} - B_{2m+1} \exp\{-(2m + 1)ik_z\} = (ik_z/\varepsilon_z) [A_{2m} \exp\{(2m + 1)ik_z\} - B_{2m}]$$

(61a)

(61b)

from which we get $(A_{2m+1}, B_{2m+1})$ in terms of $(A_{2m}, B_{2m}) m = 0, 1, \cdots M$.

$$2A_{2m+1} \exp\{(2m + 1)ik_z\} = (1 + ik_z/\varepsilon_z/k_z) \exp\{-k_z a\} A_{2m} + (1 - ik_z/\varepsilon_z/k_z) B_{2m}$$

$$2B_{2m+1} \exp\{-(2m + 1)ik_z\} = (1 - ik_z/\varepsilon_z/k_z) \exp\{-k_z a\} A_{2m} + (1 + ik_z/\varepsilon_z/k_z) B_{2m}$$

(62)

These relations take the place of (C.6) while to get $(A_{2m}, B_{2m})$ in terms of $(A_{2m-1}, B_{2m-1})$, one has to change $k_z$ into $ik_z$ in (C.3).

It is implicitly assumed that the first and last slabs of the stack have the permittivity $\varepsilon_i$. Then, the boundary conditions (45) at $z = 0$ become with evident notations

$$A_i = A_0 + B_0 \exp\{-k_z a\}$$

$$k_z/\varepsilon_0 (A_i - A_0) = -(ik_z/\varepsilon_z) [A_0 - B_0 \exp\{-k_z a\}]$$

(63)

from which we get

$$A_0 = A_i (1 + ik_z/\varepsilon_i/k_z) A_i + (1 - ik_z/\varepsilon_z/k_z) A_i$$

$$B_0 = (1 - ik_z/\varepsilon_z/k_z) A_i + (1 + ik_z/\varepsilon_z/k_z) A_i$$

(50a)

Finally, at $z = 2Ma$, output of the stack in a medium with permittivity $\varepsilon_0$, we have similarly to (34)

$$A_{2M} \exp\{-k_z a\} + B_{2M} = A_i \exp\{ik_z a\}$$

$$\{ik_z/\varepsilon_i [A_{2M} \exp\{-k_z a\} - B_{2M}] = (-k_z/\varepsilon_0) A_i \exp\{ik_z a\}$$

(64)

Using (62), (C.3) and (63a), $(A_{2m}, B_{2m})$ are obtained.
in terms of \((A, A)\), so that the relations (51) supply \(A, B\) which achieves to determine the amplitudes \(A_{2m-1}, B_{2m-1}\), \((A_{2m}, B_{2m})\) and \((A_{2m+1}, B_{2m+1})\) \(m = 1, 2, \ldots, M - 1\) by running down the sequence of amplitudes.

There is also a frequency band gap if \(c_1 > c_2\) for vertical propagation.

**Remark 3:** According to (36) the permittivity is periodic in the multilayer film \(\varepsilon(z) = \varepsilon(z + 2\alpha)\) so that, since \(\Omega^2(z) = \omega^2 c^2 \varepsilon(z) - k_z^2 > 0\) the component

\[ k_z = \Omega(z) \]

of the wave vector is periodic \(k_z = k_{z,2\alpha}\) . So, if \(\phi(z) = \exp(ik_zz)\) is a plane wave solution of the differential equation

\[ \left[ \frac{d^2}{dz^2} + \Omega^2(z) \right] \phi(z) = 0 \],

then:

\[ \phi(z + 2\alpha) = \exp[i(k_{z,2\alpha}(z + 2\alpha))] = \exp[i(k_z(z + 2\alpha))] \] (65)

and \(\phi(z)\) is periodic if \(2k_z = 2n\pi, n = 0, 1, 2\ldots\)

### 3.3. TE Plane Wave Propagation

For the TE plane wave (26), (29) impinging on a slab with the permittivity set (30), the wave Equation (28) reduces to Equation (33) so that all the calculations of Secs. (2.1), (2.2) hold valid for the TE field. One has just to change \(H_x, E_y, E_z\) into \(E_x, H_y, H_z\).

### 4. Discussion

As noticed in the introduction, the modern approach to harmonic plane wave propagation in multilayered films, made of a stack of slabs with different but constant permittivity, reposes on two principal techniques, both requiring important computational tools. The first one, mainly interested in frequency bands available for propagation, mixed solid state physics (Floquet-Bloch modes) and quantum mechanics (eigenstates of hermitian operators). The second method [5] working with the S-matrix propagation technique, an improved version of the T-matrix algorithm to analyse plane wave scattering from gratings, is mainly interested in the behaviour of high intensity lasers impinging on gratings made of 1D-photonic crystals. So, any comparison between the results supplied by the two techniques, both depending strongly on the performances of their computational tools, is difficult.

The analysis performed in Sec.2 of TM and TE waves both polarized along ox, with fields of the type \(\exp(ik_zz)\phi(z)\), suggests three comments. When these waves propagate in a slab with the permittivity set (30).

First, in this situation, the Maxwell equations have the same solutions for TM and TE waves. Second, as discussed in Remark 3, the function \(\phi(z)\) is not assumed periodic a-priori, leading to solutions of Maxwell’s equations not taken into account in [1] so that, one may ask whether these extra-solutions play some role in propagation, specially for the frequency band gaps. What kind of incident fields is able to generate these aperiodic solutions? The third point concerns the existence of analytical expressions for the electromagnetic field amplitudes in each slab of the stack so that even if numerical calculations are needed to get them, they will not take the importance they have in [1] and [5].

It is easy to transpose the analysis of Sec.3 to harmonic plane waves \(\exp(ik_zy + ik_zz)\) propagating in a photonic meta-film [9] made of alternate slabs and meta-slabs. It has been proved [10,11] that in a lossless meta-material with \(\varepsilon < 0, \mu < 0\), the solutions of Maxwell’s equations have a classical behaviour provided that the refractive index \(n\) and the impedance \(Z\) are defined as \(n = -((\omega\mu)/\varepsilon)^{1/2}\) and \(Z = (\mu/\varepsilon)^{1/2}\). As a consequence, when the TM plane wave (13), (16a) impinges on a meta-slab, one has just to change in (35), (35a) \(k\) and \(c_1\) into \(-k\) and \(-n_1^{1/2}\). Taking into account these conditions, the amplitudes of the electromagnetic field inside and outside the metaslab are still supplied in Appendix B suggesting that only minor differences exist for harmonic plane wave propagation in films and meta-films. Of course, the same properties hold valid for TE plane waves.

This result is confirmed in Appendix D where TM wave propagation in a two layered film with slabs made of dielectric or meta-dielectric is analyzed. The situation is particularly interesting when \(\omega^2 n^2 c^2 - k_z^2 < 0\) so that \(k_z\) is pure imaginary. Then, the two layered film behaves as a deforming mirror with a conjugate complex distortion factor for slabs and meta-slabs. This analysis could be generalized to a stack of alternate slabs \(2ma < z < (2(m+1)a)\) and meta-slabs \(\{(2m+1)a < z < (2(m+2)a)\text{ when }\Omega^2 < 0\}

### 5. References


Appendix A

Taking into account (10a), the expression

\[ A = d \wedge e^{-1} d \wedge H \]

becomes

\[ A = \frac{1}{2} dx \wedge \partial_x + e^{-1} \epsilon_{jk} \partial_j H_i dx_j \]  \hspace{1cm} \text{(A.1)}

and, with \( A_x, A_y \) deduced from \( A \) by a circular permutation of \( x, y, z \) a simple calculation gives

\[ A = A_x(x, y, z) + A_y(y, z, x) + A_z(z, x, y) \]  \hspace{1cm} \text{(A.2)}

with

\[ A_i(x, y, z) = \Psi_z(x, y, z)(dx \wedge dy) \]  \hspace{1cm} \text{(A.3)}

in which

\[ \Psi_z(x, y, z) = e^{-1} \left[ \partial_z (\partial_z H_x + \partial_x H_y) - \partial_{zz}^2 H_z - \partial_z \partial_z H_z \right] + e^{-2} \left[ \partial_z \partial_z H_z + \partial_z \partial_z H_x \right] \]

\[ \partial_z H_x = 0 \]  \hspace{1cm} \text{the first term of (A.4)}

Taking into account these results, \( \Psi_z(x, y, z) \) has a simple expression in terms of the Laplacian and nabla operators \( \Delta, \nabla \)

\[ \Psi_z(x, y, z) = -e^{-1} \Delta H_z + e^{-2} \nabla \partial_z H_z - e^{-2} \nabla e \cdot \partial_z H \]  \hspace{1cm} \text{(A.5)}

so that

\[ A_i(x, y, z) = \left[ -e^{-1} \Delta H_z + e^{-2} \nabla \partial_z H_z \right] dx \wedge dy \]  \hspace{1cm} \text{(A.6)}

Substituting (A.6) into (A.2) gives finally with the ad hoc circular permutations

\[ A = \left[ -e^{-1} \Delta H_z + e^{-2} \nabla \partial_z H_z \right] (dx \wedge dy) \]

\[ + \left[ -e^{-1} \Delta H_z + e^{-2} \nabla \partial_z H_z \right] (dy \wedge dx) \]

\[ + \left[ -e^{-1} \Delta H_z + e^{-2} \nabla \partial_z H_z \right] (dz \wedge dx) \]  \hspace{1cm} \text{(A.7)}

Appendix B

We get from (37a) and (37b)

\[ 2(k_1/e_1) \beta_1^* B_{2m} = \beta_1^* (k_1/e_1 - k_2/e_2) A_{2m-1} + \beta_1^* (k_1/e_1 + k_2/e_2) B_{2m-1} \]

\[ 2(k_1/e_1) \beta_1^* A_{2m} = \beta_1^* (k_1/e_1 + k_2/e_2) A_{2m-1} + \beta_1^* (k_1/e_1 - k_2/e_2) B_{2m-1} \]  \hspace{1cm} \text{(C.3)}
Similarly, according to (52) and (53a), the boundary conditions at \( z = (2m + 1) a \) are

\[
\delta_n^i A_{2m} + \delta_n^r B_{2m} = \delta_n^i A_{2m+1} + \delta_n^r B_{2m+1}
\]

\[
(k_1/e_1) \left[ \delta_n^i A_{2m+1} - \delta_n^r B_{2m+1} \right] = (k_2/e_2) \left[ \delta_n^i A_{2m} - \delta_n^r B_{2m} \right]
\]

It is real or not an d \( n \) positive implying,

\[
\delta_n^i = \exp\left[ (2m + 1) i k_1 \right],
\]

\[
\delta_n^r = \exp\left[ (2m + 1) i k_2 \right]
\]

The relation (C.4) supplies \( (A_{2m+1}, B_{2m+1}) \) in terms of \( (A_{2m}, B_{2m}) \)

\[
2(k_1/e_1) \delta_n^r A_{2m} = \delta_n^r (k_1/e_1 + k_2/e_2) A_{2m} + \delta_n^r (k_1/e_1 - k_2/e_2) B_{2m}
\]

\[
2(k_2/e_2) \delta_n^r B_{2m} = \delta_n^r (k_1/e_1 - k_2/e_2) A_{2m} + \delta_n^r (k_1/e_1 + k_2/e_2) B_{2m}
\]

\[(C.4)\]

**Appendix D**

**TM wave propagation in a two layered film**

We consider the propagation of a TM harmonic plane wave in a two layered film made of two slabs of thickness a with permittivity-permeability couples \((\varepsilon_0, \mu_0)\) outside the film, \((\varepsilon_1, \mu_1), (\varepsilon_2, \mu_2)\) in the first and second slab respectively. Then, the refractive index \(n_j = \pm (\varepsilon_j/\mu_j)^{1/2}\) is 0, 1, 2 , the plus sign for \( \varepsilon_j > 0, \mu_j > 0 \), the minus sign for \( \varepsilon_j < 0, \mu_j < 0 \).

For the TM wave (13), (16a), the components \(H_x, E_y\) of the electromagnetic field have the expressions (31)-(36) in which we use the following notations

\[
k = \omega n \rho/c, \quad n = (\varepsilon \mu)^{1/2},
\]

\[
\rho = \left[ 1 - k_2^2 c^2 / \omega^2 n^2 \right]^{1/2},
\]

\[
v = n \rho
\]

\[(D.1)\]

\(\rho\) being either positive or pure imaginary. In addition, the coefficient \(\exp(ik_x, y)\) that appears in each component is deleted. Different situations exist according to \(\rho\) is real or not and \( n \) positive implying \( k > 0 \), \( v > 0 \) or negative with \( k < 0, v < 0 \).

Then, assuming \( \rho_j > 0, n_j > 0 \), \( H_x, E_y\) have the following expressions \((j = 0, 1, 2)\)

\[
z < 0, k_1 = \omega m \rho_0/c : \text{incident and reflected fields}
\]

\[
A_x \exp(ik_x a) + B_x \exp(-ik_x a) = A_x \exp(ik_x a) + B_x \exp(-ik_x a)
\]

\[
1/v_x \left[ A_x \exp(ik_x a) - B_x \exp(-ik_x a) \right] = 1/v_x \left[ A_x \exp(ik_x a) - B_x \exp(-ik_x a) \right]
\]

\[(D.2)\]

\[
z = a:
\]

\[
A_x \exp(2ik_x a) + B_x \exp(-2ik_x a) = A_x \exp(2ik_x a)
\]

\[
1/v_x \left[ A_x \exp(ik_x a) - B_x \exp(-ik_x a) \right] = 1/v_x A_x \exp(ik_x a)
\]

\[(D.3)\]

The fields (D.3), (D.4) are invariant under the inversions \((k, n, A, B) \rightarrow (-k, -n, B, A)\) corresponding to the exchange slab \( \Rightarrow \) meta-slab so that it does not matter whether the slabs are made of dielectric or meta-dielectric, the solutions of the equations (D.6)-(D.8) will be the same in any case.

The fields (D.3), (D.4) are invariant under the inversions \((k, n, A, B) \rightarrow (-k, -n, B, A)\) corresponding to the exchange slab \( \Rightarrow \) meta-slab so that it does not matter whether the slabs are made of dielectric or meta-dielectric, the solutions of the equations (D.6)-(D.8) will be the same in any case.

We get at once from (D.8) in terms of an arbitrary amplitude \(X\)

\[
A_2 = (v + v_x) \exp(-ik_x a) X
\]

\[
B_2 = (v_0 - v_x) \exp(ik_x a) X
\]

Substituting (D.9) into (D.7) gives

\[
A_x \exp(ik_x a) + B_x \exp(-ik_x a) = A_x \exp(2ik_x a)
\]

\[
1/v_x \left[ A_x \exp(ik_x a) - B_x \exp(-ik_x a) \right] = 1/v_x A_x \exp(ik_x a)
\]

\[(D.8)\]
\[ A_i \exp(ik_1a) + B_i \exp(-ik_1a) = [(v_0 + v_2) \exp(-i(k_2a)) + (v_0 - v_2) \exp(i(k_2a))]X \]

\[ 1/v_1 \left[ A_i \exp(ik_1a) - B_i \exp(-ik_1a) \right] = 1/v_2 \left[ (v_0 + v_2) \exp(-i(k_2a)) - (v_0 - v_2) \exp(i(k_2a)) \right]X \]

(D.10)

from which we get

\[ 2A_i = \gamma(k_1, v_1; k_2, v_2)X, \quad 2B_i = \gamma(-k_1, -v_1; k_2, v_2)X \]

(D.11)

\[ \gamma(k_1, v_1; k_2, v_2) = \alpha(v_1, v_2) \exp[-i(k_1 + k_2a)] + \alpha(v_1, -v_2) \exp[-i(k_1 - k_2a)] \]

\[ \alpha(v_1, v_2) = (v_0 + v_2)(v_2 + v_1)v_2^{-1} \]

(D.11a)

\[ \alpha(v_1, v_2) = (v_0 + v_2)(v_2 - v_1)v_2^{-1} \]

(D.11b)

But we get from (D.6)

\[ 2A_i = \left(1 + v_0/v_1\right)A_i + \left(1 - v_0/v_1\right)B_i \quad \text{a)} \quad 2A_i = \left(1 - v_0/v_1\right)A_i + \left(1 + v_0/v_1\right)B_i \quad \text{b)} \]

(D.12)

and, substituting (D.11) into (D.12a) gives

\[ X = 4\left[(1 + v_0/v_1)\gamma(k_1, v_1; k_2, v_2) + (1 - v_0/v_1)\gamma(-k_1, -v_1; k_2, v_2)\right]^{-1}A_i \]

(D.13)

Once \( X \) obtained from (D.13), the relations (D.11) and (D.12b) supply respectively \((A_i, B_i)\) and \(A_i\). Substituting (D.13) into (D.9) gives \((A_i, B_i)\) and finally \(A_i\) is obtained from the first relation (D.8) which achieves to determine the amplitudes of the TM harmonic plane wave in the two layered film, a result that does not depend, as said earlier, whether one has to deal with slabs or meta-slabs.

We now assume that \( \rho \) in the first slab \( 0 < z < a \) is pure imaginary giving birth to an evanescent wave so that since \( k_1, v_1 \) are changed into \( ik_1 - iv_1 \), the components \( H_z^0, E_z^0 \) in this slab become with \( k_1, v_1 > 0 \)

\[ A_i \exp(-k_1z) + B_i = A_i \exp(ik_1a) + B_i \exp(-ik_1a) \]

\[ i/v_1 \left[ A_i \exp(-(k_1a)) - B_i \exp(ik_1a) \right] = 1/v_2 \left[ A_i \exp(ik_1a) - B_i \exp(-ik_1a) \right] \]

(D.16)

Substituting (D.9) valid for a slab or a meta-slab, into (D.16) gives

\[ 2A_i = \exp(k_1a)\gamma^*(v_1, k_2, v_2), \quad (D.17) \]

\[ 2B_i = \gamma^*(-v_1, k_2, v_2) \]

\[ \gamma^*(v_1, k_2, v_2) = \alpha(v_1, v_2) \exp(-ik_2a) \]

\[ + \alpha(v_1, -v_2) \exp(i(k_2a)) \]

(D.17a)

\[ X = 4\left[(1 + iv_0/v_1)\gamma^*(v_1, k_2, v_2) \exp(-k_1a) + (1 - iv_0/v_1)\gamma^*(-v_2, k_2, v_2) \exp(-k_1a)\right]^{-1}A_i \]

(D.17b)

But, we get from (D.15)

\[ 2A_i = \left(1 + iv_0/v_1\right)A_i + \left(1 - iv_0/v_1\right)B_i \quad (D.18a) \]

\[ 2A_i = \left(1 - iv_0/v_1\right)A_i + \left(1 + iv_0/v_1\right)B_i \quad (D.18b) \]

and, substituting (D.17) into (D.18a) gives

\[ X = 4\left[(1 + iv_0/v_1)\gamma^*(v_1, k_2, v_2) \exp(-k_1a)\right]^{-1} \exp(-k_1a) \]

(D.19)

Once \( X \) obtained from (D.19), the relations (D.17), (D.18b) supply respectively \(A_i, B_i\) and \(A_i\). Substituting (D.19) into (D.9) gives \(A_i, B_i\) and finally \(A_i\) from the first relation (D.8).

Let us explicit the amplitudes \(A_i\). Neglecting the second term that depends on \(\exp(-k_1a)\) in the coefficient of \(A_i\), the expression (D.19) reduces to

\[ A_i = 2\left(1 + iv_0/v_1\right)^{-1}A_i, B_i \simeq 0 \]

(D.21)

and, substituting (D.21) into (D.18) gives
\[ A_i = (1 - iv_0/v_i)(1 + iv_0/v_i)^{-1} A_i \]  

(D.22)

while according to (D.9) and to the first relation (D.8), the amplitudes \( A_z, B_z, A_i \), depend on \( \exp(-k_0 a) \).

Then, we get from (D.22) \( |A_i|^2 = A_i^2 \) implying that the two layered film behaves as a mirror giving a distorted image because of the factor \( (1 - iv_0/v_i)(1 + iv_0/v_i)^{-1} \) changed into its conjugate complex for a meta-slab.